4 Exponential and Logarithmic Functions and Models

One of the most basic models is that of *natural growth/decay*; the rate of change of a quantity is proportional to the quantity itself. Written using differential equations, this is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

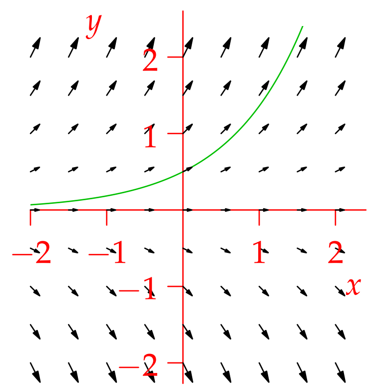
where k is a constant; if $k > 0$ this is the *natural growth equation*, if $k < 0$ the *natural decay equation*. This is commonly encountered when modelling *populations*, since the rate of growth of an otherwise unconstrained population is proportional to its size (twice the people, twice the babies. . .).

We know what solutions to the natural growth model look like because we can sketch them: if the value $y(x)$ is known, the equation tells us the value of its *derivative* $y'(x) = ky(x)$ and thus the *direction* in which the solution curve is travelling.

Exercise Consider the simplest natural growth equation with $k = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

At each point $(\frac{m}{2}, \frac{n}{2})$ where $-4 \le m, n \le 4$, we draw an arrow whose *slope* equals y. The arrows indicate the *direction* of any solution curve passing through the point. The upshot is that the arrow are *tangent* to any solution. The green curve appears to be a solution; sketch some others.



If the title of the section didn't give it away, you should recognize the above graph; it looks suspiciously exponential. . . Why does this make sense?

Definition 4.1. Let $a > 0$ be constant. The *exponential function with base a* is $f(x) = a^x$.

Recall the *exponential laws*:

$$a^{x+y} = a^x a^y, \qquad a^{x-y} = \frac{a^x}{a^y}, \qquad (a^x)^r = a^{rx}$$

For us, the crucial property is that of having a proportional derivative.

Theorem 4.2. *The rate of change of $f(x) = a^x$ is proportional to $f(x)$. Specifically,*

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

so that $f(x) = a^x$ satisfies the natural growth/decay equation $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$ with proportionality constant

$$k = f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h}$$

Exercises 1. Estimate the proportionality constant $k = \lim_{h\to 0} \frac{a^h - 1}{h}$ for several values of a by using your calculator to complete the following table to 3 d.p. for the given values of h:

a	2	2.5	2.7	2.75	3	5
$\frac{a^{0.1}-1}{0.1}$	0.718	0.960	1.044	1.065	1.161	1.746
$\frac{a^{0.01}-1}{0.01}$	0.696					
$\frac{a^{0.001}-1}{0.001}$	0.693					

What is happening to the proportionality constant as a increases?

2. Show that the proportionality constant for $\frac{1}{a}$ is negative that for a: that is,

$$\lim_{h\to 0} \frac{(\frac{1}{a})^h - 1}{h} = -\lim_{h\to 0} \frac{a^h - 1}{h}$$

If you did the exercise properly, it should appear as if there is a special number somewhere between 2.7 and 2.75 for which the proportionality constant is $k = 1$.

Definition 4.3. The value $e = 2.71828\ldots$ is the unique real number[1]such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$. The *natural exponential function* $\exp(x) = e^x$ has derivative $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$.

The green curve plotted above is the natural exponential function. Of course there are many other solutions to this equation. Indeed we can now easily verify solutions to the general equation using the chain rule: for any constants c, k,

$$y = ce^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} ce^{kx} = cke^{kx} = ky$$

In fact the converse also holds:[2]

Theorem 4.4. *The solutions to the natural growth equation* $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$ *are precisely* $y(x) = y_0 e^{kx}$ *where* $y_0 = y(0)$ *is the initial value.*

Before seeing some examples, we need one further ingredient. Since $e > 1 > 0$, the exponential function satisfies the following properties:

$$\lim_{x\to\infty} e^x = 0, \qquad \lim_{x\to\infty} e^x = \infty, \qquad \frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x > 0$$

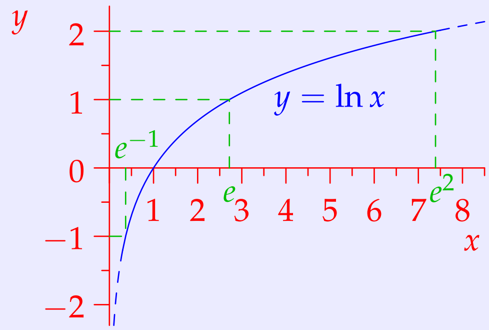
whence $\exp : \mathbb{R} \to (0, \infty)$ is an *increasing* function with $\mathrm{dom}(\exp) = \mathbb{R}$ and $\mathrm{range}(\exp) = (0, \infty)$. It is therefore *invertible.*

[1] An old joke suggests that if aliens were to land on Earth, they'd have to understand e given the technology they'd require to get here. Like π and $\sqrt{2}$, the exponential constant e is an *irrational number,* implying that its decimal representation continues forever without any eventually repeating pattern. There isn't the same geeky fascination with memorizing its digits as there is with π. Neither is there an 'e-day' (Feb 7th, Jan 27th anyone?), perhaps because the obvious related consumable is best avoided...

[2] For a discussion of the converse and why it needs a bit of thought, take a differential equations class!

Definition 4.5. The *natural logarithm* $\ln : (0, \infty) \to \mathbb{R}$ is the inverse function to the natural exponential. That is,

- For any $x > 0$, $e^{\ln x} = x$;
- For any $x \in \mathbb{R}$, $\ln e^x = x$.



By the exponential laws, every exponential function can be expressed this way

$$a^x = (e^{\ln a})^x = e^{x \ln a}$$

Exercises 1. $y = a^x$ satisfies the natural growth equation $\frac{dy}{dx} = ky$; what is the value of k?

2. If $a > 0$ and $a \neq 1$, verify that $\log_a x := \frac{\ln x}{\ln a}$ is the inverse function of $y = a^x$. This is the *logarithm with base a*; the natural logarithm has base e.

Unless the base is very simple, say $a = 2$, we typically stick to using e. The basic properties of the logarithm should be familiar to you; for instance

$$e^{\ln x + \ln y} = e^{\ln x} e^{\ln y} = xy = e^{\ln xy} \implies \ln xy = \ln x + \ln y$$

Exercises 1. Verify the remaining *logarithm laws*:

$$\ln \frac{x}{y} = \ln x - \ln y, \qquad \ln x^r = r \ln x$$

2. Logarithms were invented not for calculus but to simplify the multiplication of large numbers. In the pre-calculator era, it was common for students to carry a book of *log tables* for this purpose. Look up a log table on the internet and investigate how to use it.

3. By differentiating the expression $e^{\ln x} = x$, verify that $\frac{d}{dx} \ln x = \frac{1}{x}$.

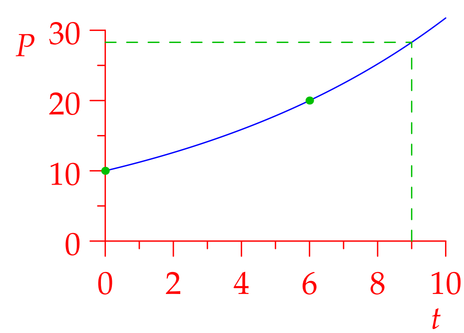
Example 4.6. A population of rabbits doubles in size every 6 months. If there are 10 rabbits at the start of the year, how many rabbits do we expect there to be after 9 months, and how rapidly is the population increasing (births/month).

Our model for the population of rabbits after t months is

$$P(t) = 10 \cdot 2^{t/6}$$

After 9 months the population will be approximately $P(9) = 10 \cdot 2^{3/2} = 20\sqrt{2} \approx 28.28$ rabbits. Moreover,

$$\frac{d}{dt} P(t) = \frac{d}{dt} 10 e^{\frac{t}{6} \ln 2} = \frac{10 \ln 2}{6} 2^{t/6}$$
$$\implies P'(9) = \frac{10 \ln 2}{6} 2^{3/2} \approx 3.27 \text{ rabbits/month}$$



Exercises 1. What mistakes might a student make if they were given the previous example? Why might they obtain the incorrect answer of 25 rabbits?

2. What clues were in the previous example that made you think an exponential model was a good idea?

Sometimes it is not so obvious that an exponential model is the way forward. One early approach to e, courtesy of Jacob Bernoulli (1683), was to think about the problem of compound interest.

Example 4.7. You deposit \$1 in an account paying 100% interest per year (nice account!). Bernoulli observed that you will end up with different sums of money depending on *when* and *how* the interest is paid.

1. If the interest is paid once at the new year, then you finish with \$2.

2. If half the interest 50¢ is paid at six months, then the balance \$1.50 earns 100% per year interest for the rest of the year, then you'll finish with $1.5 \cdot 1.50 = \$2.25$.

3. If the interest is paid in four installments we have the following table of transactions (data is rounded to the nearest cent)

Date	Interest Paid	Balance
1^{st} Jan	—	\$1
1^{st} Apr	25¢	\$1.25
1^{st} July	$\frac{1}{4} \cdot 1.25 = 31$¢	\$1.56
1^{st} Sept	$\frac{1}{4} \cdot 1.56 = 39$¢	\$1.95
Year's end	$\frac{1}{4} \cdot 1.95 = 49$¢	\$2.44

   A shorter calculation for the end of the year balance is $\$1.25^4 = \left(1 + \frac{1}{4}\right)^4$.

4. In general, if the interest is paid over n equally spaced intervals, the end of year balance would be $\$\left(1 + \frac{1}{n}\right)^n$. Here are a few examples where we've rounded things to 5 decimal places:

Frequency	End of year balance
Every month	$\$\left(1 + \frac{1}{12}\right)^{12} = 2.61304$
Every day	$\$\left(1 + \frac{1}{365}\right)^{365} = 2.71457$
Every hour	$\$\left(1 + \frac{1}{8760}\right)^{8760} = 2.71813$
Every second	$\$\left(1 + \frac{1}{31536000}\right)^{31536000} = 2.71828$

   It certainly appears as if you cannot beat \$$e$, no matter how often you pay the interest!

In fact this is a theorem:

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

While it is quite a lot of work to show that this is equivalent to the differential equation definition, it again shows that e arises very naturally. It is common in finance to compute compound interest *continuously;* a continuous interest rate of $r\%$ per year results in an initial balance of y_0 becoming

$$y(t) = y_0 e^{\frac{rt}{100}}$$

after t years. There are several 'advantages' to this.

1. The balance of an account can be found easily at any time.
2. All the power of calculus can be brought to bear.
3. A company can make an interest rate appear higher (if a savings account) or lower (if a loan) by choosing how to quote the interest rate. For instance, a loan continuously compounded at 7% has an *effective interest rate*[3] of $e^{0.07} - 1 = 7.25\%$; if you borrow \$100,000, you'll owe \$107,250 at the end of the year, not the \$107,000 you might have expected!

Exercise Which of the following would you prefer for a savings account?

- 5% interest paid continuously.
- 5.05% paid monthly.
- 5.1% paid at the end of the year.

Modifying the Natural Growth Model

An obvious modification to the model involves adding a constant to the solution:

$$y(t) = a + be^{kt} \implies \frac{dy}{dt} = kbe^{kt} = k(y - a)$$

In this case the rate of change is proportional to the *difference* between $y(t)$ and some fixed value a. If $k > 0$, then solutions will approach

Exercises 1. If c, k, M are constants, verify that $y(x) = M + ce^{-kt}$ solves the differential equation

$$\frac{dy}{dt} = k(M - y)$$

2. A commonly encountered model based on this equation is *Newton's Law of Cooling*: the temperature $T(t)$ of a body is assumed to change at a rate proportional to the difference between its temperature and that of its surroundings T_s. Find the differential equation that describes this situation.

3. A cup of coffee is left outside on a warm day when the surrounding temperature is 90°F. Suppose the initial temperature of the coffee is 200°F and that its temperature after 2 minutes is 170°F. Find the temperature as a function of time.

[3] In practice, mortgage companies often quote an interest rate which they use to compound *monthly.* For example, if the quoted rate is 7%, then the effective interest rate per year is $\left(1 + \frac{0.07}{12}\right)^{12} - 1 = 7.229\%$. By law, they have to quote this higher *effective APR* somewhere, though it is unlikely to be as prominent as the lower *simple APR*.

4. A population of 10,000 people is exposed to a novel virus. The best scientific understanding is that 1% of the susceptible population per day contracts the virus, the effects of the illness lasts ten days, after which a victim recovers and is immune from reinfection.

   (a) Model the evolution of the sick and immune populations over the next 180 days.

   (b) The health department has n vaccines available. Discuss how it should deploy these. Is the goal to eradicate the virus as rapidly as possible? What if it wants to do this while using fewer doses? Discuss the following strategies; for simplicity, assume the vaccines are 100% effective and work instantly.

      i. Use all the vaccine immediately.

      ii. Wait a while until more people are immune, then use all vaccines on at risk people.

      iii. Vaccinate a small number of at risk people per day.

      iv. Wait until there are only n at risk people left, then vaccinate them all.

This is easier using a spreadsheet. We can also approach the problem analytically.

Let $r(t)$, $s(t)$ and $i(t)$ be the at risk, sick and immune populations respectively. If t is measured in days, then

$$r(t+1) = 0.99r(t), \quad r(0) = 10000 \implies r(t) = 10000 \cdot 0.99^t$$

The sick population is the sum of the previous 10 days' decrease in the at risk population:

$$s(t) = \begin{cases} r(0) - r(t) = 10000(1 - 0.99^t) & \text{if } t \leq 10 \\ r(t-10) - r(t) = 10000(0.99^{-10} - 1)0.99^t = 1057 \cdot 0.99^t & \text{if } t > 10 \end{cases}$$

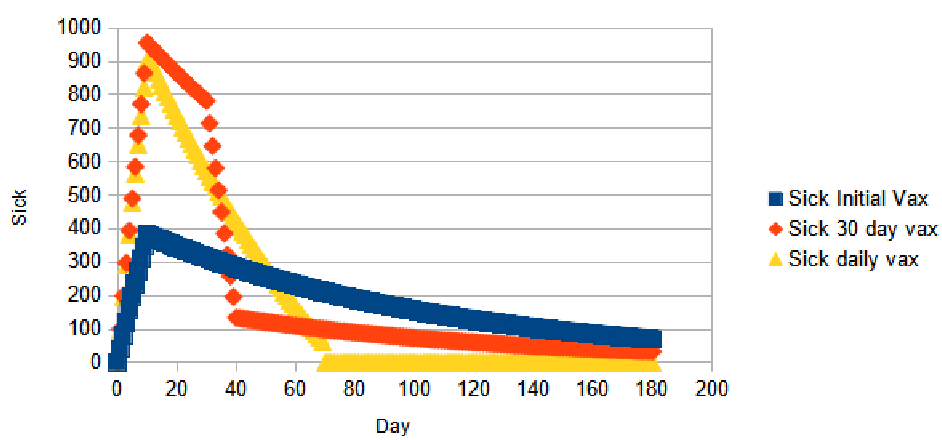
The immune population is the difference between these and the total population

$$i(t) = 10000 - r(t) - s(t) = \begin{cases} 0 & \text{if } t \leq 10 \\ 10000 - 11057 \cdot 0.99^t & \text{if } t > 10 \end{cases}$$

After 180 days, we have

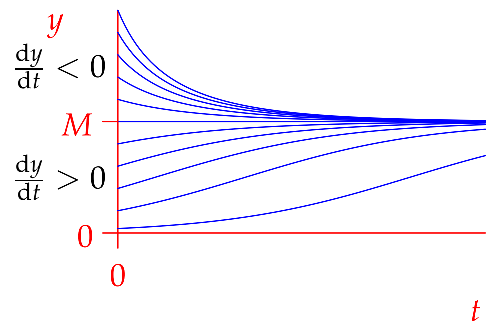
$$r(180) = 1048, \qquad s(180) = 111, \qquad i(180) = 8841$$

The other situations can also be analytically described. Here are graphs of what happens under the first three vaccination campaigns if $n = 6000$.



The Logistic Model Our final basic exponential model is one of the most commonly encountered.

- When a population y is very small, we want it to grow naturally $\frac{dy}{dt} \propto y$.
- As with Newton's law of cooling, we want y to approach a positive value M as $t \to \infty$.



The *logistic differential equation*

$$\boxed{\frac{dy}{dt} = ky(M-y)}$$

accomplishes both our requirements when k, M are positive constants. In such a situation, M is often referred to as the *carrying capacity* of the environment.

Exercise If $y_0 = y(0)$, verify that $y(t) = \dfrac{y_0 M}{y_0 + (M-y_0)e^{-kMt}}$ solves the logistic equation. You can do this directly or, if you've studied differential equations, try separation of variables!

Example 4.8. A brewer pitches 100 billion yeast cells into a starter wort with the goal of growing it to 200 billion cells. After one hour, the wort contains 110 billion cells.

(a) How long must the brewer wait if we use a natural growth model?

(b) How long must the brewer wait if we use a logistic model where we also assume that the wort contains enough sugar to grow 250 billion yeast cells?

(c) For both models, what is the maximum growth rate and at what time does it occur?

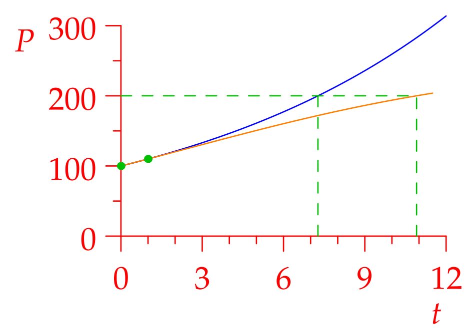
Let $P(t)$ be the yeast population in billions at time t hours. We therefore have $P(0) = 100$ and $P(1) = 110$, and want to find t such that $P(t) = 200$.

(a) The model is $\frac{dP}{dt} = ky$, which has solution

$$P(t) = P_0 e^{kt} = 100e^{kt}$$

Evaluating at $t = 1$ yields $1.1 = e^k$ whence

$$P(t) = 100(1.1)^t \implies t = \frac{\ln(0.01P)}{\ln 1.1} \approx 7.27 \text{ hours}$$



(b) The model is $\frac{dP}{dt} = ky(250-y)$, with solution

$$P(t) = \frac{25000}{100 + (250-100)e^{-250kt}} = \frac{500}{2+3e^{-250kt}}$$

Evaluating at $t = 1$ yields

$$110 = \frac{500}{2+3e^{-250k}} \implies e^{250k} = \frac{33}{28}$$

whence

$$P(t) = \frac{500}{2 + 3\left(\frac{28}{33}\right)^t} \implies t = \frac{\ln 6}{\ln \frac{33}{28}} \approx 10.91 \text{ hours}$$

(c) For the natural growth model, the growth rate is always increasing. It is maximal at the moment the population hits 200 billion and equals

$$P'(t) = (\ln 1.1)P(t) = 200 \ln 1.1 \approx 19.06 \text{ billion cells per hour}$$

For the logistic model, the maximum growth rate happens when the population is half way to carrying capacity: i.e. at 125 billion cells. We therefore have a maximum growth rate of

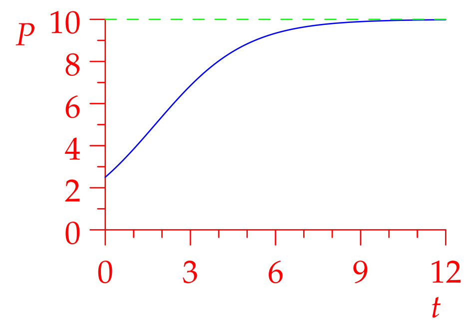
$$P'(t) = 125^2 k = \frac{125^2}{250} \ln \frac{33}{28} = \frac{125}{2} \ln \frac{33}{28} \approx 10.27 \text{ billion cells per hour}$$

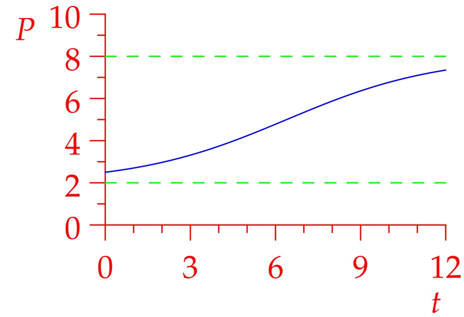
at time $t = \frac{\ln \frac{3}{2}}{\ln \frac{33}{28}} \approx 2.47$ hours.

The logistic model is easily generalized.

Example 4.9. The population $P(t)$ (in 1000's) of fish in a lake obeys the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{16}P(10 - P)$$





where t is measured in months. The first graph shows how the population recovers over a year if it starts at 2500 fish.

Now suppose 1000 fish are 'harvested' from the lake each month. The new model is then

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{1}{16}P(10 - P) - 1 = -\frac{1}{16}(P^2 - 10P + 16) \\ &= -\frac{1}{16}(P - 2)(P - 8)\end{aligned}$$

Substituting $Q = P - 2$, this is again logistic!

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{16}Q(6 - Q)$$

Provided the initial population $P(0) = Q(0) + 2$ is greater than 2000 fish, we expect the population to eventually stabilize at 8000 fish, though it takes a long time to get close to this if we start, as in the second graph, with only a little over 2000 fish.

Exponentials and real-world data

Exponentials can be combined with our method for finding best-fitting regression lines in several ways. Here are a couple of possibilities for modelling a data set $\{(t_i, y_i)\}$.

Log plots If you suspect an *exponential model* $\hat{y} = ab^t = e^{mt+c}$ then taking logarithms

$$\ln \hat{y} = mt + c$$

results in a *linear* relationship between $\ln \hat{y}$ and t.

Log-log plots If you suspect a *power function model* $\hat{y} = at^m$ then taking logarithms

$$\ln \hat{y} = m \ln t + \ln a$$

results in a *linear* relationship between $\ln y$ and $\ln t$.

Given a data set $\{(t_i, y_i)\}$, simply plot $\ln y_i$ against either t_i or $\ln t_i$ and see which looks most linear! A weakness of these approaches is that the errors are not treated equally by the logarithmic transformation. When a spreadsheet is asked to compute with such a model, the value of R^2 it typically reports is that of the underlying linear model.

Example 4.10. A population of rabbits is measured every two months resulting in the data set

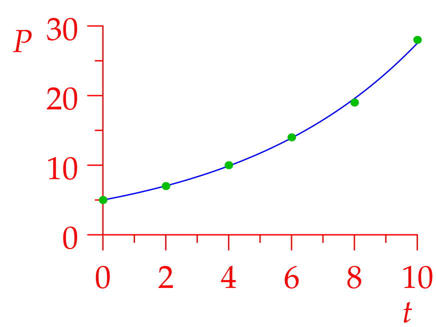
t_i	0	2	4	6	8	10
P_i	5	7	10	14	19	28

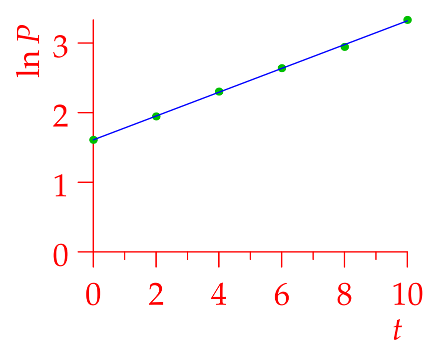
Since this is population growth, we expect an exponential model $P(t) = e^{mt+c}$. After taking logarithms of the population values, the data is very close to linear. Now perform a linear regression (this is much easier using a spreadsheet—play with it!).

	Data						average
t_i	0	2	4	6	8	10	5
t_i^2	0	4	16	36	64	100	36.67
P_i	5	7	10	14	19	28	13.83
$\ln P_i$	1.61	1.95	2.30	2.64	2.94	3.33	2.46
$t_i \ln P_i$	0	3.89	9.21	15.83	23.56	33.32	14.30

$$m = \frac{\overline{t \ln P} - \bar{t} \cdot \overline{\ln P}}{\overline{t^2} - \bar{t}^2} = \frac{14.30 - 5 \cdot 2.46}{36.67 - 5^2} = 0.171$$

$$c = \overline{\ln P} - m\bar{t} = 3.46 - 0.171 \cdot 5 = 1.609$$





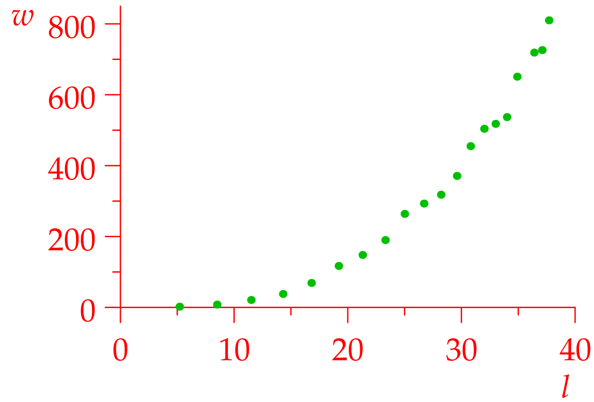
which yields the exponential model

$$\hat{y} = e^{0.171t+1.609} = 4.998(1.186)^t \approx 5(1.979^{t/4})$$

This is very close to doubling every 4 months. In fact the approximate doubling time for the population is $T = \frac{\ln 2}{m} = 4.06$ months.

Exercises The average weight and length of a fish species measured at different ages is as follows:

Age (years)	Length(cm)	Weight (g)
1	5.2	2
2	8.5	8
3	11.5	21
4	14.3	38
5	16.8	69
6	19.2	117
7	21.3	148
8	23.3	190
9	25.0	264
10	26.7	293
11	28.2	318
12	29.6	371
13	30.8	455
14	32.0	504
15	33.0	518
16	34.0	537
17	34.9	651
18	36.4	719
18	37.1	726
20	37.7	810



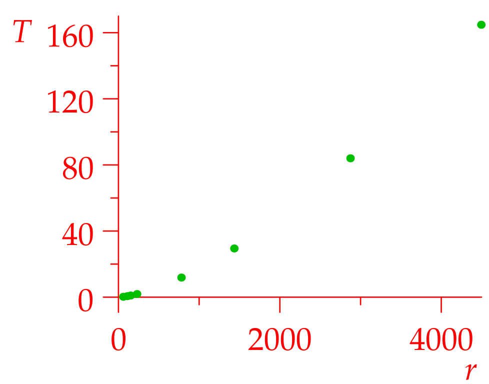
1. Do you think an exponential model is a good fit for this data? Take logarithms of the weight values and use a spreadsheet to obtain a model $\hat{w} = \exp(ml + c)$ where w, l are the weight and length respectively.

2. What happens if you try a log-log plot? Do you get a more accurate model? Why do you think this is?

Kepler's Third Law

In the early 1600's Johannes Kepler used data for the planets to empirically derive his three *laws of planetary motion*, the third of which relates the orbital period T of a planet (how long it takes to go round the sun), to its average distance r from the sun.

Planet	T (years)	r (millions km)
Mercury	0.24	57.9
Venus	0.61	108.2
Earth	1	149.6
Mars	1.88	228.0
Jupiter	11.86	778.5
Saturn	29.46	1433.3
Uranus	84.01	2872.6
Neptune	164.79	4493.6



The table shows the data for all of the planets (Kepler did not have this for Uranus and Neptune). Use a spreadsheet to analyze this data and find a model relating the period to the mean radius of a planet. Compare what you find with the statement of Kepler's third law.