Math 8 — Functions and Modeling

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Introduction

The goal of this course is connect grade-school and university-level math through problem solving and mathematical exploration, primarily for the benefit of future secondary-level teachers. The *mathematics* itself should be fairly straightforward, though you'll naturally be expected to understand the material at a higher level than would a grade-school student! The material is secondary to the goal of practicing clear explanations and considering how to foster class-discussions. This is not something you can do by reading notes or sitting silently in class...

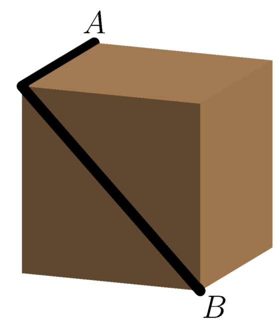
We start with two motivational problems.[1]

1. You want to travel across the surface of a cube between two opposite vertices A, B in the shortest possible manner.

   Should you follow the path indicated?

   If you think yes, explain why.

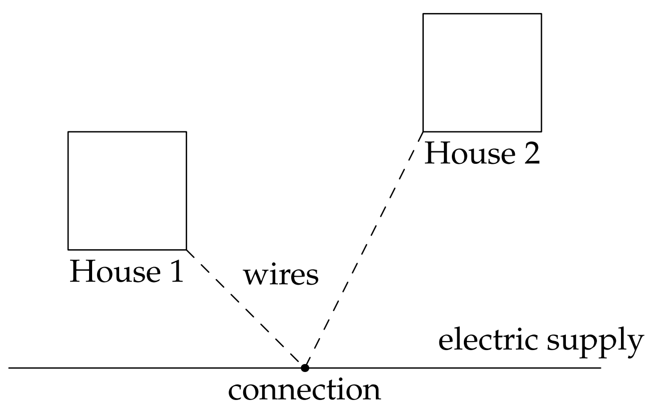
   If not, how should you find the shortest path?



2. A city wants to connect two houses to an electricity supply line using a *single* connection.

   How should we decide where to place the connection so as to minimize the required length of wire?

   What information do you require in order to find the connection point?



The point of these exercises isn't to find the right answer. Instead imagine you were to discuss these problems with grade-school students. How would you approach this? Why might calculus *not* be a sensible approach? Are there commonalities between the two problems? Brainstorm some strategies and see if you can find different approaches.

[1] We are grateful to materials from UT Austin's UTeach program for suggesting several of the examples in this course including these motivational problems.

1 Sets & Functions

In this section we refresh the notion of a function. Consider how central this is to mathematics. Consider also how you would introduce this topic to grade-school students. Do you remember when you first encountered it? How would you define *function* to someone with limited mathematical knowledge? Would you use words like *rule*, *assign*, *element*, *domain*, *vertical line test*, etc.? How helpful are these to your audience?

Examples 1.1. How would you explain the idea that the following do or do not represent functions?

1. $y = x^2$
2. Mon: fish, Tue: pork, Wed: fajitas, Thur: carbonara, Fri: pizza
3. $(3,5), (2,6), (4,2), (3,1)$.
4. $x^2 = y^2$

After trying to explain these, perhaps you settle on a semi-formal definition such as the following:

> A function f from A to B is rule which assigns to each element of a of A exactly one element $f(a)$ of B.

What parts of this definition are imprecise? Discuss. How much does this imprecision matter?

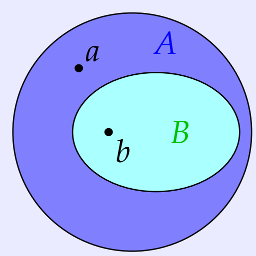
The way you teach a topic should depend on your audience. Plainly you must understand the topic to *at least* the level you teach, but this doesn't mean your students need the full story. Teaching is often a job of *selection*: from a formal palate, select enough to convey the meaning without overburdening and intimidating your students. You should also be anticipating more advanced questions and be able to dynamically offer illustrative examples. All this is to say that you have to be comfortable with a more formal definition of function. We start by refreshing some basic set notation.

Definition 1.2. A *set* A is a collection of objects or *elements*.[2] If a is an element of A, we write $a \in A$. In the abstract, sets are usually written upper case and elements lower.

A set B is a *subset* of a set A, written $B \subseteq A$, if every element of B is also an element of A. That is:

$$b \in B \implies b \in A$$

In the picture, we have $B \subseteq A$, $a \in A$, $b \in B$ and $a \notin B$.



Examples 1.3. 1. Suppose that the elements of a set A are precisely the numbers 1, 3, 5, 7 and 9. The standard way to write this is in *roster notation*:

$$A = \{1,3,5,7,9\}$$

Order doesn't matter in roster notation, so we could also write $A = \{3,9,7,1,5\}$.

The subset $B = \{1,3,5\}$ of A can also be written in *set-builder notation* as $B = \{a \in A : a < 6\}$: this is read as "The set of all elements a in A such that a is less than six."

[2] This is enough for our purposes. It typically takes a term of set theory (see Math 13) to convince you that this isn't a good definition: selection again!

2. You should be familiar with common sets of numbers: we use various combinations of roster and set-builder notation to informally define these.

Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. For instance $5 \in \mathbb{N}$ but $-3 \notin \mathbb{N}$.

Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$. For instance $-4 \in \mathbb{Z}$ but $\frac{4}{5} \notin \mathbb{Z}$.

Rational numbers or fractions: $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N}\}$. For instance $\frac{6}{7} \in \mathbb{Q}$ (here $p = 6 \in \mathbb{Z}$ and $q = 7 \in \mathbb{N}$).

Real numbers $\mathbb{R}$: for instance $\sqrt{2} \in \mathbb{R}$. A formal definition is difficult, but you should be used to visualizing the real line. *Intervals* are particularly important subsets, e.g.

$$[-4, \pi) = \{x \in \mathbb{R} : -4 \leq x < \pi\}$$

Complex numbers $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$: we'll think about these a little later in the course.

You should be comfortable with the subset relationships between these sets

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

You should also have encountered the facts that $\sqrt{2}$ and π are *irrational* (elements of $\mathbb{R}$ but not $\mathbb{Q}$), even if you have not seen proofs of these statements.

We now give a formal definition of a function.

Definition 1.4. The *Cartesian product* of sets A, B is the set of *ordered pairs*

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

A *function* from A to B is a non-empty subset $f \subseteq A \times B$ which satisfies the *vertical line test*

For all $a \in A$, there exists a *unique* $b \in B$ such that $(a, b) \in f$

If f is a function, we write $f : A \to B$ and $f(a) = b$ instead of $(a, b) \in f$.

Wow! Do you think this is the definition you should give to 10^{th} graders? Or even to freshman calculus students? Think through each of Examples 1.1 in this formal context. To do this properly, we have to carefully label the constituent sets. For instance:

2. With $A = \{\text{Mon, Tue, Wed, Thu, Fri}\}$ and $B = \{\text{fish, pork, fajitas, carbonara, pizza}\}$, we have

$$f = \{(\text{Mon,fish}), (\text{Tue,pork}), (\text{Wed,fajitas}), (\text{Thu,carbonara}), (\text{Fri,pizza})\}$$

Certainly this satisfies the vertical line test, and so f is a function.

3. This time we could *choose* $A = \{2, 3, 4\}$ and $B = \{1, 2, 5, 6\}$ from which

$$f = \{(3, 5), (2, 6), (4, 2), (3, 1)\}$$

This fails the vertical line test since

$$(3, 5) \text{ and } (3, 1) \in f, \text{ but } 5 \neq 1$$

Otherwise said, $f(3)$ is not well-defined, since it cannot equal *both* 5 and 1!

Not only is this definition too intense, it isn't even obvious how it applies to the most commonly encountered 'calculus-type' functions. It is also missing several important concepts that are often used to illuminate the discussion. We define these now, and consider a simple formula in this context.

Definition 1.5. If $f \subseteq A \times B$, then we define three associated sets:

- Domain: $\operatorname{dom} f = \{a \in A : (a,b) \in f \text{ for some } b \in B\}$
- Codomain: $\operatorname{codom} f = B$
- Range: $\operatorname{range} f = \{b \in B : (a,b) \in f \text{ for some } a \in A\}$

A *function* f might satisfy either or both of two additional properties:

- 1–1/Injectivity: Distinct inputs $a_1 \neq a_2$ produce distinct outputs $f(a_1) \neq f(a_2)$.
- Onto/Surjectivity: $\operatorname{range} f = \operatorname{codom} f$.

Example 1.6. Consider the formula $y = f(x) = \dfrac{1}{(2-x)^2}$

The *codomain* may be assumed to be $\mathbb{R}$: why even bother introducing the concept?!

The *domain* may be assumed to be the largest set of numbers for which the formula 'works'; in this case $\operatorname{dom} f = \mathbb{R} \setminus \{2\}$ consists of all real numbers except 2. Perhaps merely observing that 'we can't have $x = 2$' is enough.

The *graph* of f is precisely the set $f \subseteq \mathbb{R} \times \mathbb{R}$ viewed according to the formal definition (1.4)!

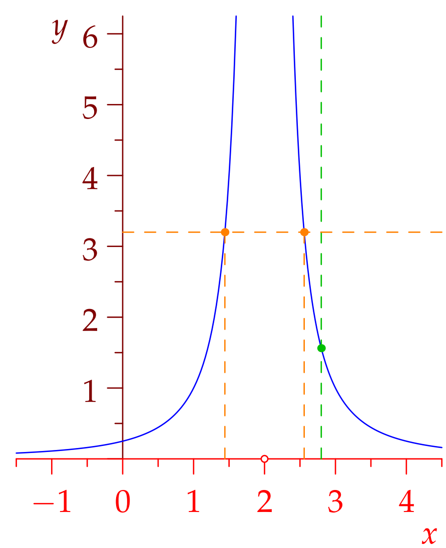
The *vertical line test* in this context is exactly what it sounds like: every vertical line $x = a$ (except $x = 2$) intersects the graph of f exactly once.

The *range* is the interval $\mathbb{R}^+ = (0, \infty)$ as can be seen by solving

$$f(x) = y \iff \frac{1}{2-x} = \pm\sqrt{y} \iff x = 2 \mp \frac{1}{\sqrt{y}}$$

Any positive output y can be obtained by evaluating $f(2 - \frac{1}{\sqrt{y}})$.

The function fails to be onto since the range does not equal to codomain.



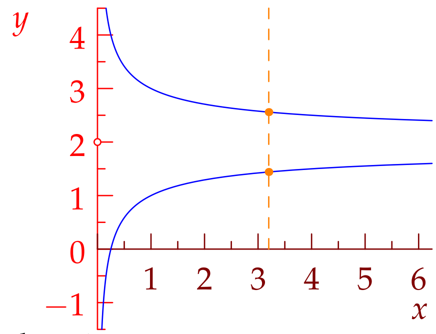
The function isn't 1–1 either. For such functions this amounts to failing the *horizontal line test*: if any horizontal line intersects the graph more than once, then there is an output which comes from more than one input.

When might we care more precisely about domains and codomains? In elementary classes this often relates to when we want to *invert* a function.

Definition 1.7. The *inverse* of a graph $f \subseteq A \times B$ is the set obtained by reversing the order of all pairs:

$$f^{-1} = \{(b,a) \in B \times A : (a,b) \in A \times B\}$$

For a function such as the above example, this corresponds to *reflecting in the line $y = x$*. There is nothing wrong with calling this graph the inverse of f.



What makes us uncomfortable is that this graph is not a function. Indeed it fails the *vertical line test* in two ways:

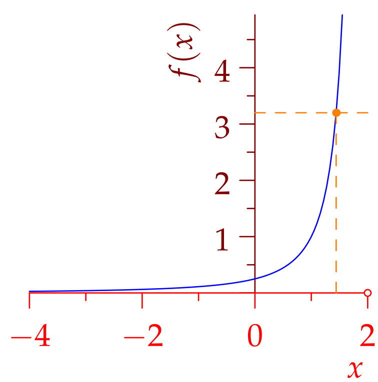
1. Some vertical lines (e.g. $x \leq 0$) do not intersect the graph.
2. Some vertical lines intersect the graph *twice*.

We can fix these two problems by restricting the domain and the codomain.

1. If we restrict the original domain to $\operatorname{dom} f = (-\infty, 2)$, then the original function passes the *horizontal line test*: it is 1–1.
2. If we also restrict the codomain to $\operatorname{range} f = (0, \infty)$ we now have an onto function.

The upshot is that we have an *invertible* function

$$f : (-\infty, 2) \to (0, \infty), \quad f(x) = \frac{1}{(2-x)^2}$$

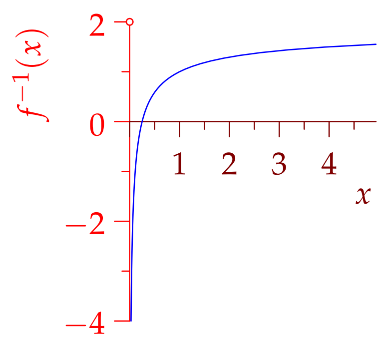


Repeating the calculation seen in Example 1.6, we see that

$$y = \frac{1}{(2-x)^2} \iff x = 2 - \frac{1}{\sqrt{y}}$$

from which f^{-1} is the *function*

$$f^{-1} : (0, \infty) \to (-\infty, 2), \quad f^{-1}(x) = 2 - \frac{1}{\sqrt{x}}$$



We knew to choose the *negative* square root since we need the output $f^{-1}(x)$ to lie in the correct range $(-\infty, 2)$.
Observe also that $\operatorname{range} f^{-1} = \operatorname{dom} f$ and $\operatorname{dom} f^{-1} = \operatorname{range} f^{-1}$.

There is a general result here.

Theorem 1.8. *The inverse of $f : A \to B$ is a function (f is invertible) if and only if f is both 1–1 and onto.*

For 'calculus' functions like the above example, it is simpler to ignore the codomain and proceed systematically:

1. Check the horizontal line test (is f 1–1?).
2. Swap x and y to write $x = f(y)$ and solve for $y = f^{-1}(x)$. This is the reflection in Definition 1.7.

Example 1.9. Let $f(x) = x^3 + 8$.

1. f passes the horizontal line test for all x:

$$x_1^3 + 8 = x_2^3 + 8 \implies x_1 = x_2$$

2. $x = y^3 + 8 \implies y = f^{-1}(x) = \sqrt[3]{x-8}$.

In this case the domain and range are both the whole real line.

Exercise Consider the function $f(x) = x^2 + 2x - 3$.

1. What do you have to do to the domain to make this invertible?
2. Compute the inverse. There are *two formulas*; think about why.

2 Polynomials

Now that we have a good dictionary for how to describe the properties and ingredients of functions it is time to consider several classes of examples. We start with the polynomials.

Linear Polynomials

Perhaps the simplest functions are the *linear polynomials*, so named because their graph is a straight line;[3] $y = f(x) = mx + c$ where m, c are constants. These functions make for very easy models: increase the input by Δx and the output changes by $\Delta y = m\Delta x$ *regardless* of the starting value x.

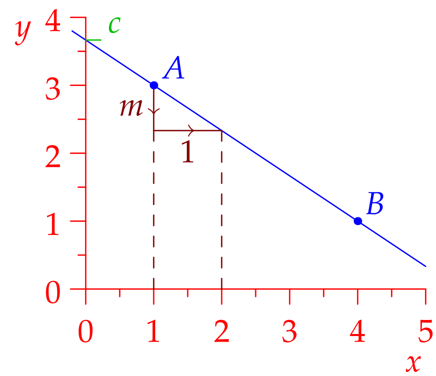
Example 2.1. Find the equation of the straight line through the points $A = (1, 3)$ and $B = (4, 1)$.

Substitute both points into the equation and solve

$$\begin{cases} 3 = m + c \\ 1 = 4m + c \end{cases} \implies -2 = 3m \implies m = -\frac{2}{3},\ c = 3 - m = \frac{11}{3}$$

The *gradient* or *slope m* represents how far one climbs/falls if one travels one unit to the right.

The *y-intercept c* is the intersection of the line with the y-axis.



There is some bookkeeping to do here: how do we know that every straight line corresponds to such a linear function? This follows fairly easily from a useful fact regarding parametrizations.

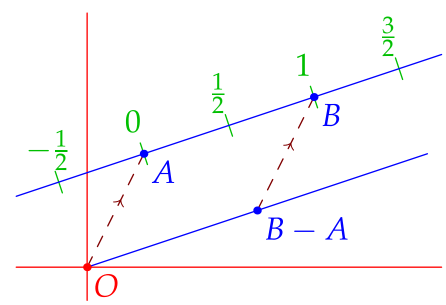
Theorem 2.2. *The set of points on the line through distinct points A and B is*

$$\ell_{A,B} = \{(1 - t)A + tB : t \in \mathbb{R}\}$$

Proof. There are several ways to think about this; we use what is essentially *vector addition*.

Certainly the set of points $\ell_{O,B-A} = \{t(B - A) : t \in \mathbb{R}\}$ describes the straight line through the origin and the point $B - A$. We simply shift this line by A...

The locations of the points corresponding to various values of t are marked.



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Example 2.3. The line through points $A = (3, 6)$ and $B = (-1, 4)$ may be parametrized by

$$(x, y) = (1 - t)(3, 6) + t(-1, 4) = \big(3 - 4t, 6 - 2t\big)$$

By solving for t in terms of x, we see that this has equation

$$y = 6 - 2t = 6 - 2 \cdot \frac{3 - x}{4} = \frac{1}{2}x - \frac{9}{2}$$

[3]Don't confuse this with the meaning of *linear* from linear algebra; in general $f(\lambda x) \neq \lambda f(x)$ for a linear polynomial.

We can repeat such an analysis in general.

- Given $y = mx + c$, let $A = (0, c)$ and $B = (1, m + c)$, the line through these consists of the points

$$(x, y) = (1 - t)A + tB = \big(t, (1 - t)c + t(m + c)\big) = (t, mt + c)$$

  all of which satisfy $y = mx + c$.

- Conversely, the line through $A = (x_0, y_0)$ and $B = (x_1, y_1)$ is parametrized by

$$(x, y) = (1 - t)A + tB = \big(x_0 + (x_1 - x_0)t, y_0 + (y_1 - y_0)t\big)$$
$$\implies y = y_0 + (y_1 - y_0)t = y_0 + (y_1 - y_0)\frac{x - x_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}x + \frac{y_0x_1 - x_0y_1}{x_1 - x_0}$$

  which has the form $y = mx + c$ with gradient $m = \frac{y_1 - y_0}{x_1 - x_0}$ given by the familiar 'rise over run.'

Quadratic Polynomials

These are functions of the form $y = f(x) = ax^2 + bx + c$ where $a \neq 0$. The simplest is $y = x^2$, the standard parabola opening upwards. That any quadratic polynomial has the same shape can be seen by *completing the square*.

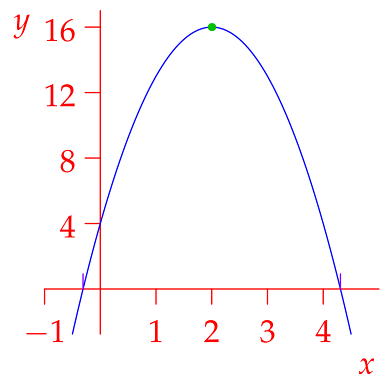
Example 2.4. Describe the parabola $y = -3x^2 + 12x + 4$.
Worry about the x terms; $-3x^2 + 12x = -3(x^2 - 4x)$ and observe that

$$-3(x - 2)^2 = -3(x^2 - 4x + 4) = -3x^2 + 12x - 12$$

gives most of what we want. Note how we *divided the x-coefficient by two*. Now tidy up,

$$y = (-3x^2 + 12x - 12) + 16 = -3(x - 2)^2 + 16$$

The parabola opens downwards with its apex at $(x, y) = (2, 16)$.



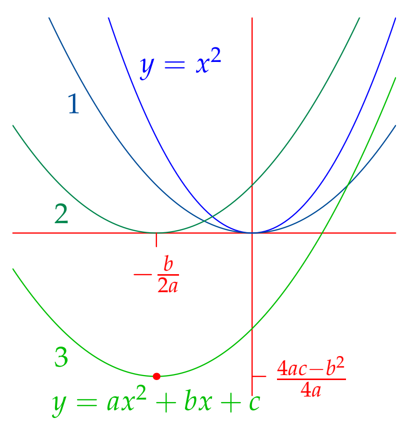
This is easy to repeat in general:

$$y = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

The graph is that of the standard parabola which has been:

1. Vertically scaled by a;
2. Shifted horizontally so that its apex is at $x = -\frac{b}{2a}$;
3. Shifted vertically by $\frac{4ac - b^2}{4a}$

Completing the square yields the famous *quadratic formula*.



Theorem 2.5. *If $a \neq 0$, then* $ax^2 + bx + c = 0 \iff x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

This formula leads immediately to the consideration of *roots/zeros*: the values x, if any, for which $f(x) = 0$.

In the above example, the roots are marked on the graph. We can find their location in two ways:

1. Quadratic formula: with $a = -3$, $b = 12$, $c = 4$, we have

$$x = \frac{-12 \pm \sqrt{12^2 - 4(-3) \cdot 4}}{2(-3)} = \frac{-12 \pm 4\sqrt{3^2 + 3}}{-6} = 2 \pm \frac{\sqrt{12}}{3} = 2 \pm \frac{2\sqrt{3}}{3}$$

   While it is really tempting to jump for the formula, it often leads to difficult surd expressions. We simplified by noticing the common factor of 4^2 inside the square root. Without this, we'd be faced with $\sqrt{144 + 48} = \sqrt{192}$!

2. Use the fact that we've already completed the square:

$$-3(x-2)^2 + 16 = 0 \iff (x-2)^2 = \frac{16}{3} \iff x = 2 \pm \frac{4}{\sqrt{3}}$$

   In many cases it is simpler to complete the square than to use the quadratic formula!

Quadratic polynomials are a good excuse to introduce several related concepts.

Definition 2.6. Let $f : I \to \mathbb{R}$ be a function defined on an interval I. We say that f is:

1. *Increasing* if $x_0 < x_1 \implies f(x_0) \leq f(x_1)$.
2. *Convex (concave up)* if $x_0 \neq x_1 \implies (1-t)f(x_0) + tf(x_1) \geq f\big((1-t)x_0 + tx_1\big)$ for all $0 < t < 1$. Otherwise said, the line segment joining any two points on the graph lies *above* the graph.

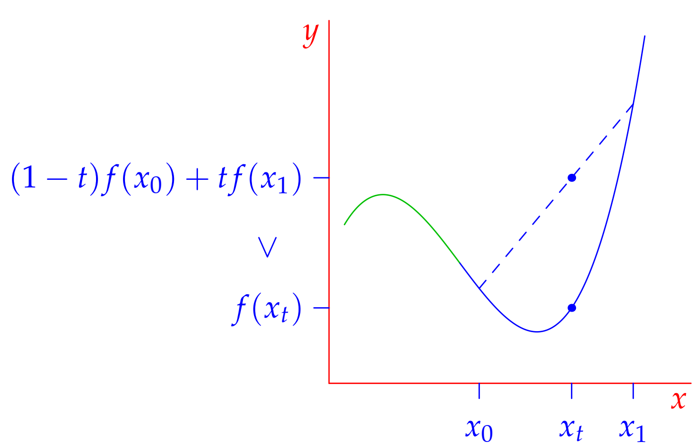
Reverse the inequalities to obtain the notions of *decreasing* and *concave down*. The inequalities can also be made strict to obtain *strictly increasing* and *strictly convex*, etc.

The function in the picture is convex where its graph is blue. If $0 < t < 1$, the value

$$x_t = (1-t)x_1 + tx_2$$

lies strictly between x_1 and x_2. At this point the function lies below the corresponding point on the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$f(x_t) < (1-t)f(x_1) + tf(x_2)$$



Theorem 2.7. *If $a > 0$, then $f(x) = ax^2 + bx + c$ is*

1. *Increasing on the interval $[-\frac{b}{2a}, \infty)$;*
2. *Decreasing on the interval $(-\infty, -\frac{b}{2a}]$;*
3. *Concave up on $\mathbb{R}$.*

The outcome is reversed if $a < 0$.

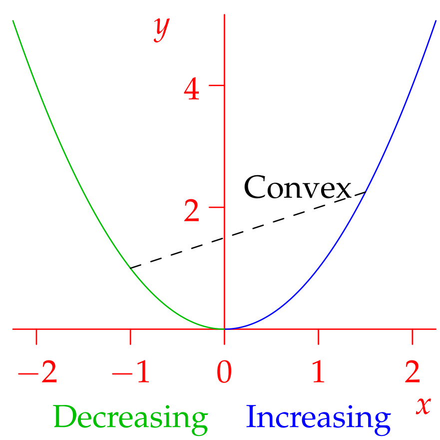
Instead of a formal argument, we consider the simplest example.

Example 2.8. $f(x) = x^2$ has apex $(0, 0)$. Simply compare

$$f(x_1) - f(x_0) = x_1^2 - x_0^2 = (x_1 - x_0)(x_1 + x_0)$$

and consider two cases:

- $0 \le x_0 < x_1 \implies x_0 + x_1 > 0 \implies f(x_1) - f(x_0) > 0$ so f is strictly increasing.
- $x_0 < x_1 \le 0 \implies x_0 + x_1 < 0 \implies f(x_1) - f(x_0) < 0$ so f is strictly decreasing.



Finally, for any $x_0, x_1 \in \mathbb{R}$ and any $0 < t < 1$, a little algebra shows that

$$\begin{aligned} f\big((1-t)x_0 + tx_1\big) - (1-t)f(x_0) + tf(x_1) &= \big(x_0 + t(x_1 - x_0)\big)^2 - (1-t)x_0^2 + tx_1^2 \\ &= t(t-1)\big(x_0 + x_1\big)^2 < 0 \end{aligned}$$

whence f is convex (concave up). An alternative argument is in the exercises.

Higher-degree Polynomials

Definition 2.9. A *degree n polynomial* is any function of the form

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_k are constants and $a_n \neq 0$.

A quadratic polynomial has degree 2 and a linear polynomial $y = mx + c$ degree one[4] (if $m \neq 0$).

The most common issues with higher-degree polynomials are related: finding roots and factorizing. In general this is very hard! The following result summarizes matters.

[4]A non-zero constant polynomial has degree zero. By convention the *zero polynomial* $y \equiv 0$ is said to have degree -1 or $-\infty$: the latter is useful in order to make the theorem $\deg fg = \deg f + \deg g$ hold for all polynomials.

Theorem 2.10 (Factor Theorem). *Suppose $f(x)$ is a polynomial of degree n whose coefficients are real numbers. Then $c \in \mathbb{R}$ is a root ($f(c) = 0$) if and only if $x - c$ is a factor:*

$$f(x) = (x - c)g(x) \quad \textit{for some degree } n - 1 \textit{ polynomial } g(x)$$

One direction of this is obvious, the other follows by what is essentially long division of polynomials. The factor theorem can be used to factorize polynomials *provided you can guess a root*! Often it is worth trying small integers; in fact if all the coefficients of $f(x)$ are integers and the leading coefficient is $a_n = \pm 1$, then the only rational roots[5] have to be integers which divide the constant term a_0!

Example 2.11. Let $f(x) = x^3 - x^2 + 3x - 10$. Any integer solutions must be divisors of 10, so we only need to try $x = \pm 1, \pm 2, \pm 5, \pm 10$. By inspection, we see that

$$f(2) = 8 - 4 + 6 - 10 = 0$$

from which $x - 2$ must be a factor. The factorization can be found in various ways; here are some options, though all are essentially versions of the same process.

1. Long division:

$$\begin{array}{r|l} & x^2 \quad + x \quad + 5 \\ x - 2 & x^3 \quad - x^2 + 3x - 10 \\ & -x^3 + 2x^2 \\ \hline & x^2 + 3x \\ & -x^2 + 2x \\ \hline & 5x - 10 \\ & -5x + 10 \\ \hline & 0 \end{array} \implies x^3 - x^2 + 3x - 10 = (x - 2)(x^2 + x + 5)$$

2. Multiplying out and solving. We know that $f(x) = (x - 2)g(x)$ where $g(x)$ is some quadratic polynomial. Thus let $g(x) = ax^2 + bx + c$ and multiply out:

$$x^3 - x^2 + 3x - 10 = (x - 2)(ax^2 + bx + c) = ax^3 + (b - 2a)x^2 + (c - 2b)x - 2c$$

   Equating coefficients, we obtain the same factorization as before,

$$a = 1, \; b = -1 + 2a = 1, \; c = \frac{-10}{-2} = 5$$

3. Work term-by-term: with practice you can do this is one line with no working!

   (a) The first term must be x^2 to create x^3

$$x^3 - x^2 + 3x - 10 = (x - 2)(x^2 + \cdots) = x^3 - 2x^2 + \cdots$$

   (b) To correct the x^2 term, add an x:

$$(x - 2)(x^2 + x + \cdots) = x^3 - x^2 - 2x + \cdots$$

   (c) To correct the x term, add an 5:

$$(x - 2)(x^2 + x + 5) = x^3 - x^2 - 2x - 10$$

[5]Rational numbers x such that $f(x) = 0$. This is a special case of the famous *rational roots theorem*.

Polynomials are often employed in modelling applications due to their simplicity and ease of evaluation. As you saw in calculus, the motion of a falling body, or of any projectile can be modelled using quadratic polynomials. Indeed this goes back to the beginning of using functions to model the natural world with Galileo's experimental observation that the distance travelled by a falling body is proportional to the *square* of the time taken: $d \propto t^2$.

Exercise A body is dropped from a height of 500 m. It's height above the ground at time t s is given by $h(t) = 500 - 5t^2$ m.

1. Over each interval of 1 s, how far does the body fall? Otherwise said, compute the values $h(n+1) - h(n)$ for each $n = 0, 1, 2, \ldots, 9$? What do you observe regarding how these values change?
2. Over an interval of time Δt, what is the *average speed* $\dfrac{h(t+\Delta t) - h(t)}{\Delta t}$ of the body?
3. What does this mean for how the speed of the falling body is related to time?

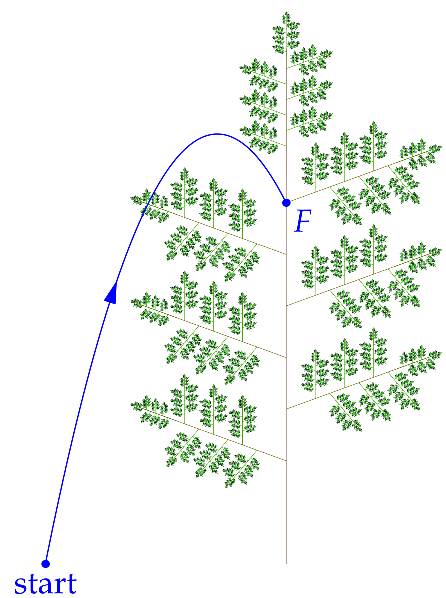
Of course you'll have seen this in calculus and will want immediately to *differentiate* to obtain the *velocity* $h(t) = -10t$ m/s and *acceleration* $h''(t) = -10\,\text{m/s}^2$. However, both historically, and in introductory calculus, this problem is what really *motivates the definition* of the derivative. Galileo's observation was essentially that the height $h(t)$ of a falling body is a solution to the differential equation

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = g$$

where g is the constant acceleration due to gravity; approximately[6] $32\,\text{ft/s}^2$ or $10\,\text{m/s}^2$.

Exercise Your frisbee is stuck 15 m up a tree. Standing 10 m away from the base, you throw a ball with the intent of knocking the frisbee out of the tree.

1. How would you model this problem? Given our earlier discussion What do you know about the required trajectory?
2. Think about why there are multiple answers to the problem: can you visualize how the required throwing speed is related to the direction in which you have to throw the ball?
3. (If you want a challenge) Find a formula linking the initial speed and gradient of the parabola. What is the *minimum* speed at which you could throw the ball if you want to dislodge the frisbee?



[6] $9.81\,\text{m/s}^2$ is more accurate, but this isn't a Physics class :)

Newton–Raphson Iteration

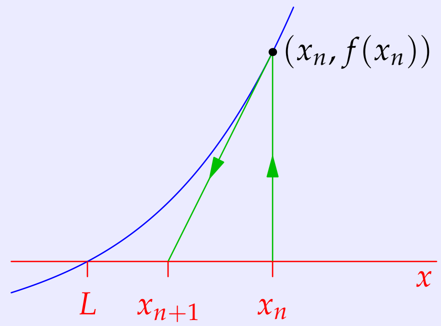
We need enormous luck if we are to find an exact root of a high-degree polynomial. Instead, we can often quickly approximate a root. The following method is particularly useful for polynomials since the computations produce sequences of rational numbers.

Definition 2.12. Given a function $f(x)$, the *Newton–Raphson iterates* of an initial value x_0 are defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

If $L = \lim_{n\to\infty} x_n$ exists, then this is a root of the function; $f(L) = 0$.

The picture explains why we might expect this to work; x_{n+1} is the x-intercept of the tangent line $y = f(x_n) + f'(x_n)(x - x_n)$ to the curve at $(x_n, f(x_n))$.

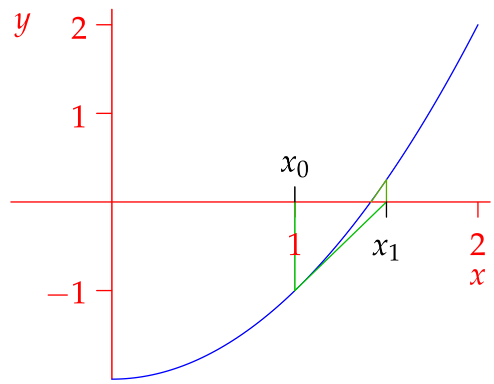


Example 2.13. To approximate $\sqrt{2}$, consider $f(x) = x^2 - 2$ and a starting value of $x_0 = 1$ since this is reasonably close to $\sqrt{2}$. Now iterate

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n^2 + 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

which yields the sequence (to 4 d.p.)

$$\left(1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots\right) = (1, 1.5, 1.4166\ldots, 1.4142\ldots)$$



This approach to approximating square roots was known to the ancient Babylonians 2500 years ago!

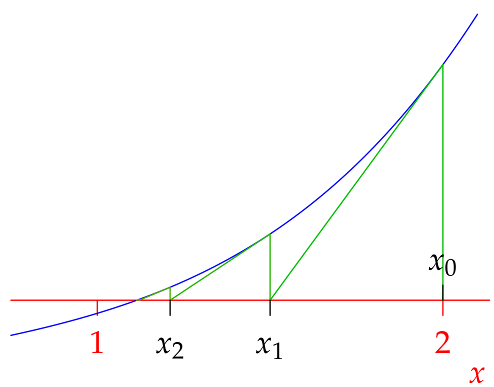
Example 2.14. To find a root of $f(x) = x^4 + 4x - 6$, start with $x_0 = 2$ and iterate

$$x_{n+1} = x_n - \frac{x_n^4 + 4x_n - 6}{4x_n^3 + 4} = \frac{3(x_n^4 + 2)}{4(x_n^3 + 1)}$$

which yields the sequence (to 4 d.p.)

$$\left(2, \frac{3}{2}, \frac{339}{280}, \ldots\right) = (2, 1.5, 1.2107, 1.1214, 1.1144, 1.1144, \ldots)$$

Since the sequence seems to be converging, we conclude that 1.1144 is approximately a root.



This method can be attempted for any differentiable function, though the sequence isn't guaranteed to converge: play with some examples such as $f(x) = x^2 - 2$, and $f(x) = x^3 - 5x$. You can find graphical interfaces online for this (for instance with Geogebra).