

6 Conic Sections

In this section we consider the traditional conic sections and relate their basic properties to modern equations.

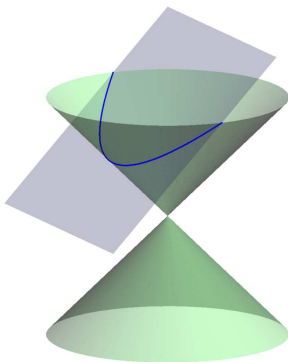
Definition 6.1. A (non-degenerate) *conic section* is the intersection of a *right circular cone*¹ with a plane not passing through the vertex.

Depending on the orientation of this plane, we obtain one of the three standard types of conic section.

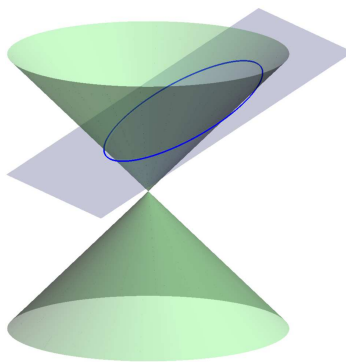
Parabola: The plane intersects in a single curve either on the upper or lower half of the cone. Equivalently, the plane is parallel to one of the lines defining the cone.

Ellipse: The intersection is a single closed curve.

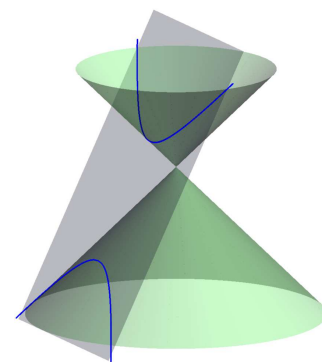
Hyperbola: The intersection comprises two curves, one each on the upper and lower halves of the cone.



Parabola



Ellipse



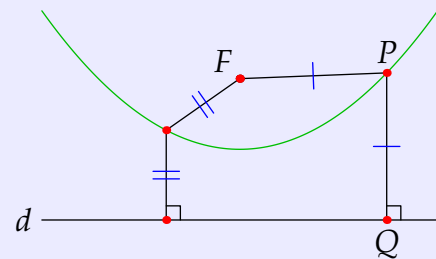
Hyperbola

Exercise Get a piece of paper, make a cone and *draw* some sections!

The conic section definitions are hard to work with because they are three dimensional. We instead make alternative definitions as curves in the plane.

Definition 6.2. Let d be a line in the plane, and F a point not on that line. The *parabola* with *directrix* d and *focus* F is the set of points P equidistant from both.

In the language of the picture, $|FP| = |PQ|$.



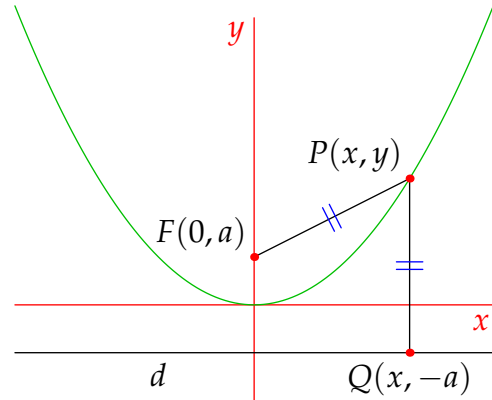
The definition is strictly *synthetic* since equidistant really means that the line segments \overline{FP} and \overline{PQ} are *congruent*. To interpret this *analytically* we choose axes and co-ordinates.

¹A *cone* is a surface obtained by joining all points on a curve with a fixed point, the *vertex*. A *right circular cone* is when this curve may be chosen to be a *circle* such that the line from its center to the vertex (the *axis*) is perpendicular (*right angles*) to the circle.

Without loss of generality, choose the origin to lie on the parabola halfway between the focus and directrix and orient things such that the focus lies on the y -axis. Otherwise said, for some a , the focus has co-ordinates $(0, a)$ and the directrix has equation $y = -a$.

Now we follow the definition: if $P = (x, y)$ is a point on the parabola, then $Q = (x, -a)$, whence

$$\begin{aligned} |FP| = |PQ| &\iff |FP|^2 = |PQ|^2 \\ &\iff x^2 + (y - a)^2 = (y + a)^2 \end{aligned}$$



If we multiply out and simplify, we recover a familiar equation:

Theorem 6.3. Given a parabola, there exist rectangular co-ordinates/axes with respect to which the parabola has equation $x^2 = 4ay$ for some $a > 0$. This is the canonical form of the parabola.

Of course parabolas are often not presented to us in canonical co-ordinates.

Example 6.4. Find the focus and directrix of the parabola with equation $y = x^2 - 6x + 7$. Complete the square!

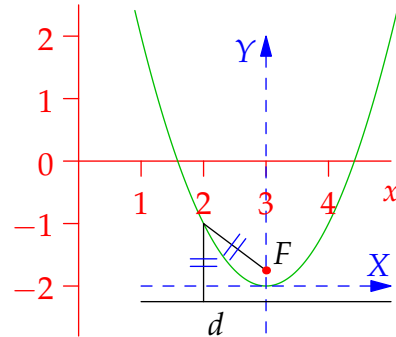
$$y = x^2 - 6x + 7 = (x - 3)^2 - 2$$

With respect to new co-ordinates $X = x - 3$ and $Y = y + 2$ we obtain the canonical form

$$X^2 = Y = 4aY \iff a = \frac{1}{4}$$

The focus has co-ordinates $(X, Y) = (0, \frac{1}{4})$ and the directrix is $Y = -\frac{1}{4}$. Translating back to our original co-ordinates we obtain

$$F = (3, -\frac{7}{4}), \quad \text{Directrix : } y = -\frac{9}{4}$$



Canonical co-ordinates can also be rotated relative to given axes, but this would take us too far afield.

Relation to the cone The simplest way to see this is to suppose that the cone is centered on the z -axis and has equation $z^2 = x^2 + y^2$: the cross-section a distance 1 up the z -axis is a circle of radius 1. The point $(x, y, z) = (0, 1, 1)$ certainly lies on the cone: any plane parallel to this has equation $z = y + k$ for some constant k . Now substitute

$$\begin{cases} z^2 = x^2 + y^2 \\ z = y + k \end{cases} \implies (y + k)^2 = x^2 + y^2 \implies x^2 = 2ky + k^2$$

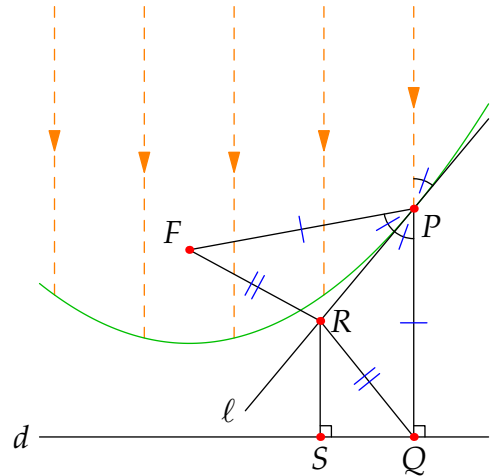
which is the equation² of a parabola!

²Strictly this is the equation of the projection of the parabola onto the xy -plane. We should really pull out a factor of $\sqrt{2}$ from y to compensate for the slope of the intersecting plane.

Why call it the focus? A *parabolic mirror* is a reflective surface obtained by rotating a parabola around its canonical y -axis. Imagine a light source at ‘vertical infinity,’ whose rays therefore arrive parallel to the y -axis. The parabolic mirror will focus these rays onto the focus of the original parabola! This is the reason that satellite dishes are parabolic, with the receiver placed at their focus. The argument requires only a little triangle geometry.

Let P lie on the parabola, Q its nearest point on the directrix and bisect angle $\angle FPQ$ with the line ℓ . Let $R \neq P$ be any other point on ℓ .

1. Why do we have $|FR| = |RQ|$?
2. Why is $|RS| < |FR|$?
3. Why does this force ℓ to be the *tangent line* to the parabola?
4. Why can we conclude that the vertical ray hitting the mirror at P is reflected towards F ?



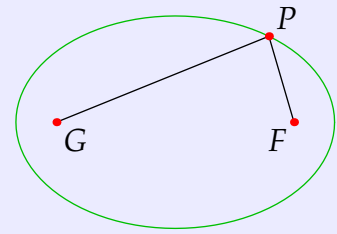
If you want a hard life, try arguing using equations and analytic geometry :)

Now we turn to the planar definition of an ellipse.

Definition 6.5. Let F, G be points in the plane and $a > \frac{1}{2} |FG|$ a positive constant. The set of points P such that

$$|FP| + |GP| = 2a$$

is an *ellipse* with *foci* F, G and *semi-major axis* a . If $F = G$, the ellipse is a circle with radius a .



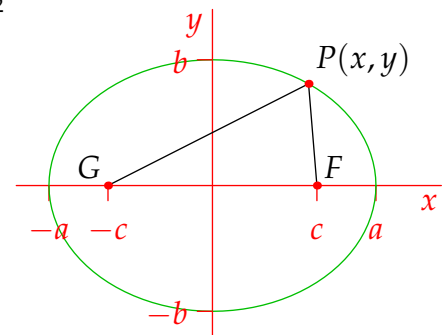
Exercises 1. To put this in canonical form, suppose the foci lie on the x -axis with the origin at their midpoint: $F = (c, 0)$ and $G = (-c, 0)$ for some c . Start by squaring

$$\begin{aligned} (x - c)^2 + y^2 &= |FP|^2 = (2a - |GP|)^2 = \left(2a - \sqrt{(x + c)^2 + y^2}\right)^2 \\ &= 4a^2 + (x + c)^2 + y^2 - 4a\sqrt{(x + c)^2 + y^2} \end{aligned}$$

then rearrange to place the remaining square-root on one side before squaring to eliminate. After simplifying, you should obtain the familiar form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where} \quad b^2 = a^2 - c^2$$

The value b is called the *semi-minor axis*. Hopefully the meaning of these terms is clear from the picture.



2. Similarly to Example 6.4, complete the square for both x and y to show that the equation

$$9x^2 + 4y^2 - 36x - 8y + 4 = 0$$

represents an ellipse. Sketch it and find the co-ordinates of its foci.

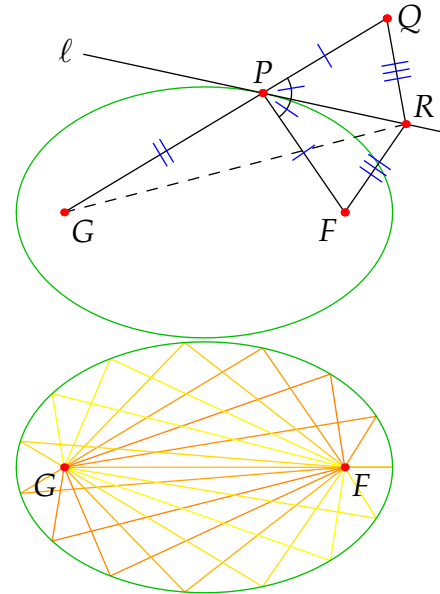
3. A reflection principle also applies to the ellipse. It is a little different from the parabola since we now have two foci.

- Extend \overline{GP} to Q such that $|PQ| = |PF|$.
- Bisect $\angle FPQ$ by the line ℓ .
- If R is any other point on ℓ , explain why

$$|FR| + |GR| > 2a$$

and thus why R lies *outside* the ellipse.

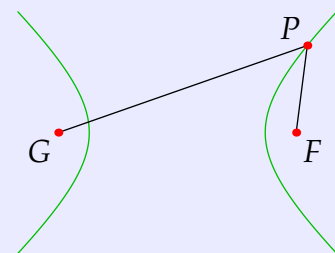
- Hence explain why a light source placed at one focus will have its rays reflected by an elliptical mirror onto the other focus.



Definition 6.6. Let F, G be points in the plane and $a < \frac{1}{2}|FG|$ a positive constant. The set of points P such that

$$|FP| - |GP| = \pm 2a$$

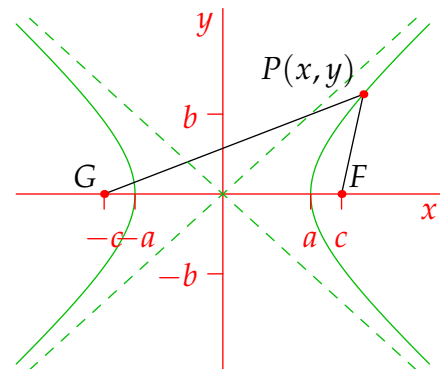
is a *hyperbola* with foci F, G .



Exercises 1. Exactly as above, verify that by choosing $F = (c, 0)$ and $G = (-c, 0)$, a hyperbola has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{where} \quad b^2 = c^2 - a^2$$

- As $x \rightarrow \infty$ show that the hyperbola approaches its *asymptotes* $y = \pm \frac{b}{a}x$.
- (Challenge!) Verify the 'reflection' principle for hyperbolæ: if a light source is placed at a focus, then the reflected rays will appear to have emanated from the *other* focus.



Eccentricity

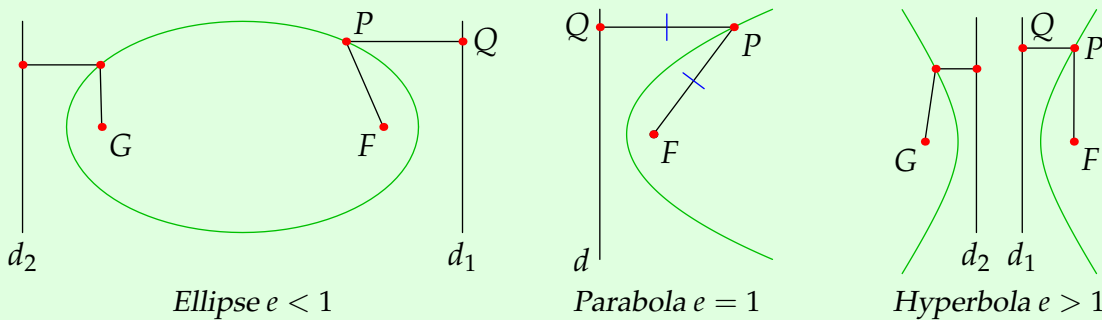
The obvious difference between our descriptions of the ellipse/hyperbola versus the parabola is the *directrix*.

Theorem 6.7. Let F be a point (the focus) not on a line d (the directrix) and choose a number $e > 0$ (the eccentricity). Consider the set of points P satisfying

$$|PF| = e|PQ|$$

where Q is the nearest point to P on the directrix.

- Depending on the value of e we obtain one of the three conic sections. For the ellipse and the hyperbola this construction only produces half of the conic; there are in fact two directrices.



- For the ellipse and hyperbola $e = \frac{c}{a}$, where $c = \frac{1}{2}|FG|$.

By convention, a circle is considered to have eccentricity $e = 0$, even though the above doesn't apply. We could imagine obtaining a circle of radius a by taking the limit $e \rightarrow 0^+$ (i.e. $c \rightarrow 0^+$). In such a limit, the point Q on the directrix must go to infinity: circles therefore do not have directrices!

Example 6.8. The canonical form ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ has $a = 5, b = 4$ and $c = \sqrt{a^2 - b^2} = 3$. Its eccentricity is therefore $e = \frac{c}{a} = \frac{3}{5}$. Since its foci have co-ordinates $(\pm\frac{3}{5}, 0)$, its right directrix must be a vertical line which intersects the x -axis when

$$e(x - a) = (a - c) \implies x = a + \frac{a - c}{e} = 5 + \frac{10}{3} = \frac{25}{3}$$

Exercises 1. Find the foci and directrices of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$. Sketch the curve.

- Verify some of the theorem.

After refreshing some trigonometry, we'll return to conics. In particular we'll see how they may be parametrized, and how the eccentricity relates to the cone/plane combination which created the conic in the first place!