Math 4 - Summary Notes

July 12, 2011

Chat: what is class about? Mathematics of Economics, NOT Economics! Lecturer NOT Economist - have some training, but reluctant to discuss strict economic models because much of my knowledge comes from non-standard places (actuarial exams, etc). In particular, the IS-LM model in 1st section of is NOT examinable and will only be covered lightly here. If you know about these things from other places, do the questions as they'll help! The first part of the class is Linear Algebra. It gets everywhere and is probably the most applicable area of undergrad math. Depending on yr background it can be applied to traffic networks, cell division models, physical motion, engineering problems, etc., etc. Why so useful? Primarily cos so easy (relatively). Linear problems can mostly be solved exactly using algorithms. Non-linear problems often cannot. Sacrifice correctness in the model for easy of solution. 'Best fitting straight line', etc. Easier then complex curve. Suppose have supply-demand model with two curves (quantity =f(price)) $q^s = f(p)$, $q^d = g(p)$ (f monotone down, g monotone up). Equilibrium price/quantity given where curves meet. Hard to find exactly, but if the curves were straight lines…

NOT a course of economic concepts! Will use some to illustrate the math, but the math is independent.

Economics is ALL models - choosing right from start how to balance accuracy and easy of solution/analysis. Math different: models approximate exactness.

Draw pic! Math fits between Reality and Explanation/Prediction of Reality. Modeling is the first link, solution is the second. Pure math is mostly concerned with the second link - economics equally with both.

Graph of model for economic growth versus interest rates. Different models - which is best?

7 Systems of Linear Equations

7.1 Solving Systems of Linear Equations

Definition 7.1. A linear equation in 2 unknowns is an equation of the form

$$\alpha x + \beta y = \gamma,$$

where α, β, γ are given constants and x, y are variables.
A system of two equations in two unknowns is a pair,

$$\begin{cases} \alpha_1 x + \beta_1 y = \gamma_1, \\ \alpha_2 x + \beta_2 y = \gamma_2. \end{cases}$$

Notice: 6 constants and two variables.
A *solution* to a system is any pair of numbers (x, y) which satisfies both equations in the system.
The *solution set* is the complete collection of solutions.
A system is *consistent* if it has at least one solution and *inconsistent* if it has no solutions.

Graph linear equations as straight lines: example $x + y = 2$, $x - y = 4$, $2x + y = 2$. Can solve all three possible systems of two equations in two unknowns. Plot and solve $((3, -1), (0, 2), (2, -2))$ by inspection.

Pictures: what can happen? Two lines either intersect at a point (most likely), are parallel and distinct (next most likely) or are the same line (least likely). Pics and examples.

$\begin{cases} 3x_1 + 4x_2 = 1 \\ 2x_1 + 2x_2 = 1 \end{cases}$, $\begin{cases} 3x_1 + 4x_2 = 1 \\ 6x_1 + 8x_2 = 2 \end{cases}$, $\begin{cases} 3x_1 + 4x_2 = 1 \\ 6x_1 + 8x_2 = 0 \end{cases}$ - geometry = lines.

We have proved the following:

Theorem 7.2. *The solution set of a system of two equations in two unknowns has either none, one or infinite solutions.*

Suppose add a third equation $\rightarrow$ system of three equations in two unknowns. Most likely no solutions. Picture.

Substitution and elimination

Generally difficult to solve systems graphically if you want numerical solutions rather than just knowing they exist.
Substitution = solve one equation for x (in terms of y), or vice versa, then substitute in. E.g. with above.
Elimination involves adding/subtracting multiples of the equations to eliminate a variable. E.g. . . .
More examples:

Market equilibrium

Examples. 1. Suppose that the supplied and demanded quantities q^s, q^d of a good are related to the price p by the following equations:

$$q^s = -5 + 2p, \qquad q^d = 40 - 3p.$$

Find the equilibrium price and quantity of goods.
Answer: at equilibrium, supply = demand, hence $-5 + 2p = 40 - 3p \Rightarrow 5p = 45 \Rightarrow p = 9 \Rightarrow q^s = q^d = 13$.

2. Two related goods have the following supply and demand equations:

$$\begin{cases} q_1^s = -6 + 4p_1, & q_1^d = 50 - 2p_1 + p_2 \\ q_2^s = -4 + 2p_2, & q_2^s = 30 - 3p_1 + p_2. \end{cases}$$

Find the equilibrium prices and quantities of the two goods.
Answer: supply = demand $q_i^s = q_i^d$ results in two equations:

$$\begin{cases} -6 + 4p_1 = 50 - 2p_1 + p_2 \\ -4 + 2p_2 = 30 - 3p_1 + p_2 \end{cases} \Longrightarrow \begin{cases} 6p_1 - p_2 = 56 \\ 3p_1 + p_2 = 34 \end{cases} \Longrightarrow 9p_1 = 90,$$

so that $p_1 = 10, p_2 = 4, q_1 = 34, q_2 = 4$.

3. Let c be a constant and consider the system of equations

$$\begin{cases} 3x - 2y & = 1, \\ cx + y & = 3. \end{cases}$$

For each value of c find how many solutions the system has and calculate them (in terms of c). Ans: $(x, y) = \frac{1}{3+2c}(7, 9 - c)$ if $c \neq -3/2$. If so, no solutions.

4. As the previous question except now

$$\begin{cases} x + 2y & = 1, \\ cx + cy & = 2. \end{cases}$$

Ans: $(x, y) = \frac{1}{c}(c + 2, -1)$ if $c \neq 0, 2$. No soln if $c = 0$. $c = 2$ gives infinite solns $((x, y) = (1 - 2y, y))$.

7.2 Linear Systems in n-variables

Definition 7.3. Linear Equation in n unknowns:

$$a_1x_1 + \cdots + a_nx_n = b.$$

Here a_i, b are constants, x_i variables (unknowns).
Linear system of m equations in n unknowns ($m \times n$-system)

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A *solution* to a system is any n-tuple $(x_1, \ldots, x_n)$ which satisfies every equation in the system.
The *solution set* is the complete collection of solutions.

Common to use x, y for $m \times 2$ and x, y, z for $m \times 3$ systems.

Definition 7.4. Two linear systems are *equivalent* if they have the same solution set.

(systems don't have to have same number of equations)

$$\begin{cases} 3x_1 + 4x_2 = 1 \\ 2x_1 + 2x_2 = 0 \end{cases} \quad \text{equivalent to} \begin{cases} x_1 = -1 \\ x_2 = 1 \end{cases} \quad \text{equivalent to} \begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 5x_2 = 2 \end{cases} \quad \text{, etc, etc.}$$

Row Operations

Three ways of changing a system to get an equivalent one:

1. Multiply both sides of an equation by a non-zero constant.
2. Add a multiple of one equation to another.
3. Change the order of two equations.

Idea - use row operations to put a system in a nice form so we can read off solutions.

Examples. 1.

$$\begin{cases} 3x + 4y - z = 9 \\ x + 2y - 3z = 9 \\ x - y + z = -2 \end{cases}$$

Get $(1, 1, -2)$.

2.

$$\begin{cases} x + 3y - 2z = 10 \\ x + 2y - 2z = 7 \\ 2x - y - 4z = -1 \end{cases}$$

Get $(1 + 2z, 3, z)$ — infinity of solutions.

Some facts:

- *All* systems of linear equations have either no, one, or infinite numbers of solutions.
- A system is *consistent* if it has at least one solution and *inconsistent* if it has no solutions.
- A system with more equations than unknowns $(m > n)$ is *overdetermined*. If $m < n$ the system is *underdetermined*.
- A consistent underdetermined system has infinite solutions.

Table of under/over/square versus expected numbers of solns.

Matrix Arrays

Definition 7.5. The *augmented matrix* of an $m \times n$ system is the matrix

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array} \right).$$

Don't need to put the line in, but it's often helpful.

Use row operations on these matrices and systems to solve them side by side. See that using augmented matrix beams that can avoid having o write down the variables x_i (saves time and makes mistakes less likely). Know how to go between a linear system and its augmented matrix.

Example:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 4 \\ -x_1 + x_3 = 1 \\ x_1 - x_2 = 0 \end{cases}.$$

Solution: (1/2,1/2,3/2) - use bottom row to eliminate x_1 terms in others first.

Row Echelon Form

Want to do row operations until matrix is a simple as possible (roughly speaking as many zeros as possible with 1's going diagonally (Reduced Row Echelon Form). Every matrix has one and it doesn't matter what row ops you do, will always get to the same RREF. Once got there can either read off the solns or easily tell there are none.

Example: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & -1 & 2 \end{array}\right)$ reduce to row of zeros has no solution: 3rd row is 0=1 contradiction.

Left part of augmented matrix isn't triangular, but is in a useful and easy to read form.

Definition 7.6. A is in Reduced Row Echelon Form if

1. Any entirely zero rows are at the bottom.
2. First $\neq 0$ entry in each non-zero row is a 1.
3. If a row isn't all zero, then the next row has more leading zeros. (not necessarily one place to the left - book mistake!)
4. Each column with a leading 1 has zeros elsewhere.

Examples. 1. $\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$ which yields $x_3 = -1$, $x_2 = 1$, $x_1 = 1$.

2. $\left(\begin{array}{cccc|c} 1 & 1 & 3 & 0 & 4 \\ 1 & 0 & 1 & 1 & 5 \\ 0 & 1 & 2 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$ which yields $(x_1, x_2, x_3, x_4) = (3 - x_3, 1 - 2x_3, x_3, 2)$.

Homogeneous Systems

A system is homogeneous if all the equations are of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0$. Clearly any homogeneous system is consistent, since $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is a (trivial) solution. If the system is underdetermined then it automatically has infinitely many solutions.

Example. Let c be a constant. Find all the solutions to the homogeneous system

$$\begin{cases} 3x + y + 2z = 0 \\ 6x - 2y + 4z = 0 \\ cx + cy - 2z = 0 \end{cases}.$$

Only trivial unless $c = -3$ in which case get line $(2\lambda, 0, -3\lambda)$.

(Ignore linear independence discussion - it's the wrong use of the phrase. Ignore free-variables too.)

8 Matrices

8.1 General Notation

Matrices are used to summarize information and to facilitate many calculations at once. Will see the latter in action later.

Definition 8.1. A *matrix* is a rectangular array of objects (usually numbers) in parentheses (square or round).
A matrix with m rows and n columns is described as an $m \times n$ matrix.
A (column) *vector* is an $m \times 1$ matrix.

Write A, B, C for matrices (capitals). Entries are a_{ij} for the ith row, jth column of A. Sometimes write $A = (a_{ij})$.
Write column vectors boldface (underlined) $\mathbf{x}$.
Scalars are usually written with lower case letters.
Sometimes write a matrix in terms of its column vectors:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = (\mathbf{a}_1, \dots, \mathbf{a}_n), \qquad \mathbf{a}_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}.$$

Definition 8.2. Two matrices are *equal* if they have the same dimensions and their corresponding entries are equal.
A matrix is *square* if $m = n$ (same number of rows and columns).
A square matrix is *diagonal* if $a_{ij} = 0$ for $i \neq j$.
The *identity matrix* I_n is the diagonal $n \times n$ matrix such that $a_{ii} = 1$.
A square matrix all of whose entries are zero is called the *zero matrix* or *null matrix*.

8.2 Basic Matrix Operations

1. Scalar Multiplication (example) αA
2. Matrix Addition (same size example) $A + B$
3. Multiplication matrix by vector $A\mathbf{x}$: need, A $m \times n$ and $\mathbf{x} \in \mathbb{R}^n$. Linear systems are matrix mult. Examples.
4. Multiplication matrix by matrix AB. $AB = (A\mathbf{b}_1, \dots, A\mathbf{b}_n)$. Need A $m \times r$, B $r \times n$ then AB is $m \times n$.

Note: AB might be defined even when BA is not. Even when they are, the two are generally not equal.

Examples. 1. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$ and anything but $B = \begin{pmatrix} p & q \\ 2q & p + 3q \end{pmatrix}$ for $AB \neq BA$.

2. Cost and profits. Suppose a grocer buys and sells apples, bananas and oranges at the following cents per fruit:

Fruit	Cost to buy	Cost to sell
Apple	40	55
Banana	20	60
Orange	50	80

Suppose the grocer buys $\mathbf{b} = (b_a, b_b, b_o)^T$ and sells $\mathbf{s} = (s_a, s_b, s_o)^T$ of each fruit. Then the profit of the grocer is

$$(s_a, s_b, s_o) \begin{pmatrix} 55 \\ 60 \\ 80 \end{pmatrix} - (b_a, b_b, b_o) \begin{pmatrix} 40 \\ 20 \\ 50 \end{pmatrix} = 55s_a - 40b_a + 60s_b - 20b_b + 80s_o - 50b_o.$$

3. Supply and demand in many markets. Suppose have a market of supplies of goods ${q^s}_1, \ldots, {q^s}_n$ and demanded quantities ${q^d}_1, \ldots, {q^d}_n$ arranged in vectors $\mathbf{q}^s$, $\mathbf{q}^d$. If $\mathbf{p}$ is the vector of the prices of these goods, then a linear model relating quantities and prices would be

$$\mathbf{q}^s = \mathbf{a} + A\mathbf{p}, \qquad \mathbf{q}^d = \mathbf{b} + B\mathbf{p},$$

where $\mathbf{a}, \mathbf{b}$ are constant vectors of length n and A, B are square $n \times n$ matrices. The equilibrium pricesd would then be a solution $\mathbf{p}$ to the equation

$$(A - B)\mathbf{p} = \mathbf{b} - \mathbf{a}.$$

We shall see how to solve this for $\mathbf{p}$ and thus $\mathbf{q}$ in chapter 9.

8.3 Matrix Transposition

Definition 8.3. The transpose of an $m \times n$ $A = (a_{ij})$ is the $n \times m$ $A^T = (b_{ij})$ where $b_{ij} = a_{ji}$.

Examples.

Rules - behave well with scalar mult and addition and $(AB)^T = B^T A^T$ (and with C) - examples. Note $(A^T)^T = A$.

Definition 8.4. A is symmetric if $A = A^T$ (note that we need A to be square).

8.4 Some Special Matrices

Some types of matrices have special properties that are useful in studying equations.

Definition 8.5. A square matrix is *idempotent* if $A^2 = A$.

Hence $A^n = A$ for all $n = 1, 2, 3, 4, 5, \ldots$

Example. $A = (4, 6// - 2, -3)$ is idempotent.

(Ignore partitioned matrices)

Definition 8.6. The *trace* of a square matrix A is the sum $a_{11} + a_{22} + \cdots + a_{nn}$.

Theorem 8.7. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ *(regardless of whether $AB = BA$ or not - proof in book).*

Example. Check for two matrices. (1,1//2,-1) and (2,-1//1,2). Observe $\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$.

9 Determinants and the Inverse Matrix

9.1 Defining the Inverse

Matrix algebra works almost like normal algebra with the exception of commutativity of multiplication and that we have no division.

Inverses: $2 \cdot \frac{1}{2} = 1$, so what is comparable idea for matrices? Only works for *most* but not all square matrices.

Definition 9.1. The inverse A^{-1} of an $n \times n$ matrix A is the $n \times n$ matrix which satisfies

$$AA^{-1} = A^{-1}A = I_n.$$

Not every square matrix has an inverse. Those that do not are known as *singular*. Only one (at most) inverse can exist for any given matrix.

Why do we care? Want to solve matrix equations such as $A\mathbf{x} = \mathbf{b}$, where $A, \mathbf{b}$ are given and $\mathbf{x}$ is unknown (of same length as $\mathbf{b}$). Solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

DON'T WRITE AS DIVISION!!! Generally $A^{-1}B \neq BA^{-1}$.

Examples - formula for 2×2. Alternative to book: if B,C both inverse to A then $B = BI = BAC = IC = C$ so only one possibility. However

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$.

If $ad - bc = 0$ then can't divide and so don't get an inverse (not proper argument).

Examples.

Definition 9.2. $ad - bc$ is the *determinant* of the matrix A and is denoted $|A|$.

Observe that A is invertible (A^{-1} exists) iff $|A| \neq 0$. Can write the inverse of a 2×2 matrix as

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{ if } |A| \neq 0.$$

Properties of the Determinant

Do example of each.

- $|A^T| = |A|$.
- Swap two rows (or columns) of A changes sign of det.
- Two identical rows (or columns) means $|A| = 0$.
- One row (column) a multiple of another then $|A| = 0$.
- Add a multiple of one row (column) to another leaves det unchanged.

- Det of triangular matrix is product of diagonal elements.
- Multiply row (column) of A by λ multiplies det by λ.
- Multiply every element of A by λ multiplies det by λ^n (here $n = 2$).
- $|AB| = |A|\,|B|$.
- $|A + B| \neq |A| + |B|$ (in general).

Example. Recall supply/demand equilibrium from eariler.

$$\mathbf{q}^s = \mathbf{a} + A\mathbf{p}, \qquad \mathbf{q}^d = \mathbf{b} + B\mathbf{p},$$

where $\mathbf{a}, \mathbf{b}$ are constant vectors of length n and A, B are square $n \times n$ matrices. The equilibrium prices would then be a solution $\mathbf{p}$ to the equation

$$(A - B)\mathbf{p} = \mathbf{b} - \mathbf{a},$$

which yields $\mathbf{p} = (A - B)^{-1}(\mathbf{b} - \mathbf{a})$, provided $A - B$ is invertible.

Put in some numbers: suppose (as in earlier example)

$$\begin{cases} q_1^s = -6 + 4p_1, & q_1^d = 50 - 2p_1 + p_2 \\ q_2^s = -4 + 2p_2, & q_2^s = 30 + p_1 - 3p_2. \end{cases}$$

Then

$$\mathbf{q}^s = \begin{pmatrix} -6 \\ -4 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{p}, \qquad \mathbf{q}^d = \begin{pmatrix} 50 \\ 30 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} \mathbf{p}.$$

Little of the geometry of 2 by 2 changing triangle area.

9.2 Determinants and Inverses of 3×3 Matrices

Definition 9.3. The *ijth minor* M_{ij} of a 3×3 matrix A is the determinant of the 2×2 matrix obtained by deleting the ith row and jth column of A.
The *ijth cofactor* of A is the $C_{ij} = (-1)^{i+j}M_{ij}$. (do cofactor pattern)

Examples.

Definition 9.4. The *determinant* of A is any of the 6 sums:

$$|A| = \sum_{i=1}^{3} a_{ij}C_{ij} = \sum_{j=1}^{3} a_{ij}C_{ij}.$$

The other possible sums are always zero.
Examples + quick chat about volumes.
To find the inverse of a 3×3 matrix with det$\neq 0$,

1. Write down matrix of cofactors.
2. Define adjoint adj(A) as transpose of matrix of cofactors.
3. A^{-1} is divide by determinant.

Example.

9.3 The Inverse of an $n \times n$ matrix and its properties

Can use cofactors to find the det and adjoint of higher order matrices just as before. The cofactors are the determinants of the minors (up to sign), etc. Becomes computationally time consuming however.

Easier (sometimes) method is to use two tricks:

1. Diagonal method for 3×3 determinant.

2. $(A|I)$ method for inverse.

Example. $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 4 & -1 \\ 0 & 3 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 7 & 8 & -13 \\ 1 & 2 & -1 \\ -3 & -6 & 9 \end{pmatrix}$ — do both ways.

Theorem 9.5. *A, B non-singular $\Rightarrow$ AB non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.*
$(A^{-1})^{-1} = A$.
$(A^{-1})^T = (A^T)^{-1}$.
$\det(A^{-1}) = (\det A)^{-1}$

Proof. Calculate $B^{-1}A^{-1}(AB)$ and $(AB)B^{-1}A^{-1}$.
$A^{-1}A = I \ldots$, etc. ■

Example. Firm produces three outputs y_1, y_2, y_3 with three inputs z_1, z_2, z_3. The input-requirements matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 0 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

relates the outputs to inputs according to $\mathbf{z} = A\mathbf{y}$. If firm wants to produce $y_1 = 2, y_2 = 3, y_3 = 1$ it needs $\mathbf{z} = (9, 7, 16)$. Suppose instead that firm has resources $\mathbf{z} = (17, 16, 29)$, then can produce $\mathbf{y} = A^{-1}\mathbf{z} = (5, 3, 2)$. Here

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 6 & 0 & -3 \\ -1 & -2 & 2 \\ -4 & 1 & 2 \end{pmatrix}.$$

Talk about complexity of say an administrator shutting down a company and trying to drain stock to zero while maximising profit.

9.4 Cramer's rule

Inverse matrices are useful for solving the equation $A\mathbf{x} = \mathbf{b}$ where A is square and invertible. Cramer's rule is an alternative method that allows you to obtain as many or as few entries of $\mathbf{x}$ without having to compute all the others.

Simply use the formula:

$$\mathbf{x} = A^{-1}\mathbf{b}\frac{1}{|A|}\begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{pmatrix}.$$

The upshot is that

$$x_i = \frac{1}{|A|}b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}$$

for each $i = 1, \dots, n$. Otherwise said, x_i is the determinant of the matrix A with its ith column replaced with $\mathbf{b}$.

Example. Calculate x_2 if $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 0 & 3 \\ 3 & 2 & 4 \end{pmatrix}$ and $\mathbf{b} = (2, 3, -5)^T$. Ans -6. Could have obtained from A^{-1} earlier. Sim $x_1 = 9$, $x_3 = -5$.

Original method is usually best when need 2 or more values of x_i. Also has advantage that once you have A^{-1} can change to any $\mathbf{b}$ and still solve. Finding all solutions with Cramer is same number of calculations as by finding inverse.

Example. Open Leontief Input-Output model. Model economy as a basket of related goods, the outputs of each being required for the others. For example suppose there are three sectors to consider; electronics, mining, automotive. If these sectors are related by the following: Each \$ 1 of production in each of the industries in the left column requires the corresponding level of production in the top row.

	Electronics	Mining/Drilling	Automotive
Electronics	0.4	0.2	0
Mining/Drilling	0.2	0.2	0.4
Automotive	0.3	0.1	0.1

Let matrix of above coefficients be A: a_{ij} denotes the monetary input required into industry j in order to produce one unit of industry i's output. Let $\mathbf{x}$ be the vector of money outputs of each industry, then $A\mathbf{x}$ is the vector of total production demand. Let $\mathbf{d}$ be the demand of consumers from the industries. Thus total demand on the industries is $A\mathbf{x} + \mathbf{d}$. In a closed economy this has to equal the supply demand, hence $\mathbf{x} = A\mathbf{x} + \mathbf{d}$. The result is $\mathbf{x} = (I - A)^{-1}\mathbf{d}$. For our example

$$(I - A)^{-1} = \begin{pmatrix} 0.6 & -0.2 & 0 \\ -0.2 & 0.8 & -0.4 \\ -0.3 & -0.1 & 0.9 \end{pmatrix}^{-1} = \begin{pmatrix} 1.1954 & 0.517 & 0.230 \\ 0.862 & 1.552 & 0.690 \\ 0.747 & 0.345 & 1.264 \end{pmatrix}.$$

If the public demands $\mathbf{d} = (5000, 1000, 10000)$ then the total output of the industries must be $\mathbf{x} = (12586, 12758, 16724)$.
Of course could have used Cramer's rule if just one variable was required.

10 Advanced topics in Linear Algebra

10.1 Vector Spaces

Recall: a vector is a matrix with n rows and 1 column. Written $\mathbf{v}$. $\mathbf{v}^T$ is a row vector.

Think of vectors geomterically as arrows to points. Thus $\mathbf{v} = \binom{1}{2}$ is the vector pointing from the origin in $\mathbb{R}^2$ to the point with co-ordinates $(1, 2)$.

Example of adding/subtracting vectors graphically + scaler multiplication.

Definition 10.1. The *inner product* of two vectors $\mathbf{v}, \mathbf{w}$ is the product $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w} = \sum v_i w_i$.
The *length* of a vector is $|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

Ex of length calc.

Definition 10.2. A list of n-dimensional vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *linearly dependent* iff $\det(\mathbf{v}_1, \ldots, \mathbf{v}_n) = 0$. Equivalently we have theat there exist scalers $\lambda_1, \ldots, \lambda_n$ not all zero such that $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n = 0$. Vectors are linearly independent otherwise. I.e. the only scalar solns to above eqn are all zero.

In 2-dim a pair of vectors is linearly independent if they point in separate directions (span parallelogram with non-zero area). In 3-dim need to span parallelepiped with non-zero volume (3 vectors don't lie in a common plane).

Examples.

1. $(1,2)^T$ and $(3,1)^T$ are lin ind.
2. $(1,3,4)^T, (1,3,0)^T$ and $(0,-1,1)^T$ are lin ind (calc det), but replace $\mathbf{v}_3$ with $(0,0,1)^T$ to get trio that is not lin ind.

Definition 10.3. A basis for a vector space V is a list of linearly independent vectors so that any vector in V can be generated as a linear combination of vectors in the basis.

Example. See above.

Observe that in $\mathbb{R}^n$ a basis contains exactly n linearly indep vectors.

How to solve? Suppose have a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$. Given a vector $\mathbf{u}$, how to we find the coefficients? Take dot products of $\mathbf{u}$ with each basis vector to get n eqns in n variables:

$$\begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \cdots & \mathbf{v}_1 \cdot \mathbf{v}_n \\ \vdots & \ddots & \vdots \\ \mathbf{v}_n \cdot \mathbf{v}_1 & \cdots & \mathbf{v}_n \cdot \mathbf{v}_n \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \mathbf{u} \cdot \mathbf{v}_1 \\ \vdots \\ \mathbf{u} \cdot \mathbf{v}_n \end{pmatrix}.$$

Point is that it can be done!

Example. Let $\mathbf{u} = (4,-2)$, $\mathbf{v}_1 = (1,2)^T$ and $\mathbf{v}_2 = (3,1)^T$. We want to find λ_1, λ_2 such that $\mathbf{u} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$. Get $-2, 2$.

Easiest when a basis is *orthonormal*: i.e. $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$. In such cases,

$$\mathbf{u} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \Longrightarrow \lambda_i = \mathbf{u} \cdot \mathbf{v}_i.$$

Angles between vectors: The angle between two vectors ($-\pi/2 < \theta \leq \pi/2$) is related to the vectors by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta.$$

Clear that vectors are orthogonal iff they have zero inner product.

Standard basis of $\mathbb{R}^n$ is orthogonal - ex.

There are lots of vector spaces that are not $\mathbb{R}^n$. E.g. sets of functions, etc. Bases/dimension defined similarly: dimension is simply the maximum number of linear indep vectors in the space.

Definition 10.4. The *rank* of a matrix is the maximum number of linearly independent columns. It always equals the maximum number of linearly independent rows.

Example of a 4by5 of rank 3.

Theorem 10.5. *An $n \times n$ matrix is non-singular iff* $\det A \neq 0 \iff A^{-1}$ *exists* $\iff \operatorname{rank}(A) = n$.

10.2 The Eigenvalue Problem

Eigenvalues and eigenvectors are convenient concepts to describe what a matrix does when it multiplies vectors. You can say that a matrix stretches by a factor of 3 in one direction and 2 in another - this is exactly the eigendescription of a matrix.

Definition 10.6. Let A be $n \times n$. A non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ if $A\mathbf{x} = \lambda\mathbf{x}$.

A stretches $\mathbf{x}$ by a factor of λ but doesn't change its direction.

Example: If $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, then

$$A\mathbf{x}_1 = \mathbf{x}_1. \qquad A\mathbf{x}_2 = 5\mathbf{x}_2.$$

Characteristic equation: Write $A\mathbf{x} = \lambda\mathbf{x}$ as $(A - \lambda I)\mathbf{x} = 0$. We're looking for non-zero $\mathbf{x}$ solving this. But such an $\mathbf{x}$ exists iff $A - \lambda I$ is singular: iff $\det(A - \lambda I) = 0$.

$p(\lambda) = \det(A - \lambda I) = 0$ is degree n polynomial. If λ_1 is a solution then $A - \lambda_1 I$ is singular and so there exists a non-zero vector $\mathbf{v}_1$ with $(A - \lambda_1 I)\mathbf{v} = 0$. Thus all solutions to $p(\lambda) = 0$ are eigenvalues and to each there corresponds at least one eigenvector.

Since $p(\lambda)$ degree n, there are at most n eigenvalues - repeated roots mean that many matrices do not have this many. Eigenvalues may also be complex...

Note that evectors are defined only up to non-zero scale.

Theorem 10.7. *λ is an eigenvalue of A iff*
$\iff$ $(A - \lambda I)\mathbf{x} = 0$ *has anon-zero solution* $\mathbf{x}$
$\iff$ $A - \lambda I$ *is singular*
$\iff$ $\det(A - \lambda I) = 0$.

Examples.

1. $A = 7\text{Id}$ has $\lambda = 7$ $\mathbf{v} =$ anything.
2. $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ has $\lambda = 4, 1$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
3. $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ has $\lambda = 3$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Only one eigenvalue and one eigenvector.
4. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ has $\lambda = 1, 2, 3$, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ , \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
5. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has $\lambda = 2, -1$, $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^{\perp}$.
6. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ has $\lambda = 1$, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Diagonalization

Diagonalizing a square matrix is about finding a similar diagonal matrix. I.e. given A we want to find an invertible M and diagonal D such that $A = MDM^{-1}$.

Theorem 10.8. *Suppose that an $n \times n$ matrix A has n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $A = MDM^{-1}$ where*

$$M = (\mathbf{v}_1, \ldots, \mathbf{v}_n), \qquad D = \text{diag}(\lambda_1, \ldots, \lambda_n).$$

1. $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ is diagonalizable.
2. $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ is diagonalizable.
3. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is diagonalizable.
4. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is diagonalizable.

$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ is not diagonalizable, neither is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Definition 10.9. A square matrix is *orthogonal* if it satisfies $Q^T Q = I$.

Theorem 10.10. *If A is a symmetric matrix then it has n independent orthonormal eigenvectors. Putting these as the columns of a matrix Q we have Q being orthogonal. Moroever $Q^{-1}AQ$ is diagonal with e'values down diagonal.*

10.3 Quadratic Forms

Useful in calculus especially - will use later. All need to know is how to make them out of matrices.

Definition 10.11. A *quadratic form* is a scalar expresiion $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is a square matrix.

Don't need A to be symmetric but can always choose it to be so.
Examples:

Definition 10.12. A quadratic form $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ is

$$\begin{cases} \text{positive definite} \\ \text{positive semi-definite} \\ \text{negative definite} \\ \text{negative semi-definite} \end{cases} \text{if } q(\mathbf{x}) \begin{cases} > 0 \\ \geq 0 \\ < 0 \\ \leq 0 \end{cases} \quad \text{for all } \mathbf{x} \neq 0.$$

In the case that A is symmetric the above can be translated into statements about eigenvalues: A quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is

$$\begin{cases} \text{positive definite} \\ \text{positive semi-definite} \\ \text{negative definite} \\ \text{negative semi-definite} \end{cases} \quad \text{if the eigenvalues of } A \text{ are all } \begin{cases} > 0 \\ \geq 0 \\ < 0 \\ \leq 0. \end{cases}$$

Theorem 10.13. *Can write all quadratic forms using symmetric matrices A. $+/-$ (semi-) definiteness relates to signs of eigenvalues.*

Definition 10.14. A principal minor of a square matrix A is the determinant of a matrix obtained by deleting some rows and columns from A - the rows and columns must be identical.
The leading principal minors of a square matrix A are the determinants of the matrices $A_1 = a_{11}$, $A_2 = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, $A_3 = \cdots, \ldots A_n = A$ - i.e. delete the last $n-k$ rows and columns.

Theorem 10.15. *A symmetric matrix A is positive definite iff all leading principal minors are positive.*
A symmetric matrix A is negative definite iff the leading principal minors alternate -/+/-/+ etc.
A symmetric matrix A is positive semi-definite iff some principal minors are zero and the rest are positive.
A symmetric matrix A is negative semi-definite iff $-A$ is positive semi-definite.

Examples.
1. $q(\mathbf{x}) = 7x^2 - 2xy + y^2$ is positive definite: leading principal minors 7, 6.
2. $q(\mathbf{x}) = 2y^2 + 2xy$ is positive semit-definite: principal minors 0, 2, 0.
3. $q(\mathbf{x}) = 2x^2 + 2y^2 + z^2 + 4xy + 2yz$ is positive semi-definite: principal minors 2,2,1, 0,2,1, 0 (1by1, 2by2, 3by3).

11 Calculus for functions of n-variables

11.1 Partial Differentiation

Use $(x_1, \ldots, x_n)$ to denote a point in $\mathbb{R}^n$ and $y = f(\mathbf{x}) = f(x_1, \ldots, x_n)$ denotes a function of these variables. Example $y = 7x_1 + 3\sin(x_1 x_2^2)$.

Continuity/differentiability is a little harder than in 1-dimension. Give rough verbal descriptions of why.

Definition 11.1. The *partial derivative* of the function $f(x_1, \ldots, x_n)$ with respect to x_i is the limit

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h},$$

if the limit exists. Often write f_{x_i} as a shorthand.

Idea: hold all variables constant and look at small changes in function related to small changes in one variable at a time. Picture of surface and slopes.

Examples.
1. Let $f(x_1, x_2, x_3) = 7x_1 + 3x_2 x_3^{-1}$, then $\frac{\partial f}{\partial x_1} = 7$, $\frac{\partial f}{\partial x_2} = 3x_3^{-1}$ and $\frac{\partial f}{\partial x_3} = -3x_2 x_3^{-2}$.
2. Let $f(x_1, x_2) = 7x_1 + 3\sin(x_1 x_2^2)$, then $\frac{\partial f}{\partial x_1} = 7 + 3x_2^2 \cos(x_1 x_2^2)$ and $\frac{\partial f}{\partial x_2} = 3x_1 \cos(x_1 x_2^2)$.

Notice how the chain-rule still applies.

Marginal-product functions

Suppose that $y = f(x_1, \ldots, x_n)$ represents the output of an company/industry/economy which depends on n inputs $x_1, \ldots, x_n$. The partial derivative f_{x_i} represents (approximately) the change in output that results from an increase of 1 in the input x_i, if all other inputs are unchanged. This approximation will be better if the units are smaller. f_{x_i} is thus the marginal product function with respect to the variable x_i.

Examples. 1. Let $y = 18x_1^{1/3}x_2^{1/2}$ be the production function (typically decreasing powers gives decreasing returns from more input - law of decreasing marginal productivity). Then the marginal production functions are

$$f_{x_1} = 6x_1^{-2/3}x_2^{1/2}, \qquad f_{x_2} = 9x_1x^{-1/2}.$$

2. General Cobb–Douglas production function $y = f(K, L) = AK^\alpha L^\beta$ where $A > 0, 0 < \alpha, \beta < 1$ are constants dependent on the technology/industry. Here K represents available capital and L available labor. Then

$$f_K = \alpha AK^{\alpha-1}L^\beta, \qquad f_L = \beta AK^\alpha L^{\beta-1}.$$

Observe: as quantity of labor increases, the marginal benefit of the increase reduces; as capital increases the marginal benefit of it decreases. (I.e. doubling K does not result in twice the output). Conversely, increasing K leads to higher marginal product of L - labor benefits from extra capital. Thus the inputs are complimentary.

3. The constant elasticity of substitution (CES) production function is the following

$$y = A[\delta x_1^{-r} + (1-\delta)x_2^{-r}]^{-1/r}],$$

where $A > 0, 0 < \delta < 1, r > -1$. After some nasty calcs, observe that

$$y_{x_1} = \frac{\delta}{A^r}\left(\frac{y}{x_1}\right)^{r+1}, \qquad y_{x_2} = \frac{\delta}{A^r}\left(\frac{y}{x_2}\right)^{r+1}.$$

Time-dependence

Suppose that the inputs $x_1, \ldots, x_n$ are dependent on time. The chain rule then says that

$$y'(t) = f_{x_1}x_1' + \cdots + f_{x_n}x_n'.$$

Examples. 1. $y = 7x_1 + 3x_2^2$ where $x_1 = 2t^2$, $x_2 = t^3$. Then calc. . .

2. Etc

11.2 Second-order partial derivatives

Can differentiate multiple times. The partial derivative $f_{x_ix_j}$ is simply $\frac{\partial}{\partial x_j}f_{x_i}$. Supposing that all differentials exist, one may keep differentiating forever.

Example. Find derivs up to order 3 of $f(x_1, x_2) = x_1^2x_2^3 - 4x_1x_3^2$.

Observe that the mixed partials satisfy Young's theorem: if the derivatives are continuous then the order one calculates derivatives is irrelevant; e.g. $f_{x_1x_2} = f_{x_2x_1}$.

Definition 11.2. The *gradient* vector of a function of n variables $f(\mathbf{x})$ is the vector (of length n) of partial derivatives $\nabla f = \ldots$

Fact: gradient vector points orthogonally away from surface.
Example.

Definition 11.3. The *Hessian* of a function $f(\mathbf{x})$ is the $n \times n$ matrix of second order partial derivatives

$$\nabla_2 f = \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix}.$$

Notice a function of n variables has n partial deriavtives of order 1 and n^2 of order 2 (indeed n^k of order k).
Example.
Interpretation: $f_1 x_1 + \frac{1}{2} f_{11} x_1^2$ is approx of what f looks like in the x_1 direction. I.e. f_1 is the slope, and the second derivative indicates whether the function curves up or down.

Examples. The Cobb-Douglas production function $y = f(x_1, x_2) = Ax_1^\alpha x_2^\beta$ with two inputs has Hessian

$$\nabla_2 f \begin{pmatrix} \alpha(\alpha-1)Ax_1^{\alpha-2}x_2^\beta & \alpha\beta Ax_1^{\alpha-1}x_2^{\beta-1} \\ \alpha\beta Ax_1^{\alpha-1}x_2^{\beta-1} & \beta(\beta-1)Ax_1^\alpha x_2^{\beta-2} \end{pmatrix}.$$

Give $0 < \alpha, \beta < 1$, the signs are thus $\left(\begin{smallmatrix} - & + \\ + & - \end{smallmatrix}\right)$. That f_{11} and f_{22} are negative says that as one increases one input, the marginal product of that variable decreases. Conversely $f_{12} > 0$ says that as one increases on variable, the marginal product of the other increases.

11.3 First order total differential

Used to writing $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ from which we get $\mathrm{d}f = f'(x)\mathrm{d}x$. This is useful for chain rule calculations, etc.; $\int \mathrm{d}f = \int f'(x)\mathrm{d}x$ is the change of variable formula for integrals. We think about this as $\mathrm{d}f$ measuring the size of a small change in f resulting from the small change $\mathrm{d}x$ in x. An analogous thing can be done in multi-variable calculus:

Definition 11.4. The first order *total differential* of a function $y = f(x_1, \ldots, x_n)$ is

$$\mathrm{d}y = \mathrm{d}f = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n = f_1\mathrm{d}x_1 + \cdots + f_n\mathrm{d}x_n.$$

The total derivative allows us to calculate (approximately) the resulting small change in the value of the function f given small changes $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ in the inputs. The approx is better if the changes are small compared to the size of f itself.

Example. Let $y = f(x_1, x_2) = 60x_1^{1/2}x_2^{1/3}$. Then

$$\mathrm{d}y = 30x_1^{-1/2}x_2^{1/3}\mathrm{d}x_1 + 20x_1^{1/2}x_2^{-2/3}\mathrm{d}x_2.$$

Suppose that the inputs are currently $x_1 = 9, x_2 = 8$ and that we increase each by x_1 by 1 and x_2 by 2 ($\mathrm{d}x_1 = 1,\ \mathrm{d}x_2 = 2$). Then the approximate change in y is

$$\mathrm{d}y = 30 \cdot \frac{1}{3} \cdot 2 \cdot 1 + 20 \cdot 3 \cdot \frac{1}{4} \cdot 2 = 20 + 30 = 50.$$

Note $f(9,8) = 360$ and $f(10,10) = 408.775 \approx 360 + 50$, so approximation is quite good. Advantage is that the approx is easy to calculate once you have the partial derivative values, can plug in any changes in the inputs after this.

Implicit Functions

Can use total derivatives to calculate slopes of implicitly defined functions.
$F(x,y) = 0 \Rightarrow F_x \mathrm{d}x + F_y \mathrm{d}y = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{F_x}{F_y}$.

Examples. 1. $x^2 + y^3 = 5$ at $(2,1)$ — do as a function.

2. $x \sin y + y \sin x = \pi$ at $(\pi/2, \pi/2)$ — can't repeat as a function.

Level curves

Let $y = f(x_1, \ldots, x_n)$. The set of points $(x_1, \ldots, x_n)$ for which y is a constant is a *level curve* $(n = 2)$ or *level set* $n \geq 3$.

Suppose $y = f(x_1, x_2)$ is constant. Then $0 = \mathrm{d}f = f_1 \mathrm{d}x_1 + f_2 \mathrm{d}x_2$. The level curve thus has slope $-f_1/f_2$.

Examples. 1. Find the slopes of the level curves of the function $f(x_1, x_2) = x_1^2 x_2^3$. Here

$$0 = f_1 \mathrm{d}x_1 + f_2 \mathrm{d}x_2 = 2x_1 x_2^3 \mathrm{d}x_1 + 3x_2^2 x_2^2 \mathrm{d}x_2 \Rightarrow \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = -\frac{2x_2}{3x_1}.$$

Isoquants

If $y = f(x_1, x_2)$ is a production function for inputs x_1, x_2, then the level curves of y are described as *isoquants* — curves of 'equal quantity'. I.e. the level curve $100 = f(x_1, x_2)$ is the curve whose points describe all the possible input combinations that will product 100 output.

It is common to want to substitute one input for another in order to produce the same outcome. The rate at which one can do this is called the *Marginal Rate of Technical Substitution* and is defined as follows:

$$\text{MRTS } = -\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_1}{f_2}.$$

Example. Find the MRTS for the production function $y = f(x_1, x_2) = x_1^{2/3} x_2^{1/3}$. Here

$$\text{MRTS } = \frac{2x_2}{x_1}.$$

If the inputs are currently $x_1 = x_2 = 1000$, then MRTS $= 2$. Notice $(1001^2 \times 998)^{1/3} = 999.99 = (999^2 \times 1002)^{1/3}$. I.e. if delete one of x_1 need to replace with two of x_2.

Notion carries over to more variables. Suppose have three variables $y = f(x_1, x_2, x_3)$. Then $\mathrm{MRTS}_{1,3} = -\frac{\mathrm{d}x_3}{\mathrm{d}x_1} = \frac{f_1}{f_3}$ is the marginal rate at which input 1 can be substituted for input 3.

Example. If $y = f(x_1, x_2, x_3) = 12x_1^{1/4}x_2^{1/3}x_3^{1/2}$, then the partial derivatives are

$$f_1 = \frac{3x_2^{1/3}x_3^{1/2}}{x_1^{3/4}}, \qquad f_2 = \frac{4x_1^{1/4}x_3^{1/2}}{x_2^{2/3}}, \qquad f_3 = \frac{6x_1^{1/4}x_2^{1/3}}{x_3^{1/2}},$$

from which we get the marginal rates of technical susbstitution

$$\begin{aligned} \mathrm{MRTS}_{1,2} &= \frac{f_2}{f_1} = \frac{4x_1}{3x_2}, & \mathrm{MRTS}_{1,3} &= \frac{f_3}{f_1} = \frac{2x_1}{x_3}, & \mathrm{MRTS}_{2,3} &= \frac{f_3}{f_2} = \frac{3x_2}{2x_3}, \\ \mathrm{MRTS}_{2,1} &= \frac{f_1}{f_2} = \frac{3x_2}{4x_1}, & \mathrm{MRTS}_{3,1} &= \frac{f_1}{f_3} = \frac{x_3}{2x_1}, & \mathrm{MRTS}_{3,2} &= \frac{f_2}{f_3} = \frac{2x_3}{3x_2}. \end{aligned}$$

I.e. if have equal quantities of all three inputs going into the system, then can replace one of x_2 by $4/3$ of x_1, or 3 of x_2 with 4 of x_1.