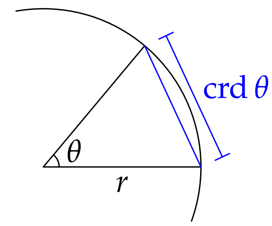
7 Trigonometry and Polar Co-ordinates

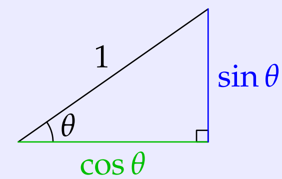
In this section we review trigonometry and periodic functions and discuss their relation to polar co-ordinates. Some of this will be non standard.

The trigonometric functions date back at least 2000 years. Ancient mathematicians were interested in the relationship between the *chord* of a circle and the central angle, often for the purpose of astronomical measurement. It wasn't until 1595 that the term *trigonometry* (literally *triangle measure*) was coined, and the functions were considered primarily related to triangles.



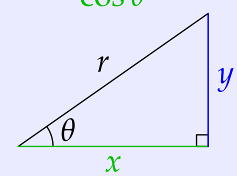
Here are three related definitions of sine, cosine and tangent, the first two based on a right triangle, and the other on a circle.

Definition 7.1. 1. (a) Given a right triangle with *hypotenuse* (longest side) 1 and angle θ, define $\sin\theta$ and $\cos\theta$ to be the side lengths *opposite* and *adjacent* to θ. Also define $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to be the slope of the hypotenuse.

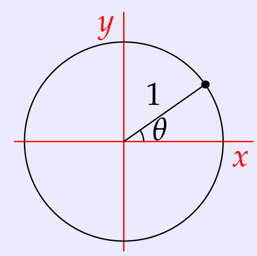


(b) Given a right triangle with angle θ, *hypotenuse* r, *adjacent* x and *opposite* y, define

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$



2. Define $(\cos\theta, \sin\theta)$ to be the co-ordinates of a point on the unit circle relative to its *polar angle* θ measured counter-clockwise from the positive x-axis. Provided $\cos\theta \neq 0$, we define $\tan\theta = \frac{\sin\theta}{\cos\theta}$.



Exercises 1. Discuss the advantages and disadvantages of these definitions. Consider your audience; what prerequisites are you assuming? Is it easier to think about *lengths* rather than *ratios*? Are there contexts where one definition is more useful than the other?

2. Explain why the two triangle definitions are equivalent to one another. Where do you need basic facts from Euclidean geometry such as *congruent* and *similar* triangles. Why do the triangle definitions follow from the circle definition? What is missing if you try to use the triangle definition to justify the circle version?

3. The identity $\cos^2\theta + \sin^2\theta = 1$ is simply Pythagoras' Theorem in disguise. Why?

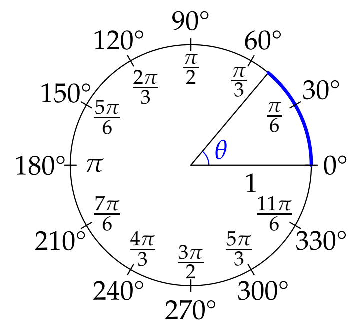
4. The word *sine* is the result of a long line of translations and transliterations from an ancient Sanskrit term meaning *half-chord.* For the chord picture at the top of the page, how does the length of the chord $\operatorname{crd}\theta$ relate to modern trigonometric functions?

Measuring Angles There are two standard ways to measure angles.

Degrees A full revolution has 360° and a right angle 90°. Degree measure dates back to ancient Babylon. The choice of 360 is possibly related to the coincidence that 360 has lots of integer divisors and is approximately the number of days in a year.

Radians The radian measure of an angle is the length of the arc subtending the angle in a circle of radius 1. Since the circumference of a unit circle is 2π, we have the following identifications.

Degrees	Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	n/a
180°	π	0	-1	0



Exercises 1. It is conventional not to state units when using radians since they are effectively a ratio and therefore *unitless*. Think this through: if the central angle in a circle of radius r is subtended by an arc with arc-length ℓ, what is the radian measure of the angle?

2. Why do we get the values of sine and cosine in the above table?

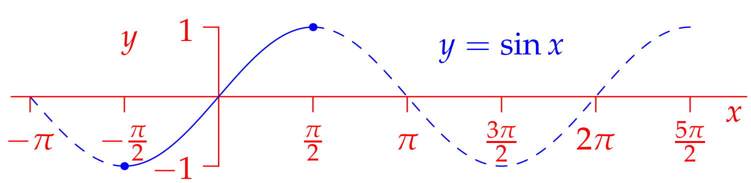
3. Using the pictures, explain why we have the relations

$$\sin(\tfrac{\pi}{2}-\theta)=\cos\theta=\sin(\theta+\tfrac{\pi}{2}),\qquad \sin(-\theta)=-\sin\theta,\qquad \sin(\pi-\theta)=\sin\theta$$

Periodicity, Graphs, Inverses & Solving Equations For our purposes, the circle definition is superior because it allows us to easily consider non-acute angles and *periodicity*. We can measure the polar angle counter-clockwise as many times round the origin as we like, whence, for any integer n,

$$\sin(\theta+2n\pi)=\sin\theta,\quad \cos(\theta+2n\pi)=\cos\theta$$

Otherwise said, sine and cosine have period 2π. Armed with these facts, it is easy to draw the graphs; just think about the circle when $0\le\theta<2\pi$ and continue periodically! Due to periodicity, sine and cosine are *not invertible* unless we choose a domain on which they are 1–1.



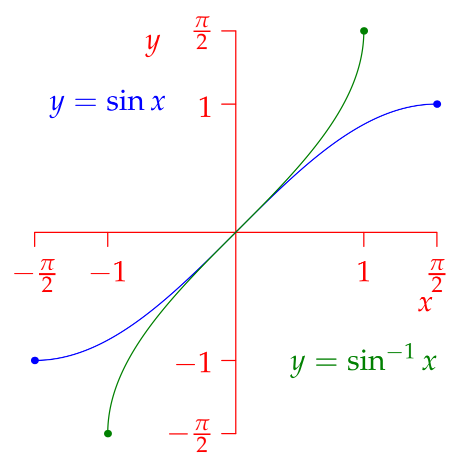
$f(x)=\sin x$ is 1–1 on domain $[-\frac{\pi}{2},\frac{\pi}{2}]$

Inverse function $f^{-1}(x)=\arcsin x=\sin^{-1}x$

Domain $\operatorname{dom}(\arcsin)=[-1,1]=\operatorname{range}(\sin)$

Range $\operatorname{range}(\arcsin)=[-\frac{\pi}{2},\frac{\pi}{2}]=\operatorname{dom}(\sin)$

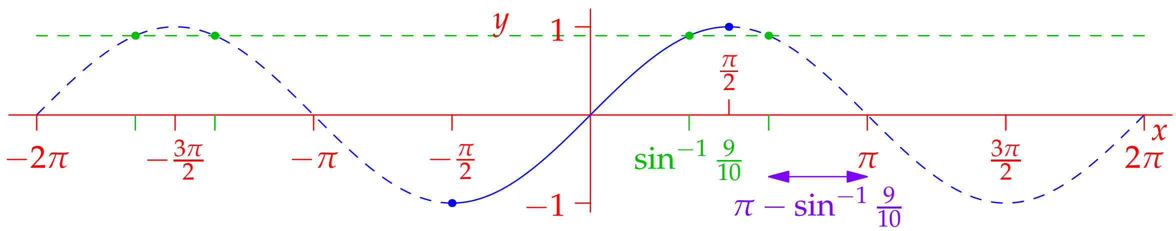
This is why your calculator always returns a value in the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ when you hit the $\sin^{-1}$ button!



Example 7.2. If you know the graphs, then symmetry helps you solve equations. For example, if $\sin x = \frac{9}{10}$ then all solutions are given by

$$x = \sin^{-1}\frac{9}{10} + 2\pi n \quad \text{and} \quad \pi - \sin^{-1}\frac{9}{10} + 2\pi n$$

where n is any integer.



Exercises 1. Draw the graph for cosine and observe that it is invertible if we restrict the domain to the interval $[0, \pi]$. Draw the graph of $\cos^{-1}$.

2. Describe all solutions to the equation $\cos x = -0.2$.

3. Explain why the tangent function has period π; that is $\tan(\theta + n\pi) = \tan\theta$. What facts are we using about sine and cosine and why are they obvious from the definition?

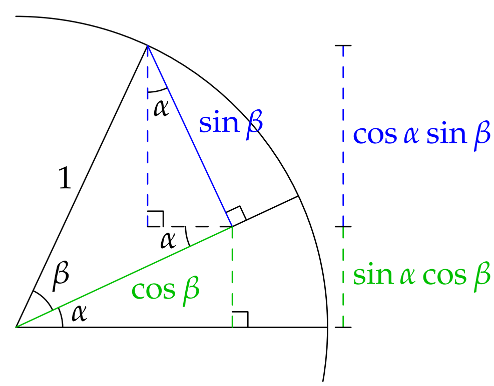
4. Describe all solutions to the equation $\tan x = 5$.

Multiple-angle Formulae

It is often useful to be able to combine arguments for sine and cosine. Here is a simple justification of the sine formulae

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

at least when $0 < \alpha + \beta < \frac{\pi}{2}$.



Exercises 1. How might you go about extending this argument to general angles?

2. The cosine multiple angle formula is also easily seen in the same picture. How?

3. Derive the familiar expressions for $\sin 2\theta$ and $\cos 2\theta$.

4. Find an expression for $\tan(\alpha + \beta)$ purely in terms of $\tan\alpha$ and $\tan\beta$.

5. Two wooden wedges with slope $\frac{1}{4}$ are placed on top of each other to make a steeper slope. What is the gradient of the new slope?

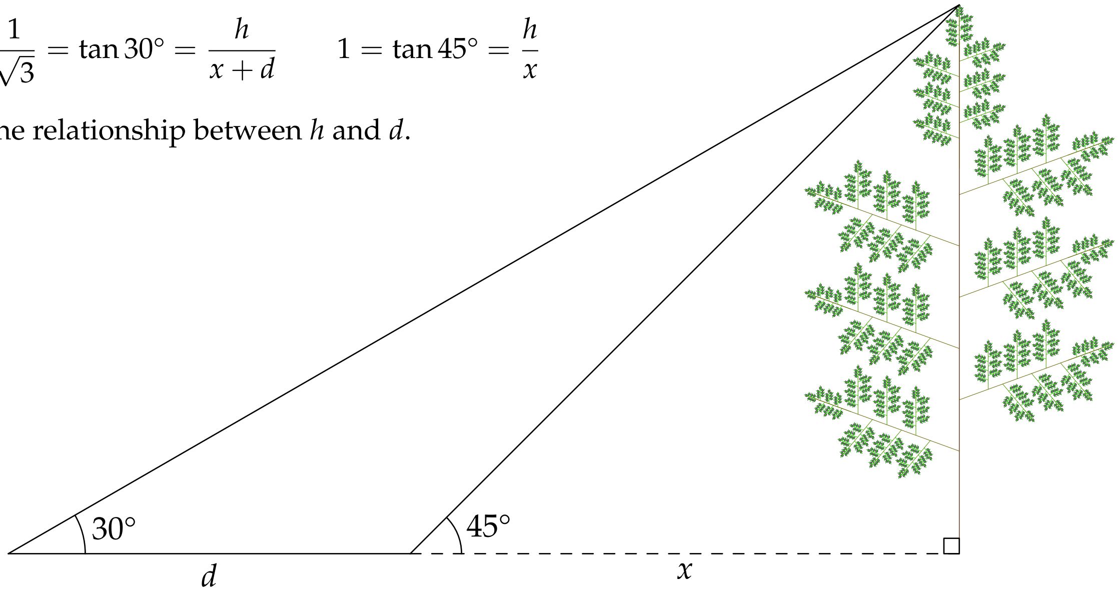
Solving Triangles & the Euclidean Congruence Theorems

Basic applications of trigonometry often involve finding unknown edges and angles of triangles given incomplete information. Here is a classic example.

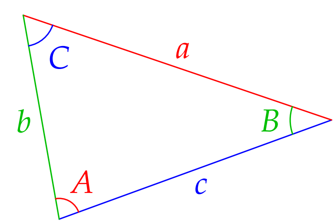
Exercise To find the height h of a tall tree, two angles of elevation are measured a distance d apart along a straight line from the base of the trunk. If these angles are 45° and 30° respectively, and the distance (unknown) from the base of the tree to the nearer measurement is x, then

$$\frac{1}{\sqrt{3}} = \tan 30^\circ = \frac{h}{x+d} \qquad 1 = \tan 45^\circ = \frac{h}{x}$$

Find the relationship between h and d.



In general, by *solving a triangle* we mean computing all six sides and angles given three of them. The Euclidean triangle congruence theorems tell us which combinations are sufficient to determine all the others. Finding these values explicitly typically requires some combination of the *sine* and *cosine* rules.



Theorem 7.3. *For any triangle,*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

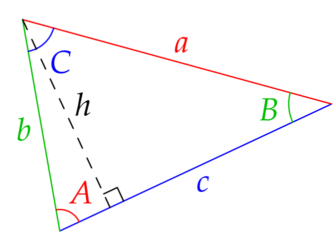
The cosine rule is simply Pythagoras' for non-right triangles.

The object of the previous exercise was the height or *altitude* of a triangle. It turns out that this is the key to solving triangles in general!

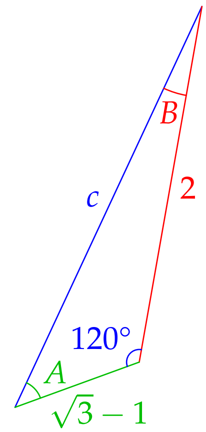
Exercise If h is the altitude with respect to the base c, then

$$h = a\sin B = b\sin A, \qquad c = a\cos B + b\cos A$$

Why is the sine rule now obvious? The cosine rule follows by squaring the second expression and using some identities including the multiple angle formula. Try it!



Examples 7.4. 1. A triangle has side lengths 2 and $\sqrt{3}-1$, and the angle between them is 120°. Find the remaining sides and angles.



- Apply the cosine rule with $a = 2$, $b = \sqrt{3}-1$ and $C = 120°$,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 2^2 + (\sqrt{3}-1)^2 - 2\cdot 2(\sqrt{3}-1)\cos 120° \\ &= 4 + 3 + 1 - 2\sqrt{3} + 2(\sqrt{3}-1) = 6 \end{aligned}$$

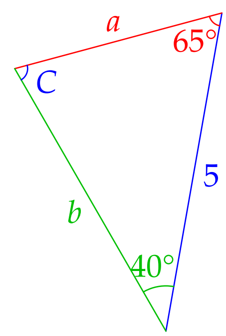
- We now have an opposite pair $(c, C) = (\sqrt{6}, 120°)$, so we can apply the sine rule,

$$\sin A = \frac{2}{\sqrt{6}}\sin 120° = \frac{2\sqrt{3}}{2\sqrt{6}} = \frac{1}{\sqrt{2}} \implies A = 45°$$

  We chose the acute angle since $A = 180° - B - C = 60° - B < 90°$.

- The final angle is then $B = 180° - 45° - 120° = 15°$.

2. A triangle has one side with length 5 and its two adjacent angles are 40° and 65°. Find the remaining data.



- Write $c = 5$, $A = 40°$ and $B = 65°$, The remaining angle is plainly $C = 180° - 40° - 65° = 75°$.
- This gives us an opposite pair, so we can apply the sine rule

$$a = c\frac{\sin A}{\sin C} = 5\frac{\sin 40°}{\sin 75°} \approx 3.327$$

  A second application yields

$$b = c\frac{\sin B}{\sin c} = 5\frac{\sin 65°}{\sin 75°} \approx 4.691$$

Exercises The two examples above are analytic versions of the side-angle-side (SAS) and angle-side-angle (ASA) triangle congruence theorems from Euclidean geometry.

1. There are two further triangle congruence theorems. What are they, and explain how you can use the sine/cosine rules to find the remaining data in both cases.

2. SSA isn't a triangle congruence theorem. There are *two* non-congruent triangles with data $a = 1$, $b = \sqrt{3}$ and $A = 30°$. Find them.

Polar Co-ordinates

Definition 7.1 essentially includes this definition. If θ is the polar angle of a point (x, y), then its polar-coordinates are (r, θ) where $r = \sqrt{x^2 + y^2} \geq 0$. There are really only two things to be careful of when computing these:

1. Every point has a unique radius r, but the polar angle is non-unique! If θ is a polar angle, so is $\theta + 2\pi n$ for any integer $n \in \mathbb{Z}$.

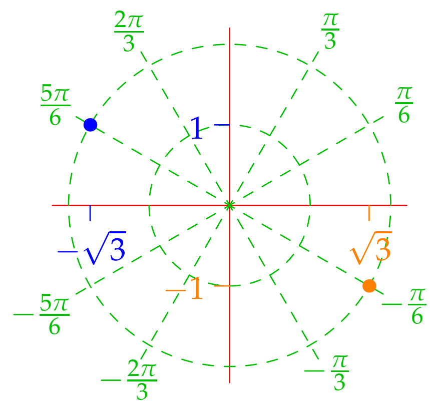
2. Whenever $x \neq 0$,

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x}$$

however, this *doesn't* guarantee that $\theta = \tan^{-1}\frac{y}{x}$. Why?

Example 7.5. If $(x, y) = (-\sqrt{3}, 1)$, then $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ is unique. For the polar angle,

$$\tan\theta = -\frac{1}{\sqrt{3}} = \tan\left(-\frac{\pi}{6}\right) \not\Longrightarrow \theta = -\frac{\pi}{6}$$



We really need to draw the picture! Remember that arctan has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, so will always return an angle in quadrants 1 or 4. Our point is in the second quadrant so we need to adjust.

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Of course we could also add any integer multiple of 2π.

Curves in Polar Co-ordinates

The simplest curves in polar co-ordinates are circles centered at the origin: if $a > 0$, then $r = a$ describes the circle of radius a. Indeed this provides the natural parametrization of a circle:

$$x = a\cos\theta, \quad y = a\sin\theta$$

For this reason, sine and cosine are sometimes called *circular functions.*

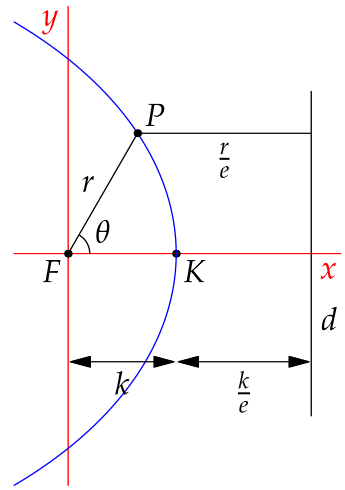
Exercises 1. If $a > 0$, describe the curve with polar equation $r = 2a\cos\theta$.

2. Try to sketch the following curves.

   (a) $r = \theta$ (b) $r = \theta(\theta - 2)$ (c) $r = (\theta - 1)^2 + 1$ (d) $r = (\theta - 1)^2 - 1$

   When in doubt, sketch the curve first as if it is rectangular (i.e. 4. is $y = (x-1)^2 - 1$) and see if you can transfer over. What happens to example 4 when $\theta = 1$? Once you've tried these, use a computer to see if you're right!

Conics in Polar Co-ordinates A fun application is to recover polar forms for the standard conics. We use the focus/directrix/eccentricity definition, place the origin at a focus F and directrix d vertically to the right of the origin. Recall that a conic with eccentricity e may be described as the set of points P such that $|PF| = e\,|Pd|$. Exactly one point on the conic (K) lies between the focus and directrix. If we let $|KF| = k$, it should be clear from the picture that



$$\frac{r}{e} + r\cos\theta = k + \frac{k}{e} \implies r = \frac{k(1+e)}{1+e\cos\theta}$$

This is *independent* of the type of conic, and even works when $e = 0$; $r = 2k$ is a circle with radius k!

- If $e < 0$, then $1 + e\cos\theta > 0$ for all θ and r is always defined. We obtain a closed curve: an ellipse.
- If $e = 1$, then $1 + \cos\theta = 0$ when $\theta = \pi$. We obtain a parabola opening leftwards.
- If $e > 1$, then $1 + e\cos\theta = 0$ at two values $\theta = \pm\cos^{-1}\frac{1}{e}$. These lines are parallel to the asymptotes of the hyperbola. Permitting $r < 0$, the remaining values of θ give the other branch of the hyperbola!

Exercises 1. If $e = 1$, find the *rectangular* equation of the parabola described by the above.

2. (If you know a little Physics...) Let $\hat{\mathbf{r}} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ be the unit vector pointing away from the origin, and $\hat{\boldsymbol{\theta}} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ be the unit vector obtained by rotating $\hat{\mathbf{r}}$ counter-clockwise by 90°.

   (a) Check that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \dot{r}\hat{\boldsymbol{\theta}}$, where $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}$ means differentiation with respect to time. What is $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{\theta}}$ in terms of $\hat{\mathbf{r}}$?

   (b) Verify that

   $$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\hat{\mathbf{r}}) = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\right]\hat{\boldsymbol{\theta}}$$

   (c) Note that $r\hat{\mathbf{r}}$ is the position vector of a point with polar co-ordinates (r, θ). If an inverse-square attractive *force* acts on a particle, Newton's second law tells use that

   $$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r\hat{\mathbf{r}}) = -\frac{b}{r^2}\hat{\mathbf{r}}$$

   for some positive constant b. This is Newton's model of gravity.
   Check that *every conic* is a solution to this differential equation.
   (*Hint: use the fact that $r^2\dot{\theta}$ is constant! This is essentially conservation of angular momentum*)

This explains Kepler's 1st law, that planets orbit the sun in ellipses with the sun at one focus. The other conical orbits are indeed possible: comets regularly pass the sun along hyperbolic orbits, never to return to the solar system.