3 Exponential and Logarithmic Functions & Models

Introducing exponential functions without calculus presents a significant challenge. The simplest approach is as a short-hand notation for *repeated multiplication*: for instance

$$a^5 = a \cdot a \cdot a \cdot a \cdot a$$

analogous to how multiplication represents repeated addition

$$5a = a + a + a + a + a$$

The problem with this approach is that it doesn't help you understand what should be meant by, say, $a^{3/4}$ or $a^{\sqrt{2}}$: multiplying something by itself '$\sqrt{2}$ times' sounds[9] insane!

Rigorously addressing this problem requires *continuity* and other ideas which you'll encounter in upper-division analysis; topics unsuitable for this course. Instead, we assume some familiarity with exponential functions in the context of introductory calculus, and offer two ways to introduce exponential functions and e via modeling.

3.1 The Natural Growth Model and the Natural Exponential

A basic model for any variable quantity is that its *rate of change be proportional to the quantity itself.* This idea necessarily needs some calculus; as a differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

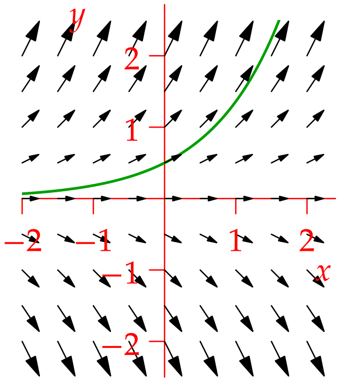
where k is a constant; if $k > 0$ this is the *natural growth equation*, if $k < 0$ the *natural decay equation.* This is commonly encountered when modelling *population growth*; an otherwise unconstrained population seems like its growth rate should be proportional to its size (twice the people, twice the babies...). This model is hugely applicable, since *population* can refer to essentially any quantifiable value: people, bacteria, money, reagents in a chemical/nuclear reaction, etc.

Example 3.1. The simplest natural growth equation has $k = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

If a point (x, y) lies on a solution curve, then the differential equation tells us the solution's *direction of travel*. Visualize this by drawing an arrow with slope $\frac{\mathrm{d}y}{\mathrm{d}x} = y$; the arrows are *tangent* to any solution.[10] You should easily be able to sketch some other solution curves.

You should, of course, recognize the graph...



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[9] The same issue arises for multiplication: $3\sqrt{2} = \sqrt{2} + \sqrt{2} + \sqrt{2}$ is relatively easy to understand, but how would you convince someone what $\pi\sqrt{2}$ means?

Definition 3.2. Let $a > 0$ be constant. The *exponential function with base a* is $f(x) = a^x$.

Recall the *exponential laws*, which are very natural when x, y, r are positive integers:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^r = a^{rx}$$

These hold for all exponents, with the same continuity caveats mentioned previously. For modeling, the crucial property of exponential functions is that they have proportional derivatives.

Theorem 3.3. *The rate of change of $f(x) = a^x$ is proportional to $f(x)$. Specifically,*

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = \left(\lim_{h\to 0} \frac{a^h - 1}{h}\right) a^x$$

Otherwise said, $f(x) = a^x$ satisfies the natural growth/decay equation $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$ with proportionality constant

$$k = f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h}$$

It is, again, a matter for a more advanced course to see that the limit $\lim\limits_{h\to 0} \frac{a^h-1}{h}$ always exists. For our purposes, estimation is enough.

Example 3.4. We estimate the proportionality constant $k = \lim\limits_{h\to 0} \frac{a^h-1}{h}$ to 3 d.p. using a calculator for four values of h:

a	2	2.5	2.7	2.75	3	5
$\frac{a^{0.1}-1}{0.1}$	0.718	0.960	1.044	1.065	1.161	1.746
$\frac{a^{0.01}-1}{0.01}$	0.696	0.921	0.998	1.017	1.105	1.622
$\frac{a^{0.001}-1}{0.001}$	0.693	0.917	0.994	1.012	1.099	1.611
$\frac{a^{0.0001}-1}{0.0001}$	0.693	0.916	0.993	1.012	1.099	1.610
$\frac{a^{0.00001}-1}{0.00001}$	0.693	0.916	0.993	1.012	1.099	1.609

What appears to be happening to the proportionality constant as a increases?

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[10] A similar approach is available for any first-order differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y)$: the equation defines its *slope field* (arrows), to which solution curves must be tangent.

The Natural Exponential Function

It certainly appears as if there is a special number somewhere between 2.7 and 2.75 for which the proportionality constant is precisely $k = \lim_{h\to 0} \frac{a^h-1}{h} = 1$.

Definition 3.5. The value $e = 2.71828\ldots$ is the unique real number such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$. The *natural*[11] *exponential function* $\exp(x) = e^x$ has derivative $\frac{d}{dx}e^x = e^x$.

The function $f(x) = \frac{1}{2}e^x$ is plotted in Example 3.1. Of course there are many other solutions to the natural growth equation $\frac{dy}{dx} = y$: for any constants c, k,

$$y = ce^{kx} \implies \frac{dy}{dx} = \frac{d}{dx}ce^{kx} = cke^{kx} = ky$$

In fact the converse also holds.

Theorem 3.6. *The solutions to the natural growth equation* $\frac{dy}{dx} = ky$ *are precisely the functions*

$$y(x) = y_0e^{kx} = y_0\exp(kx) \qquad (\exp(x) = e^x \text{ is alternative notation})$$

where $y_0 = y(0)$ *is the initial value.*

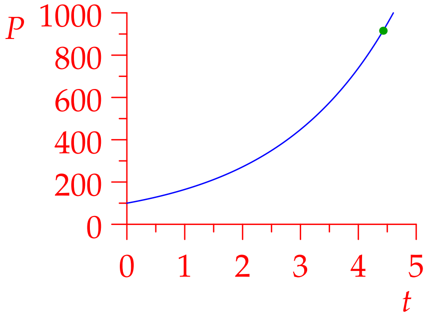
The converse can be verified using the standard technique of separation of variables as seen in any differential equations course.[12]

Example 3.7. A Petri dish contains a population $P(t)$ of bacteria satisfying the natural growth equation $\frac{dP}{dt} = 0.5P$ where time is measured in weeks from the start of the year.

If $P(0) = 100$ bacteria, then $P(t) = 100e^{0.5t}$. Specifically, at the end of January ($4\frac{3}{7}$ weeks) one expects there to be

$$P(\tfrac{31}{7}) = 100\exp\tfrac{31}{14} = 915 \text{ bacteria}$$

The exponential doesn't return 915 exactly; this is only an approximation. Models like this work best for large populations where integer rounding errors are of minimal concern.



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[11] *Natural* here means *unavoidable*. It is now cliché to suggest that if aliens were to land on Earth, they'd have to understand e given the technology required to get here. Of course they'd likely use a different symbol; ours comes from Leonhard Euler around 1728. Like π and $\sqrt{2}$, the constant e is an *irrational number*: its decimal representation contains no repeating pattern. There isn't the same geeky fascination with memorizing the digits of e as there is with π, neither is there an 'e-day' (Feb 7th at 6:28 p.m. perhaps?).

[12] Ignoring the trivial case when $y(x) \equiv 0$, the substitution rule from integration says

$$\frac{dy}{dx} = ky \implies \int \frac{1}{y}\frac{dy}{dx}\,dx = \int k\,dx \implies \int \frac{1}{y}\,dy = \int k\,dx \implies \ln|y| = kx + C\ldots$$

Compound Interest and the Discovery of e

The first description of e as a number came in 1683 when Jacob Bernoulli tried to model the growth of money in a hypothetical bank account. Here is a modernized version of his approach.

Example 3.8. $1 is deposited in an account paying 100% interest per year (nice!). Bernoulli observed that account balance at the end of the year depends on *when* the interest is paid.

- If the interest is paid once at the end of the year (*simple* interest), the final balance is $2.
- If half the interest (50¢) is paid at six months, then $1.50 earns $\frac{1}{2}\cdot 1.50 = 75$¢ interest for the rest of the year; at year's end, the account contains $2.25.
- If the interest is paid in four installments, we have the following table of transactions (data is rounded to the nearest cent)

Date	Interest Paid	Balance
1^{st} Jan	—	$1
1^{st} Apr	25¢	$1.25
1^{st} July	$\frac{1}{4}\cdot 1.25 = 31$¢	$1.56
1^{st} Oct	$\frac{1}{4}\cdot 1.56 = 39$¢	$1.95
New Year	$\frac{1}{4}\cdot 1.95 = 49$¢	$2.44

  In brief, the year-end balance is $\left(1+\frac{1}{4}\right)^4 = \$2.44$.

- In general, if the interest is paid over n equally spaced intervals, the account balance at year's end is $\$\left(1+\frac{1}{n}\right)^n$. Here are a few examples rounded to 5 d.p.

Frequency	Balance at year's end ($)
Every month	$\left(1+\frac{1}{12}\right)^{12} = 2.61304$
Every day	$\left(1+\frac{1}{365}\right)^{365} = 2.71457$
Every hour	$\left(1+\frac{1}{8760}\right)^{8760} = 2.71813$
Every second	$\left(1+\frac{1}{31536000}\right)^{31536000} = 2.71828$

  As the frequency of payment increases, it appears as if the balance is approaching $$e$...

In fact this is a theorem, though its proof requires significant work (beyond this class):

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{and more generally} \quad e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

This again shows that e arises very naturally as the limit of a process of repeated multiplication.

Simple, Monthly & Continuous Interest In finance, interest is typically computed in one of three ways. In each case we describe the result of investing \$1 at an annual interest rate of $r\% = \frac{r}{100}$.

Simple interest The account pays $\$\frac{r}{100}$ at year's end. Your invested dollar becomes $\$\left(1+\frac{r}{100}\right)$.

Monthly interest The account pays simple interest of $\frac{r}{12}\%$ each month, so that each month's interest itself earns interest in future months. At year's end, the account balance[13] will be $\$(1+\frac{r}{1200})^{12}$.

Continuous interest After t years (any fraction of a year) the dollar-balance of the account will be

$$e^{\frac{rt}{100}} = \exp\frac{rt}{100} = \lim_{n\to\infty}\left(1+\frac{rt}{100n}\right)^n$$

Example 3.9. A bank account pays 6% annual interest in equal monthly installments. To what simple annual interest rate does this correspond? Would you perfer an account paying 6% continuously?

Investing \$1 for a year in this account, results in a dollar-balance

$$\left(1+\frac{0.06}{12}\right)^{12} = 1.005^{12} \approx 1.06168\ldots$$

This corresponds to a simple interest rate (one payment at year's end) of 6.17%.

By contrast, 6% continuous interest result in your dollar becoming $\$e^{6/100} \approx 1.06184$, corresponding to a (marginally higher) simple interest rate of 6.18%. This is plainly preferable, particularly if you have a lot of money to invest! The difference is more noticable with an investment of \$1,000,000 over ten years:

$$1{,}000{,}000 \times 1.005^{120} = \$1{,}819{,}396.73 \quad \text{versus} \quad 1{,}000{,}000e^{0.6} = \$1{,}822{,}118.80$$

There are several reasons for these varying approaches, not all of them consumer-friendly:

1. Simple interest is simple! It is easy to understand and compute, but hard to decide how or even whether to compute interest if an account is closed part-way through a year.
2. Monthly interest fits with the frequency of most paychecks, and is therefore commonly how basic savings accounts, loans and mortgages compute interest.
3. Continuous interest allows the balance of an account to be computed easily at any time. It is also much easier to apply mathematical analysis (calculus).
4. A bank can make an interest rate appear *higher* (if a savings account) or *lower* (if a loan) by choosing different ways to quote interest rates.

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[13] The period need not be monthly, though this is most common in practice: if interest is paid in n equally-spaced installments, the final balance is $\$(1+\frac{r}{100n})^n$.

Example 3.10. A bank quotes you a loan with an interest rate of 7% (continuously compounded). If you borrow \$100,000, then at the end of the year you'll owe

$$100000e^{0.07} = \$107{,}250.82$$

not the \$107,000 you might have expected! This corresponds to a simple interest rate (one payment at the end of the year) of 7.25%.[14]

Exercises 3.1. *Key concepts: Exponentials are naturally related to calculus and limits, Compound interest (simple, regular, continuous)*

1. Draw a slope field for the natural decay equation $\frac{dy}{dx} = -\frac{1}{3}y$ and use it to sketch the solution curve with initial condition $y(0) = 6$. What is the *function* $y(x)$ in this case?
2. Which of the following would you prefer for a savings account? Why?
   - 5% interest paid continuously.
   - 5.05% compounded monthly.
   - 5.1% paid at the end of the year.
3. You invest \$1000 in an account that pays 4% simple interest per year.
   (a) How much money will you have after 5 years?
   (b) If you close the account after 2 years and 3 months, the bank needs to decide how much interest to credit you with. Do this in two ways (the answers will be different!):
      i. Compute using the simple interest rate for 2.25 years ($(1 + \frac{r}{100})^{2.25}$).
      ii. Suppose that interest is paid at 4% for all completed years and then at 4% paid monthly for any completed months of an incomplete year. Find the balance of the account at closing.
4. Explain why the proportionality constant for $\left(\frac{1}{a}\right)^x$ is *negative* that for a^x: that is,

   $$\lim_{h\to 0} \frac{(\frac{1}{a})^h - 1}{h} = -\lim_{h\to 0} \frac{a^h - 1}{h}$$

   Try to find both an *algebraic* reason and a *pictorial* one.
5. Sketch the function $f(x) = e^{-x^2}$. Where have you seen this before and what uses does this function have?

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[14] In the US, loan companies typically quote an interest rate which they use to compound *monthly*. For example, if the quoted rate is 7%, then the effective annual (simple) interest rate is $\left(1 + \frac{0.07}{12}\right)^{12} - 1 = 7.229\%$. By law, this higher *effective APR* must be quoted somewhere, though it is unlikely to be as prominently posted.

3.2 Logarithmic Functions

Since $e > 1 > 0$, the natural exponential function satisfies several properties:

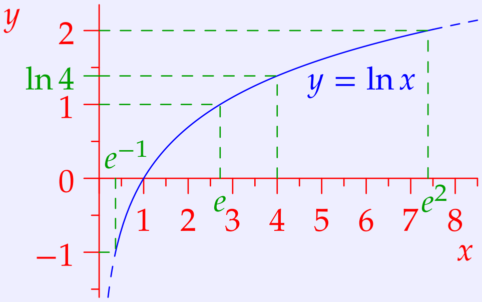
$$\lim_{x\to-\infty} e^x = 0, \qquad \lim_{x\to\infty} e^x = \infty, \qquad \frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x > 0$$

It follows that $\exp : \mathbb{R} \to (0,\infty)$ is a *differentiable* (thus continuous), *increasing* (thus 1–1) function onto its range $(0,\infty)$. It is therefore *invertible.*

Definition 3.11. The *natural logarithm* $\ln : (0,\infty) \to \mathbb{R}$ is the inverse function to the natural exponential. That is,

- If $x > 0$, then $e^{\ln x} = x$;
- If $y \in \mathbb{R}$, then $\ln e^y = y$.

Its graph is obtained from that of the natural exponential in the usual manner: reflect in the line $y = x$.



Since exp and ln are inverse functions, we can solve equations in the usual way: for instance,

$$e^{3x+1} = 100 \implies 3x + 1 = \ln 100 \implies x = \frac{1}{3}(\ln 100 - 1) \approx 1.202$$

One of the great advantages of logarithms is that they allow every exponential function to be expressed in terms of the natural exponential: by the exponential laws (page 44),

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$

This identity is crucial for interpreting and analyzing natural growth models.

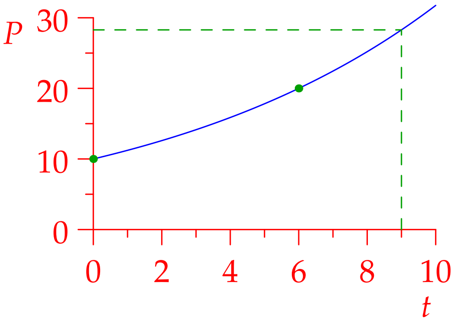
Example 3.12. A population of rabbits doubles in size every 6 months. If there are 10 rabbits at the start of the year, how many rabbits do we expect there to be after 9 months, and how rapidly is the population increasing (births/month).

We are told that the population of rabbits after t months is

$$P(t) = 10 \cdot 2^{t/6}$$

After 9 months the population will be approximately

$$P(9) = 10 \cdot 2^{3/2} = 20\sqrt{2} \approx 28.28 \text{ rabbits}$$



Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = \frac{\mathrm{d}}{\mathrm{d}t}10e^{\frac{t}{6}\ln 2} = \frac{10\ln 2}{6}2^{t/6} \implies P'(9) = \frac{10\ln 2}{6}2^{3/2} \approx 3.27 \text{ rabbits/month}$$

If you ask students this question, what do you expect to be the most common *incorrect* answers?

The Logarithm Laws and General Logarithms

The logarithm laws should be familiar. They follow immediately from Definition 3.11 and the exponential laws (page 44)

$$e^{\ln x+\ln y} = e^{\ln x}e^{\ln y} = xy = e^{\ln xy} \implies \boxed{\ln xy = \ln x + \ln y}$$

$$\text{Similarly} \quad \boxed{\ln \tfrac{x}{y} = \ln x - \ln y \quad \text{and} \quad \ln x^r = r\ln x} \tag{$*$}$$

These laws allow us to solve more general exponential equations by taking logs of both sides:

$$2^x = 5 \implies x\ln 2 = \ln 5 \implies x = \frac{\ln 5}{\ln 2} \approx 2.322$$

More generally, if $a > 0$ and $a \neq 1$, then the exponential function with base a is invertible:

$$y = f(x) = a^x = e^{x\ln a} \implies \ln y = x\ln a \implies x = \frac{\ln y}{\ln a} \implies f^{-1}(x) = \frac{\ln x}{\ln a}$$

Definition 3.13. Given $a > 0$ and $a \neq 1$, the *logarithm with base a* is the function

$$\log_a x := \frac{\ln x}{\ln a}$$

As the inverse of the base a exponential function $y = a^x$, the base a logarithm satisfies

- If $x > 0$, then $a^{\log_a x} = x$;
- If $y \in \mathbb{R}$, then $\log_a a^y = y$.

The natural logarithm has base e. Unless the base is very simple (e.g. $a = 2$ or 10), we typically stick to using the natural logarithm. On a calculator, the 'log' button means $\log_{10}$.

Modifying the Natural Growth Model

Remember that *modeling* always involves some guesswork and assumptions, which necessarily come with trade-offs. Simpler assumptions & models are easier to analyze, but tend to be less accurate. Modeling is therefore always a part of a feedback loop:

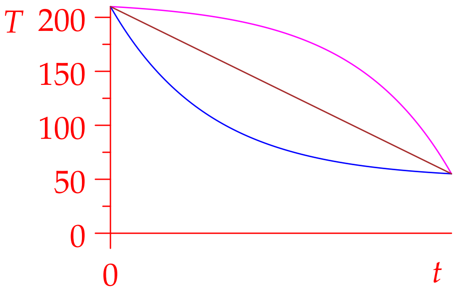
1. Model: Data & theory suggest a mathematical model.
2. Solution: The model is solved, whether exactly using algebra or approximately using numerical techniques.
3. Test: The solutions provide predictions which may be tested against real-world experiments. Perhaps this suggests a tweak to the model…

Applied mathematicians typically desire a 'Goldilocks' model: complicated enough to make accurate predictions without being too difficult to analyze and apply.

Newton's Law of Cooling Here is a standard example of an exponential model motivated by a real-world situation.

Example 3.14. A just-poured cup of coffee has temperature 210°F. The coffee is left outside where the air temperature is 50°F. It seems obvious that the coffee will cool slowly *towards* 50°F; but how?

To help decide how to model this, ask yourself some questions:

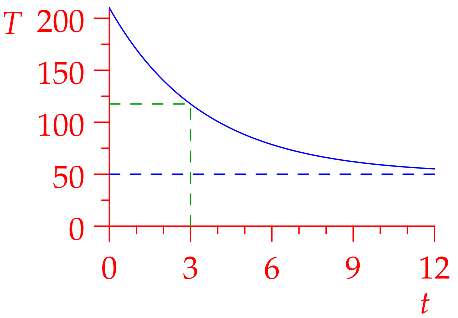


1. When should the rate of cooling be most rapid?
2. What happens to the rate of change in the long run?
3. Which of the three graphs drawn seems most likely?
4. Can you suggest a family of functions which behave in this manner?

Hopefully it seems reasonable to model this problem using a shifted exponential function, where the temperature $T(t)$ of the coffee at time t satisfies

$$T(t) = 50 + 160e^{-kt}$$

for some positive constant k. This satisfies all our criteria:



- $T(0) = 210$°F.
- As t increases, e^{-kt} decreases to zero, so $T(t)$ decreases towards 50°F.
- The rate of cooling $\left|\frac{dT}{dt}\right| = 160ke^{-kt}$ is largest at $t = 0$ and decreases as t increases.

To complete the model (find k), it is enough to supply one further data point. We continue the problem:

> Suppose after 2 minutes that the temperature of the coffee is 140°F. How long does it take for the coffee to cool to 100°F?

We know that $140 = T(2) = 50 + 160e^{-2k}$, whence

$$e^{-2k} = \frac{140-50}{160} = \frac{9}{16} \implies e^{-k} = \frac{3}{4} \implies T(t) = 50 + 160\left(\frac{3}{4}\right)^t$$

When $T(t) = 100$, we see that

$$\left(\frac{3}{4}\right)^t = \frac{100-50}{160} = \frac{5}{16} \implies t = \frac{\ln\frac{5}{16}}{\ln\frac{3}{4}} = \frac{\ln 16 - \ln 5}{\ln 4 - \ln 3} \approx 4.043 \text{ minutes}$$

Of course, all this should really be compared against experimental data!

The above is an example of a general model called *Newton's law of cooling*. This model arises theoretically by asserting a sensible answer to the first two of our motivating questions:

> The rate of temperature change of a body is proportional to the *difference* between the body and its surroundings.

This theoretical approach results in a simple modification of Theorem 3.6.

Corollary 3.15. *If M and k are constant, then*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(M - y) \iff y(t) = M + (y_0 - M)e^{-kt}$$

where $y_0 = y(0)$ is the initial value.

Many other modifications of the natural growth model exist, though they are more appropriate for other courses.

Exercises 3.2. *Key concepts: Exponentials are naturally related to calculus and limits, Compound interest (simple, regular, continuous), Newton's Law of Cooling*

1. Find the solution to the equation $4^{2-\sqrt{x}} = 10$.
2. Find the value of x which satisfies the equation $4^{6x} = 8$.

   (*Your answer should not contain any logarithms*)
3. Over one year, find the continuous interest rate $s\%$ corresponding to a simple rate of 5%.
4. The function $y = a^x$ satisfies the natural growth equation $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$. How does the value of k depend on a (use logarithms, not limits)?
5. Verify the remaining logarithm laws $(*)$ on page 50.
6. By differentiating the expression $e^{\ln x} = x$, verify that $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$.
7. Sketch graphs of the functions $f(x) = \log_2 x$ and $g(x) = \log_{0.5} x$. How are they related? What happens to the graph of $\log_a x$ as a changes?
8. A cup of coffee is left outside on a warm day when the surrounding temperature is 90°F. Suppose the initial temperature of the coffee is 200°F and that its temperature after 2 minutes is 170°F. Find the temperature as a function of time.
9. (a) Check that

   $$y(x) = M + (y_0 - M)e^{-kt}$$

   solves the differential equation in Corollary 3.15.

   (b) Suppose a student believes Theorem 3.6. Use a substitution to convince them that the *only* solution to $y' = k(M - y)$ is as given in part (a).

10. A body with initial temperature T_0 is placed in an environment with constant temperature $M < T_0$. Suppose that the temperature $T(t)$ obeys Newton's law of cooling, and that it takes t_1 seconds for the body to cool to $\frac{1}{2}(T_0 + M)$.

    Find the temperature at time t, and determine how long it takes for the body to cool to $\frac{1}{3}(T_0 + 2M)$.

11. (a) Sketch the graph of the function

    $$y(t) = \frac{100}{1 + 9e^{-t}}$$

    What happens as $t \to \infty$, and when is the slope maximal?

    (b) Check that $y(t)$ satisfies the differential equation

    $$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{100}(100 - y)$$

    For what values of y is the derivative positive? Negative? Zero? Comment on how the graph of the solution in part (a) fits with this.

    (c) Can you think of a real-world situation that might be modeled by such and equation?

12. Logarithms were originally invented not for calculus but to simplify the multiplication of large numbers. In the pre-calculator era, students commonly consulted a book of *log tables* for this purpose. Investigate log tables and how to use them.