Math 4 Homework 5 Answers

§12.1.

1(c) $\nabla y = \begin{pmatrix} 2-6x_1+x_2 \\ 1+x_1-8x_2 \end{pmatrix} = 0 \iff \begin{pmatrix} 6 & -1 \\ -1 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{47}\begin{pmatrix} 8 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{47}\begin{pmatrix} 17 \\ 8 \end{pmatrix}$

$\therefore$ One stationary value $(x_1, x_2) = \left(\frac{17}{47}, \frac{8}{47}\right)$.

(j) $\nabla y = \begin{pmatrix} 4(x_1-x_2)-4x_1^3 \\ -4(x_1-x_2)-4x_2^3 \end{pmatrix} = 0 \iff x_1 - x_2 = x_1^3 = -x_2^3$

$\iff x_2 = x_1$ and $x_2 = -x_1 \iff (x_1, x_2) = (0,0)$.

3/ $\pi(q_1,q_2) = p_1q_1 + p_2q_2 - C = 60q_1 + 100q_2 - q_2^2 - (q_1+q_2)^2$

$= 60q_1 - q_1^2 + 100q_2 - 2q_2^2 - 2q_1q_2$

$\nabla\pi = \begin{pmatrix} 60-2q_1-2q_2 \\ 100-2q_1-4q_2 \end{pmatrix} = 0 \iff \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 60 \\ 100 \end{pmatrix}$

$\iff \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 60 \\ 100 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 30 \\ 50 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$

$\therefore$ profit max sol^n is $(q_1,q_2) = (10,20)$, $(p_1,p_2) = (60,80)$, $\pi = \$1300$

New $\tilde{\pi} = 10q_1 - q_1^2 + 100q_2 - 2q_2^2 - 2q_1q_2$

$\nabla\tilde{\pi} = \begin{pmatrix} 10-2q_1-2q_2 \\ 100-2q_1-4q_2 \end{pmatrix} = 0 \iff \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 100 \end{pmatrix} = \begin{pmatrix} -40 \\ 45 \end{pmatrix}$

not a physical sol^n, since require $q_1, q_2 \geq 0$.

If $q_1 = 0$ have $\tilde{\pi}(q_2) = 100q_2 - 2q_2^2$ which has max at $q_2 = 25$

$\Rightarrow \tilde{\pi} = 1250$

If $q_2 = 0$ have $\tilde{\pi}(q_1) = 10q_1 - q_1^2$ which has max at $q_1 = 5$

$\Rightarrow \tilde{\pi} = 50$

$\therefore$ max is now \$1250, \$50 down from before.

7/ $\pi(q_1,q_2) = p(q_1+q_2) - C_1 - C_2 = 100q_1 + 100q_2 - q_1^2 - q_2^2 - 2q_1q_2 - 2q_1^2 - 3q_2^2$

$= 100q_1 + 100q_2 - 3q_1^2 - 4q_2^2 - 2q_1q_2$

$\nabla\pi = \begin{pmatrix} 100-6q_1-2q_2 \\ 100-2q_1-8q_2 \end{pmatrix} = 0 \iff \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 100 \end{pmatrix} = \frac{50}{11}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

At equilibrium have $\frac{\partial C_1}{\partial q_1} = 4q_1 = \frac{600}{11}$, $\frac{\partial C_2}{\partial q_2} = 6q_2 = \frac{600}{11}$

These are equal. For constant output q_1+q_2 any increase in q_1 must be accompanied by corresponding decrease in q_2.

In balance, $dq_1 = -dq_2$, Thus $d\pi = d[(100-(q_1+q_2))(q_1+q_2)] - dC_1 - dC_2$

$= -\frac{dC_1}{dq_1}dq_1 - \frac{dC_2}{dq_2}dq_2 = \left(\frac{dC_1}{dq_1} - \frac{dC_2}{dq_2}\right)dq_2$

Hence $d\pi = 0 \iff \frac{dC_1}{dq_1} = \frac{dC_2}{dq_2}$.

§12.2/ 1/c/ $\nabla_2 y = \begin{pmatrix} -6 & 1 \\ 1 & -8 \end{pmatrix}$ which has leading principal minors $-6, \begin{vmatrix} -6 & 1 \\ 1 & -8 \end{vmatrix} = 47$

$\therefore$ -ve def^{n} $\Rightarrow \left(\frac{17}{47}, \frac{8}{47}\right)$ is a maximum

(j) $\nabla_2 y = \begin{pmatrix} 4-12x_1 & -4 \\ -4 & 4-12x_2 \end{pmatrix} \Rightarrow \nabla_2 y(0,0) = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$ which is neither +ve nor -ve definite.

Need to be a little cleverer:

let $x_2 = x_1$, then $y = -2x_1^4$ which has a maximum at 0.

now let $x_2 = -x_1$, then $y = 8x_1^2 - 2x_1^4$ which has a minimum at 0

$\therefore$ (0,0) is a saddle point of y.

4/ Qn in §12.1 has $\nabla_2\pi = \begin{pmatrix} -6 & -2 \\ -2 & -8 \end{pmatrix}$ which is -ve definite

(leading principal minors are $-6, \begin{vmatrix} -6 & -2 \\ -2 & -8 \end{vmatrix} = 44$).

$\therefore$ critical point is a maximum.

6/ $\frac{\partial \pi}{\partial L} = 0 = \frac{\partial \pi}{\partial K}$ reads (with $\alpha = 1/2, \beta = 3/4, w = 2, r = 4, p = 64$)

$$\begin{cases} \frac{1}{2} \cdot 64 \cdot L^{-1/2} K^{3/4} = 2 \\ \frac{3}{4} \cdot 64 \cdot L^{1/2} K^{-1/4} = 4 \end{cases} \Rightarrow \begin{cases} L^{-1/2} K^{3/4} = \frac{1}{16} \\ L^{1/2} K^{-1/4} = \frac{1}{12} \end{cases}$$

$$\Rightarrow \quad K^{1/2} = \frac{1}{16 \cdot 12} \Rightarrow K = \frac{1}{36864}$$

$$\& \; L/K = \frac{16}{12} = \frac{4}{3} \Rightarrow L = \frac{1}{27648}$$

Problem is that the Hessian is not ~~+ve~~ definite, and so this critical point is not (necessarily) a maximum.

(Here $1 - \alpha - \beta < 0$)

§12.3/

1/b/ $y = 2x_1 + x_2 - 3x_1^2 - 4x_2^2 + x_1 x_2$ s.t. $0 \leq x_1, x_2 \leq 1$

$$\nabla y = \begin{pmatrix} 2 - 6x_1 + x_2 \\ 1 - 8x_2 + x_1 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -1 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{47} \begin{pmatrix} 8 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 17/47 \\ 8/47 \end{pmatrix} \quad (\text{note } 0 \leq x_1, x_2 \leq 1)$$

$y\left(\frac{17}{47}, \frac{8}{47}\right) = \frac{21}{47}$ is the max value of y without constraint, since $\nabla_2 y = \begin{pmatrix} -6 & 1 \\ 1 & -8 \end{pmatrix}$ is $-$ve definite.

$\therefore$ don't need to check boundary conditions.

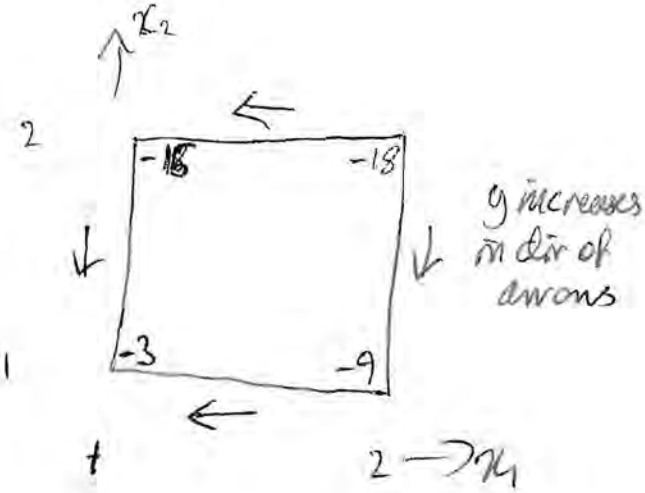
c/ Here, critical point not in range $1 \leq x_1, x_2 \leq 2$.
Check boundaries...

$x_1=1: y = -1+2x_2-4x_2^2$

$x_1=2: y = -8+3x_2-4x_2^2$

$x_2=1: y = -3+3x_1-3x_1^2$

$x_2=2: y = -14+4x_1-3x_1^2$



$\therefore$ max at $(1,1)$ is $y=-3$.

2(a) Now have
$$\begin{cases} p_1 = 200-(q_1+q_2) \\ p_2 = 80-2q_2 \\ C=(q_1+q_2)^2 \\ 0\le q_2 \le 4 \\ 0 \le q_1 \end{cases}$$

$$\pi(q_1,q_2) = p_1q_1+p_2q_2 - C = 200q_1 - q_1^2 - q_1q_2 + 80q_2 - 2q_2^2 - q_1^2 - q_2^2 - 2q_1q_2$$
$$= 200q_1 + 80q_2 - 2q_1^2 - 3q_2^2 - 3q_1q_2$$

$$\therefore \nabla\pi = \begin{pmatrix} 200 - 4q_1 - 3q_2 \\ 80 - 3q_1 - 6q_2 \end{pmatrix} = 0 \iff \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 200 \\ 80 \end{pmatrix}$$
$$= \frac{1}{15}\begin{pmatrix} 6 & -3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 200 \\ 80 \end{pmatrix} = \begin{pmatrix} 64 \\ -56/3 \end{pmatrix}$$

not a physical sol^n, since $q_2<0$.

Check boundaries:

$q_1=0: \pi(q_2) = 80q_2 - 3q_2^2$ has max at $q_2 = \frac{80}{6} = 13\tfrac{1}{3}$

so max in range is $\pi(4) = 272$.

$q_2=0: \pi(q_1) = 200q_1 - 2q_1^2$ has max at $q_1 = \frac{200}{4} = 50$

$\Rightarrow \pi(50) = 5000$

$q_2 = 4: \pi(q_1) = 188q_1 - 2q_1^2 + 272$ has max at $q_1 = \frac{188}{4} = 47$

$\pi(47) = 4690$

$\therefore \max \pi = \pi(50,0) = 5000$.

4/ If max at $x=a$ with $f'(a)=0$, then both parts of thm 12.7

~~i) $(x-a)f'(a)=0$~~ hold simply because $f'(a)=0$

~~ii) $f'(a)\geq$~~ and i.e. (i) $f'(a)\leq 0$ ✓ and $(x-a)f'(a)=0$

(ii) $f'(a)\geq 0$ ✓ and $(b-x)f'(a)=0$

~~For thm~~ Result is identical for $f'(b)=0$.

Result for thm 12.8 is similar, for same reason.

§13.1

1/d/ $y=(x_1+2)(x_2+1)$ s.t. $x_1+x_2-21=0$

$$\mathcal{L}=(x_1+2)(x_2+1)+\lambda(x_1+x_2-21)$$

$$\therefore \left.\begin{aligned}\frac{\partial\mathcal{L}}{\partial x_1}&=x_2+1+\lambda=0\\ \frac{\partial\mathcal{L}}{\partial x_2}&=x_1+2+\lambda=0\\ \frac{\partial\mathcal{L}}{\partial\lambda}&=x_1+x_2-21=0\end{aligned}\right\}\Rightarrow \left.\begin{aligned}x_2+1&=x_1+2\\ x_1+x_2&=21\end{aligned}\right\}\Rightarrow x_1=10, x_2=11.$$

2/a/ $\mathcal{L}=x_1^2+x_2^2+\lambda\left(x_1^2/25+x_2^2/4-1\right)$

$$\left.\begin{aligned}\frac{\partial\mathcal{L}}{\partial x_1}&=2x_1+\frac{2}{25}\lambda x_1=0\\ \frac{\partial\mathcal{L}}{\partial x_2}&=2x_2+\frac{2}{4}\lambda x_2=0\\ \frac{\partial\mathcal{L}}{\partial\lambda}&=x_1^2/25+x_2^2/4-1=0\end{aligned}\right\}\Rightarrow x_1\left(2+\frac{2}{25}\lambda\right)=0=x_2\left(2+\frac{2}{4}\lambda\right)$$

$\Rightarrow x_1=0$ & $\lambda=-4$

or $x_2=0$ & $\lambda=-25$

$$\left.\begin{aligned}x_1=0&\Rightarrow x_2^2=4\Rightarrow y=4\\ x_2=0&\Rightarrow x_1^2=25\Rightarrow y=25\end{aligned}\right\}\Rightarrow \max y=25.$$

2d) $\left(0.2x_1^{-1/2}+0.8x_2^{-1/2}\right)^{-2}=1$ is constraint.

$$\therefore \mathcal{L} = 2x_1+x_2+\lambda\left(\left(0.2x_1^{-1/2}+0.8x_2^{-1/2}\right)^{-2}-1\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2 - 2\cdot\lambda\cdot\left(-\tfrac{1}{2}\right)\cdot 0.2x_1^{-3/2}\left(0.2x_1^{-1/2}+0.8x_2^{-1/2}\right)^{-3} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - 2\lambda\left(-\tfrac{1}{2}\right)\cdot 0.8x_2^{-3/2}\left(0.2x_1^{-1/2}+0.8x_2^{-1/2}\right)^{-3} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \left(0.2x_1^{-1/2}+0.8x_2^{-1/2}\right)^{-2}-1 = 0$$

$$\therefore \quad 2\lambda\left(-\tfrac{1}{2}\right)\left(0.2x_1^{-1/2}+0.8x_2^{-1/2}\right)^{-3} = \frac{2}{0.2}x_1^{3/2} = \frac{1}{0.8}x_2^{3/2}$$

$$\therefore x_2 = 4x_1$$

$$\therefore 1 = \left(0.2x_1^{-1/2}+0.8\,(4x_1)^{-1/2}\right)^{-2}$$

$$= \left(\left(\frac{2}{10}+\frac{8}{10}\cdot\frac{1}{2}\right)x_1^{-1/2}\right)^{-2} = \left(\frac{6}{10}x_1^{-1/2}\right)^{-2} = \left(\frac{5}{3}\right)^2 x_1 = \frac{25}{9}x_1$$

$$\therefore x_1 = \frac{9}{25} \Rightarrow x_2 = \frac{36}{25}$$

$$\therefore \min f = \frac{2\cdot 9+36}{25} = \frac{54}{25}.$$