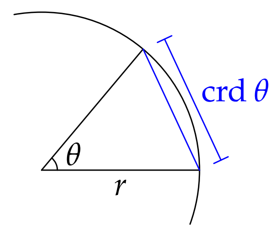
2 Trigonometric Functions and Polar Co-ordinates

In this chapter we review trigonometry and periodic functions and discuss their relation to polar co-ordinates. Some of this will be non-standard.

2.1 Definitions & Measuring Angles



Trigonometric functions date back at least 2000 years. Ancient mathematicians were interested in the relationship between the *chord* of a circle and the central angle, often for the purpose of astronomical measurement. It wasn't until 1595 that the term *trigonometry* (literally *triangle measure*) was coined and the functions were considered as coming from triangles.

As with a lot of grade-school mathematics, it is likely that you know several ways of getting at the basic trigonometric functions.

Definition 2.1. Here are several related definitions.

1. (a) Given a right triangle with *hypotenuse* (longest side) 1 and angle θ, define $\sin\theta$ and $\cos\theta$ to be the side lengths *opposite* and *adjacent* to θ.

   Define $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to be the slope of the hypotenuse.

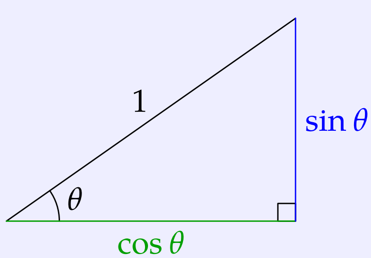
   (b) Given a right triangle with angle θ, *hypotenuse* r, *adjacent* x and *opposite* y, define

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

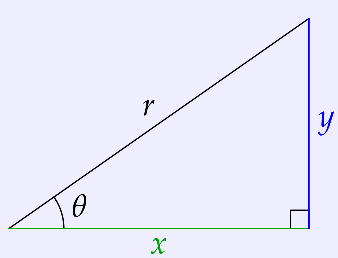
2. (a) The co-ordinates $(\cos\theta, \sin\theta)$ describe a point on the unit circle where θ is the *polar angle* measured counter-clockwise from the positive x-axis.

   (b) The co-ordinates $(r\cos\theta, r\sin\theta)$ describe a point on the circle of *radius* $r \geq 0$ centered at the origin, where θ is again the polar angle. We call (r, θ) the *polar co-ordinates* of the point.

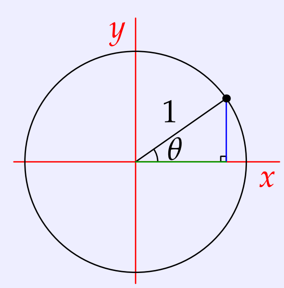
   In either case, if $\cos\theta \neq 0$, we also define $\tan\theta = \frac{\sin\theta}{\cos\theta}$.



Definition 1(a)



Definition 1(b)



Definition 2(a)

Discuss some of the advantages and weaknesses of these definitions:

- What prerequisites are you assuming in each case?
- Is it easier to think about *lengths* rather than ratios?
- Where do you need basic facts from Euclidean geometry such as *congruent*/*similar* triangles?
- Convince yourself that that the triangle definitions follow from the circle definitions. What is missing if you try to use the triangle definition to justify the circle version?
- If you were introducing trigonometry for the first time, which approach would you use?

You might even have met other definitions, for instance using power (Maclaurin) series or differential equations. Plainly these are not suitable for grade-school, but have the great benefit of making the calculus relationship $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta = \cos\theta$ very simple. Establishing this using the triangle definition is a somewhat tricky!

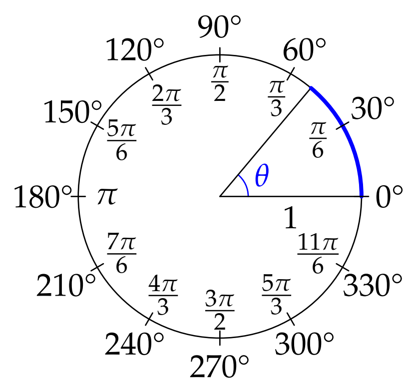
Measuring Angles

There are two standard ways to measure angles (to sensibly associate a *number* to each angle).

Degrees A full revolution has 360° and a right-angle 90°. Degree-measure dates back to ancient Babylon 2–4000 years ago.[8]

Radians The radian measure of an angle is the length of the arc subtending the angle in a circle of radius 1. Since the circumference of a unit circle is 2π, we have the following identifications.

Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	n/a
180°	π	0	−1	0



In elementary mathematics, degrees are the most common way to measure angles. Do you know any other methods?

[8]It is not known why they chose 360, but it fits nicely with their *base-60* system of counting (decimals are base-10). The traditional subdivisions of a degree are also base-60. For instance, 34°12′45″ is 34 degrees, 12 (arc)minutes and 45 (arc)seconds; converted to decimal notation, this becomes

$$34°12'45'' = 34 + \frac{12}{60} + \frac{45}{60^2} = 34.2125°$$

The standard hour-minute-second measurement of time has the same origin.

Exercises 2.1. *Key concepts: Multiple definitions of sine and cosine, Degrees, Radians*

1. The identity $\cos^2\theta + \sin^2\theta = 1$ is the Pythagorean Theorem in disguise. Why?

2. The word *sine* is the result of a long list of translations and transliterations from an ancient Sanskrit term meaning *half-chord.* For the chord picture on page 29, how does the length of the chord $\text{crd}\,\theta$ relate to modern trigonometric functions?

3. It is conventional *not* to state units when using radians since they are effectively a ratio and therefore *unitless.* Think this through: if the central angle in a circle of radius r is subtended by an arc with length ℓ, what is the radian measure of the angle? What facts from basic geometry justify this observation?

4. Explain how to get the values of sine and cosine in the above table.

   (*Hint: Draw some triangles and use Pythagoras*!)

5. Using one of more pictures in Definition 2.1, explain why the following relations are true

   $$\sin(\tfrac{\pi}{2} - \theta) = \cos\theta = \sin(\theta + \tfrac{\pi}{2}), \qquad \sin(-\theta) = -\sin\theta, \qquad \sin(\pi - \theta) = \sin\theta$$

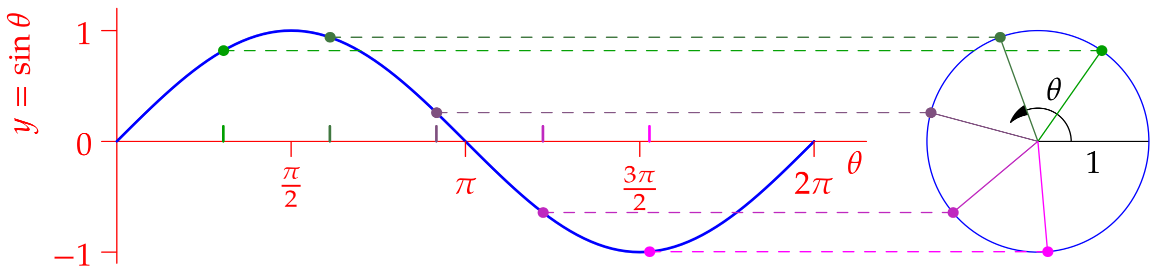
   (*You cannot use graphs or the multiple-angle formulæ for this!*)

6. Consider the triangles in Definition 2.1. If you know the value of $\sin\theta$, what do you know about the values of $\cos\theta$ and $\tan\theta$?

   To be specific, if $\sin\theta = \frac{12}{13}$, what can you say about $\cos\theta$ and $\tan\theta$?

2.2 Periodicity, Graphs & Inverses

One advantage of the circle definition is that it makes sketching the graph of sine very easy. Simply draw axes next to a unit circle and transfer the height across!



You *could* try the same trick for the cosine graph (plot the x-co-ordinate), though it is tricky to get the picture correct. Alternatively, by Exercise 2.1.5, the graph of $\cos\theta = \sin(\theta + \frac{\pi}{2})$ is simply that of sine shifted $\frac{\pi}{2} = 90°$ to the *left*.

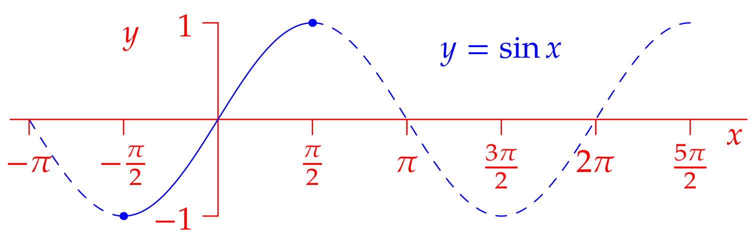
The circle definition also allows us to extend trigonometric functions *periodically* since we can measure the polar angle by looping as many times round the origin as we like: that is, for any integer n,

$$\sin(\theta + 2n\pi) = \sin\theta, \quad \cos(\theta + 2n\pi) = \cos\theta$$

Otherwise said, sine and cosine have period 2π radians (360°).

Inverse Trigonometric Functions

Sine and cosine are *non-invertible* unless we choose a domain on which they are 1–1. The standard convention is to restrict to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$: the graph should make it clear why.

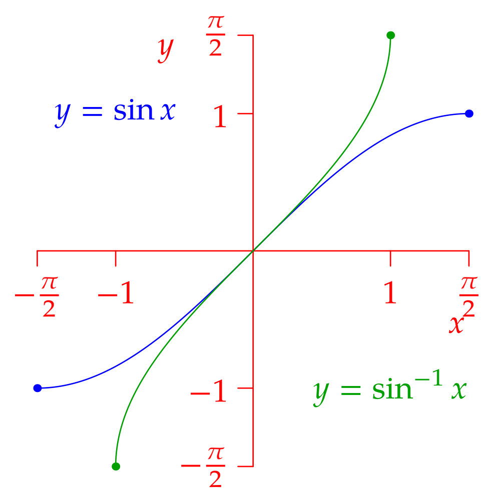


$f(x) = \sin x$ is 1–1 on the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Denote the inverse function $f^{-1}(x) = \arcsin x = \sin^{-1} x$

Domain $\operatorname{dom}(\arcsin) = [-1, 1] = \operatorname{range}(\sin)$

Range $\operatorname{range}(\arcsin) = [-\frac{\pi}{2}, \frac{\pi}{2}] = \operatorname{dom}(\sin)$

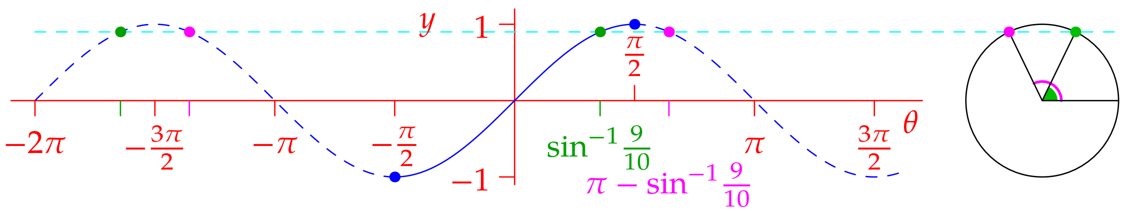


These choices are why your calculator returns a value in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}] = [-90°, 90°]$ whenever you hit the $\sin^{-1}$ button.

Thinking about the inverse cosine and

Example 2.2. If you know the graphs, then symmetry and periodicity help you solve equations. For example, if $\sin\theta = \frac{9}{10}$ then all solutions are given by

$$\theta = \sin^{-1}\frac{9}{10} + 2\pi n \quad \text{or} \quad \pi - \sin^{-1}\frac{9}{10} + 2\pi n \qquad (n \text{ is any integer})$$



Alternatively, we could use the circle definition directly: $\sin\theta = \frac{9}{10}$ means we want angles θ corresponding to the intersections of the unit circle with the *horizontal line* $y = \frac{9}{10}$.

Periodic Models

Trigonometric functions find applications in modeling precisely because they are *periodic*. In general, a function has period T if

$$f(x+T) = f(x) \text{ for all } x$$

If k is constant, it is easy to find the period of the function $f(x) = \sin kx$ just by considering what we have to add to the input x to increase the argument kx by 2π:

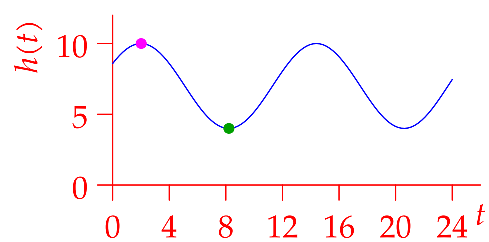
$$T = \frac{2\pi}{k} \implies f(x+T) = \sin(kx + 2\pi) = \sin kx = f(x)$$

We may therefore obtain a simple periodic model regardless of what period is required.

Example 2.3. On a given day at the local pier, high tide occurs at 02:00 with a water depth of 10 ft and low tide at 08:12 with a depth of 4 ft.

We might model the water depth using a periodic function. Since it takes 6 hours and 12 minutes to go from high to low tide, the function should have period $T = 2\times 6\frac{12}{60} = \frac{62}{5}$ hours. In this case

$$h(t) = 7 + 3\cos\left(\frac{5\pi}{31}(t-4)\right)$$



would do the trick, where t is measured in hours after midnight (is it clear why? How did we get this?).

In reality, tidal height is very close to being periodic, but the magnitude of the high and low tides are somewhat variable.

Fourier Series (non-examinable) In fact *any* periodic function may be approximated using trigonometric functions. This is easy to state if you are familiar with integration. If $f(x)$ has period T and we define constants

$$a_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos\frac{2\pi nx}{T}\,\mathrm{d}x, \qquad b_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\sin\frac{2\pi nx}{T}\,\mathrm{d}x \tag{$*$}$$

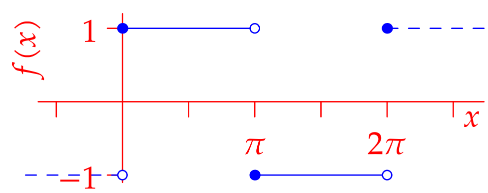
then

$$f(x) \approx \frac{a_0}{2} + a_1\cos\frac{2\pi x}{T} + b_1\sin\frac{2\pi x}{T} + a_2\cos\frac{4\pi x}{T} + b_2\sin\frac{4\pi x}{T} + \cdots \tag{$\dagger$}$$

This expression is called the *Fourier series* of $f(x)$. It often takes only a small number of terms to obtain a pretty good approximation. Modern data-compression algorithms often employ Fourier series. Given a periodic function $f(x)$, one estimate (say) the first 10 Fourier coefficients ($*$)—for a real-world function, this would likely be done by computer. The Fourier coefficients are transmitted to the receiver, who quickly recovers an approximation to the original function using ($\dagger$).

Example 2.4. A square-wave function with period $T = 2\pi$ is given by

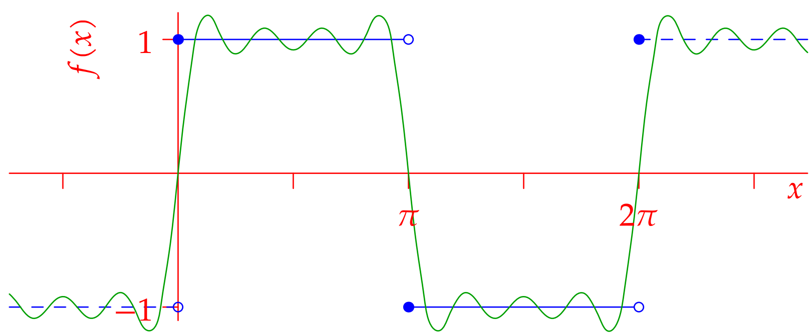
$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \pi \\ -1 & \text{if } \pi \le x < 2\pi \end{cases}$$



extended periodically to the real line. If you are comfortable with basic integrals, it is easily checked that the Fourier coefficients are

$$a_n = \int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x = 0, \qquad b_n = \int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Use a graphics tool to see how the first few terms of the series approximate the function. The graph obtained by taking the first four non-zero terms is shown below.



The graph of $y = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x\right)$

Exercises 2.2. *Key concepts: Finding the sine graph from a circle, Periodicity, Choosing a domain for inversion, Solving algebraic trigonometric equations, Modeling using periodic functions*

1. $f(x) = \sin x$ is also 1–1 on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Sketch the graph of its corresponding inverse function.

2. Draw the graph for cosine and observe that it is invertible if we restrict the domain to the interval $[0, \pi]$. Draw the graph of $\cos^{-1}$.

3. Describe all solutions to the equation $\cos x = -0.2$.

4. (a) Explain why the tangent function has period π; that is $\tan(\theta + n\pi) = \tan\theta$. What facts are we using about sine and cosine and why are they obvious from the definition?

   (b) Graph the tangent function. Find an interval on which the tangent function is 1-1. Draw its inverse function.

5. Describe all solutions to the equation $\tan x = 5$.

6. (a) Suppose $\theta = \cos^{-1}\frac{9}{41}$. Find the exact values for $\sin\theta$ and $\tan\theta$.
   (*Hint: draw a right-triangle. Assume* $\cos^{-1}$ *means what your calculator thinks it does...*)

   (b) What changes if $\theta = \cos^{-1}\frac{-9}{41}$?

7. Let $f(x) = \csc x = \frac{1}{\sin x}$ be the cosecant function. Describe a domain on which this function is 1–1 and sketch the graph of its inverse $y = f^{-1}(x)$.

8. Use a computer to sketch the curve

$$y = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x\right)$$

   What simple periodic function do you think this is approximating?

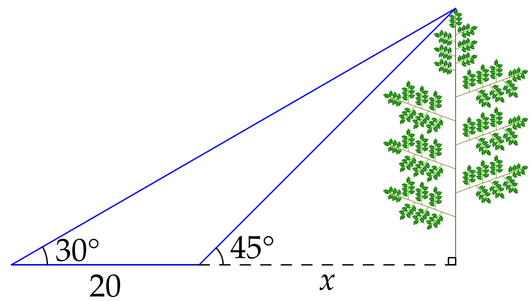
2.3 Solving Triangles

Basic trigonometric calculations typically involve finding the edges and angles of a triangle given partial data.

Example 2.5. To find the height h of a tall tree, two angles of elevation 45° and 30° are measured a distance 20 ft apart along a straight line from the base of the trunk.

This problem is easily attacked by drawing a picture and observing that the tree-trunk completes two right-triangles. If the (unknown) distance from the base of the tree to the nearer measurement is x, then

$$\frac{1}{\sqrt{3}} = \tan 30° = \frac{h}{x+20} \qquad 1 = \tan 45° = \frac{h}{x}$$



Substitute the second equation into the first to obtain

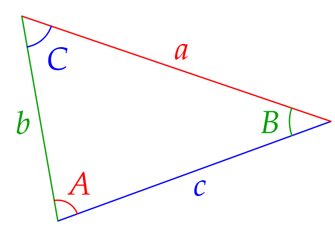
$$h = \frac{20}{\sqrt{3}-1} \approx 27.32\,\text{ft}$$

While this certainly answers the original question, there is in fact enough data in the problem to recover everything about the original triangle.

- The second base angle is $(180° - 45° = 135°)$.
- The third (summit) angle is $180° - 30° - 135° = 15°$.
- Two applications of Pythagoras' Theorem compute the remaining sides of the triangle

$$\sqrt{x^2+h^2} = \sqrt{2}h = \frac{20\sqrt{2}}{\sqrt{3}-1} \approx 38.64\text{ ft}$$
$$\sqrt{h^2+(x+20)^2} = \sqrt{h^2+3h^2} = 2h = \frac{40}{\sqrt{3}-1} \approx 54.64\text{ ft}$$

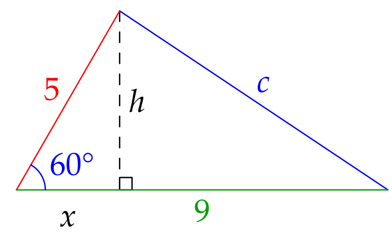
The example is a disguised version of *solving a triangle*: given three pieces of data (from the three sides and angles), compute the remaining three. The Euclidean triangle congruence theorems tell us which combinations are sufficient to determine all the others. The example is the ASA congruence: angle-side-angle data (30°–20–135°) is enough to compute everything else about the triangle.



When in doubt, basic trigonometry problems can always be attacked as in the example: drop a perpendicular (the tree trunk!) to create a right-triangle, then use the definitions of sin/cos/tan and/or Pythagoras.

Example 2.6. Given the SAS (side-angle-side) combination 5–60°–9, find the third side of the triangle.

The altitude h creates two right-triangles, from which



$$\begin{aligned} &h = 5\sin 60°, \quad x = 5\cos 60° \\ \implies & c^2 = (9-x)^2 + h^2 = 9^2 + (x^2+h^2) - 18x \\ &\quad = 9^2 + 5^2 - 18 \cdot 5\cos 60° = 61 \\ \implies & c = \sqrt{61} \approx 7.81 \end{aligned}$$

Since we now know c and $9 - x = \frac{13}{2}$ the remaining angles could also be easily found.

The Sine and Cosine Rules

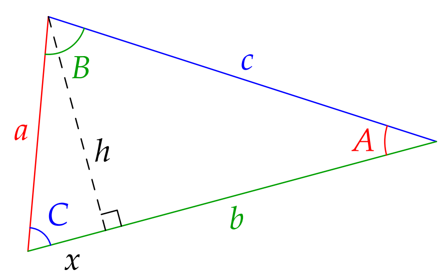
In elementary situations it is typically easier to have students drop the perpendicular as we've done. However, once comfortable with the method, it is helpful to have short-cuts which skip the need to work with the perpendicular at all.

Theorem 2.7. *For any triangle (labeled so that a is the side opposite angle A, etc.),*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

The cosine rule is just Pythagoras' Theorem with a correction for non-right triangles. Both rules follow straightforwardly by dropping an altitude as before!

Proof. Consider the picture where we dropped an altitude and defined



$$h = a\sin C = c\sin A, \quad x = a\cos C, \quad b - x = c\cos A$$

The first equation rearranges to

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

Symmetry (drop a second altitude) for the rest.

The cosine rule comes from the same picture via two applications of Pythagoras,' first on the right half, then the left:

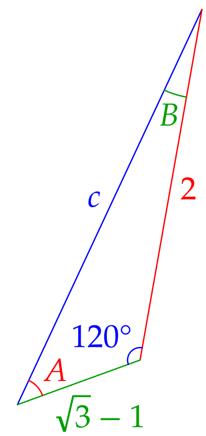
$$\begin{aligned} c^2 &= h^2 + (b-x)^2 = h^2 + x^2 + b^2 - 2bx \\ &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

The other versions of the cosine rule are obtained by choosing other altitudes. ■

Examples 2.8. We use the rules instead of explicitly drawing an altitude, though the latter approach is always acceptable.

1. A triangle has sides 2 and $\sqrt{3}-1$, and the angle between them is $120°$. Find the remaining sides and angles.

   We apply the cosine rule with $a=2$, $b=\sqrt{3}-1$ and $C=120°$

   $$\begin{aligned}c^2&=a^2+b^2-2ab\cos C\\&=2^2+(\sqrt{3}-1)^2-2\cdot 2(\sqrt{3}-1)\cos 120°\\&=4+3+1-2\sqrt{3}+2(\sqrt{3}-1)=6\end{aligned}$$

   We have an opposite pair $(c,C)=(\sqrt{6},120°)$, so the sine rule may be applied

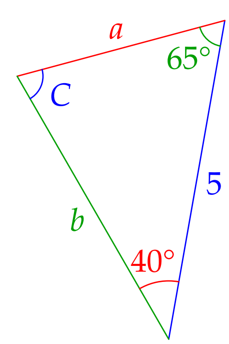
   $$\sin A=\frac{2}{\sqrt{6}}\sin 120°=\frac{2\sqrt{3}}{2\sqrt{6}}=\frac{1}{\sqrt{2}}\implies A=45°$$

   We chose the acute angle since $A=180°-B-C=60°-B<90°$.
   The final angle is then $B=180°-45°-120°=15°$.

   You could instead drop a perpendicular, say from the vertex A to the *extension* of the side of length 2. Think about why the perpendicular has to be *outside* the triangle...

2. A triangle has one side with length 5 and its two adjacent angles are $40°$ and $65°$. Find the remaining data.

   This time the initial data is ASA. Writing $c=5$, $A=40°$ and $B=65°$, the remaining angle is plainly

   $$C=180°-40°-65°=75°$$

   This gives us an opposite pair (c,C), so we can apply the sine rule

   $$a=c\frac{\sin A}{\sin C}=5\frac{\sin 40°}{\sin 75°}\approx 3.327$$

   A second application yields

   $$b=c\frac{\sin B}{\sin c}=5\frac{\sin 65°}{\sin 75°}\approx 4.691$$

3. (Courtesy of an 8 year-old contributor) Suppose the Earth is modeled as a sphere of radius 3960 miles. Suppose identically tall Giants stood in California and Scotland, 5150 miles apart by great circle; how tall would they (their eyes) have to be to 'see' each other?

   What does this problem have to do with triangles?

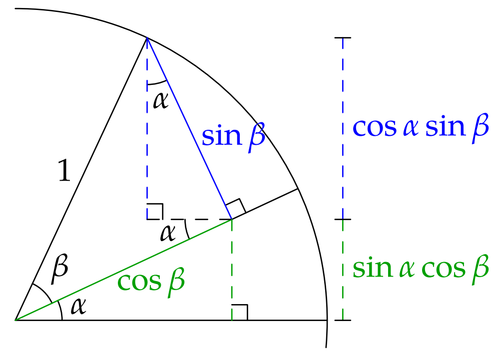
Multiple-angle Formulae

Also useful in the context of basic trigonometry is the ability to sum angles. The picture provides a simple justification of

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

at least when $0 < \alpha + \beta < \frac{\pi}{2}$. If you look carefully, you should be able to see how the same picture establishes

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$



Exercises 2.3. *Key concepts: Solving triangles means finding sides and angles given three pieces of data, Relation to the Euclidean triangle congruence theorems* (*SAS, SSS, ASA, SSA*)

1. Find the remaining angles in the triangle in Example 2.6.
2. The other Euclidean congruence theorems are SSS and SAA. Explain how to solve triangles using these minimal data in two ways:

   (a) By drawing an altitude.
   (b) Using the sine/cosine rules.
3. SSA isn't a triangle congruence theorem. For instance, there are *two* non-congruent triangles with data $a = 1$, $b = \sqrt{3}$ and $A = 30°$. Find them.
4. Use the multiple-angle formulae to derive the familiar expressions for $\sin 2\theta$ and $\cos 2\theta$.
5. Find the exact value of $\sin 105°$.
6. (a) Find an expression for $\tan(\alpha+\beta)$ purely in terms of $\tan\alpha$ and $\tan\beta$.

   (b) Two wooden wedges with slope $\frac{1}{4}$ are placed on top of each other to make a steeper slope. What is the gradient of the new slope?
7. Calculate the required heights of the Giants in Example 2.8.3

2.4 Polar Co-ordinates

Definition 2.1 provides an alternative way to describe points in the plane. If θ is the polar angle of a point with Cartesian (rectangular) co-ordinates (x, y), then its polar-coordinates are precisely the values (r, θ) in the definition!

Example 2.9. Computing $x = r\cos\theta$ and $y = r\sin\theta$ is easy given r and θ. For instance, the point with polar co-ordinates $(r, \theta) = (2, \frac{5\pi}{6})$ has Cartesian co-ordinates

$$(x, y) = \left(2\cos\tfrac{5\pi}{6}, 2\sin\tfrac{5\pi}{6}\right) = \left(-\sqrt{3}, 1\right)$$

Computing polar co-ordinates from Cartesian is harder, requiring at least some visualization.

1. Every point (x, y) has a unique radius $r = \sqrt{x^2 + y^2}$ but non-unique polar angle. If θ is a polar angle, so also is $\theta + 2\pi n$ for any integer n. The origin $(x, y) = (0, 0)$ is even stranger; certainly $r = 0$, but *any* θ provides a legitimate polar angle!

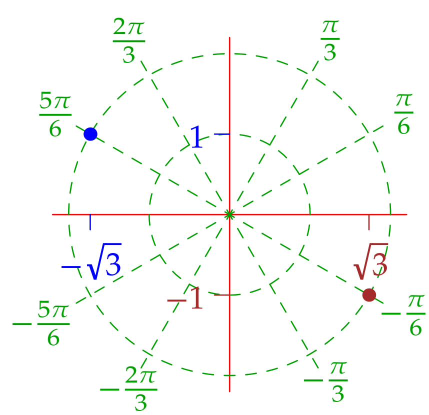
2. Whenever $x \neq 0$ (away from the y-axis),

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x}$$

however, this doesn't guarantee that $\theta = \tan^{-1}\frac{y}{x}$. Continuing the example shows us why...

Example (2.9, cont). If $(x, y) = (-\sqrt{3}, 1)$, then the radius is easy

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$



For the polar angle,

$$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right) \not\Rightarrow \theta = -\frac{\pi}{6}$$

however we cannot (and should not!) conclude that $\theta = -\frac{\pi}{6}$. Arctan has range $(-\frac{\pi}{2}, \frac{\pi}{2})$ and so always returns an angle in quadrants 1 or 4. Our point is in the *second* quadrant ($x < 0 < y$) so we need to adjust. We use the fact that tan is π-periodic:

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} = 150°$$

We could alternatively add any integer multiple of 2π.

The example wasn't too tricky since the polar angle was exactly computable. When you have to rely on a calculator, it is much easier to make a mistake.

Example 2.10. The point $(x, y) = (-8, -15)$ has polar co-ordinates (quadrant 3!)

$$r = \sqrt{8^2 + 15^2} = 17, \qquad \theta = \pi + \tan^{-1}\frac{15}{8} \approx 242°$$

We could summarize with a formula describing precisely how to compute θ dependent on quadrant (the signs of x, y), though it is better to get in to the habit of drawing a picture.

Curves in Polar Co-ordinates

Polar co-ordinates are well-suited to describing curves that encircle the origin. Indeed circles centered at the origin with radius $a > 0$ have the very simple polar form $r = a$. Converting to rectangular co-ordinates recovers the the natural parametrization of a circle:

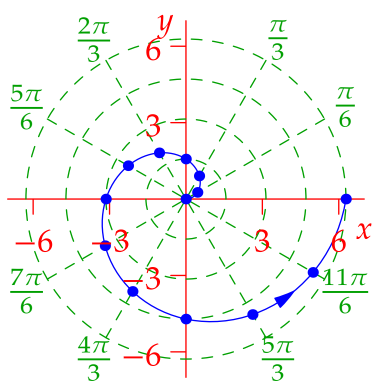
$$x(\theta) = a\cos\theta, \quad y(\theta) = a\sin\theta$$

This partly explains why mathematicians call sine and cosine *circular functions.*

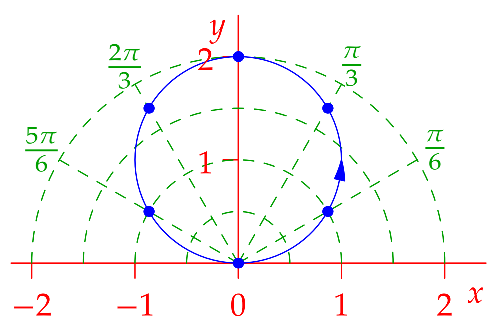
General polar graphs are harder to visualize, though the major reason is lack of familiarity. Have a little empathy: whatever discomfort you feel graphing *polar* functions is similar to what grade-school students feel when asked to sketch Cartesian curves for the first time.

Examples 2.11. We plot two examples of polar curves. Try drawing out polar grid lines and making a table of points for nice values of θ.

1. The curve $r = \theta$ is easy to plot since r increases at exactly the same rate as the angle; we therefore have a *spiral.* We've plotted points for θ a multiple of $\frac{\pi}{6}$ (30°) from 0 to 2π.



Example 1: $0 \le \theta \le 2\pi$



Example 2: $0 \le \theta \le \pi$

2. The curve $r = 2\sin\theta$ is a little easier to work with, since we know exact values for sine, assisted by $\sqrt{3} \approx 1.73$.

   This looks like a circle! To confirm, multiply both sides by r and complete the square:

θ	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	$\frac{4\pi}{3}$	$\frac{5\pi}{6}$	π
$2\sin\theta$	1	1.73	1	1.73	1	0

$$r^2 = 2r\sin\theta \implies x^2 + y^2 = 2y \implies x^2 + (y-1)^2 = 1$$

   This describes the set of points with distance 1 from the point $(0, 1)$: a circle!

You should think about what happens in both examples if we extend the domain:

- What would $r = \theta$ look like if θ were allowed to be *negative*?
- What happens to $r = 2\sin\theta$ when $\theta > \pi$?

Exercises 2.4. *Key concepts: Converting to and from polar co-ordinates, Graph sketching*

1. Convert the following points to polar co-ordinates.
   (a) $(-5, 5)$
   (b) $(3, -4)$
   (c) $(-5\sqrt{3}, -15)$
   (d) $(-1, \tan 3)$ (careful—this is 3 radians!)
2. Define a function $\theta(x, y)$ returning the polar angle $\theta \in [0, 2\pi)$ for *any* non-origin point $(x, y) \neq (0, 0)$.
   (*The answer is not pleasant: think about arctan, and don't divide by zero*...)
3. If $a > 0$, describe the curve with polar equation $r = 2a\cos\theta$.
   (*Be careful with* $\theta > \frac{\pi}{2}$ *since cosine goes negative*...)
4. The algebraic trickery in the Example 2.11.2 sometimes bears fruit, though you have to be lucky! By multiplying both sides by $1 - \sin\theta$ and converting to rectangular co-ordinates, show that the polar function
   $$r(\theta) = \frac{2a}{1 - \sin\theta}$$
   is a parabola in disguise and sketch it when $a = 1$. How does the graph depend on a?
5. In a similar vein to Exercise 4, sketch the curve $r = \frac{2}{2+\sin\theta}$. What type of curve is this?
6. Try to sketch the following curves.
   (a) $r = \theta(\theta - 4)$
   (b) $r = (\theta - 1)^2 + 1$
   (c) $r = (\theta - 1)^2 - 1$

   If you're unsure what to do, plot some points as in the examples. Perhaps also sketch the curve first on rectangular axes (e.g., (a) corresponds to $y = x(x - 4)$). What happens to (c) when $\theta = 1$?

   Once you've tried these, use some technology to see how close you got to the real thing.