3 Exponential and Logarithmic Functions & Models

Introducing exponential functions without calculus presents a significant challenge. The simplest approach is as a short-hand notation for *repeated multiplication*: for instance

$$a^5 = a \cdot a \cdot a \cdot a \cdot a$$

analogous to how multiplication represents repeated addition

$$5a = a + a + a + a + a$$

The problem with this approach is that it doesn't help you understand what should be meant by, say, $a^{3/4}$ or $a^{\sqrt{2}}$: multiplying something by itself '$\sqrt{2}$ times' sounds[8] insane!

To rigorously address this problem requires *continuity* and other ideas surrounding the foundations of calculus which you'll encounter in upper-division analysis; topics unsuitable for this course. Instead, we assume some familiarity with exponential functions via introductory calculus, where they are unavoidable and offer two ways to introduce exponential functions and e via modelling.

3.1 The Natural Growth Model

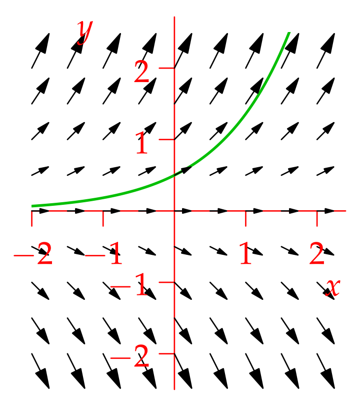
A basic model for any variable quantity is that its *rate of change be proportional to the quantity itself.* This idea necessarily needs some calculus; as a differential equation,

$$\frac{dy}{dx} = ky$$

where k is a constant; if $k > 0$ this is the *natural growth equation*, if $k < 0$ the *natural decay equation.* This is commonly encountered when modelling *population growth*; an otherwise unconstrained population seems like its growth rate should be proportional to its size (twice the people, twice the babies...). This model is hugely applicable, since *population* can refer to essentially any quantifiable value: people, bacteria, money, reagents in a chemical/nuclear reaction, etc.

Example 3.1. The simplest natural growth equation has $k = 1$:

$$\frac{dy}{dx} = y$$



If a point (x, y) lies on a solution curve, then the differential equation tells us the *direction of travel* of the solution. We may visualize this by drawing an arrow with slope $\frac{dy}{dx} = y$; the arrows are *tangent* to any solution.[9] You should easily be able to sketch some other solution curves.

You should, of course, recognize the graph...

[8] The same issue arises for multiplication: $3\sqrt{2} = \sqrt{2} + \sqrt{2} + \sqrt{2}$ is relatively easy for grade-school students to understand, but how would you convince someone what $\pi\sqrt{2}$ means?

[9] A similar approach is available for any first-order differential equation $\frac{dy}{dx} = F(x, y)$: the equation defines its *slope field* (arrows), to which solution curves must be tangent.

Definition 3.2. Let $a > 0$ be constant. The *exponential function with base a* is $f(x) = a^x$.

Recall the *exponential laws*:

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^r = a^{rx}$$

For modelling, the crucial property of exponential functions is that they have proportional derivative.

Theorem 3.3. *The rate of change of $f(x) = a^x$ is proportional to $f(x)$. Specifically,*

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

so that $f(x) = a^x$ satisfies the natural growth/decay equation $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$ with proportionality constant

$$k = f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h}$$

Example 3.4. We estimate the proportionality constant $k = \lim_{h\to 0} \frac{a^h - 1}{h}$ to 3 d.p. using a calculator for four values of h:

a	2	2.5	2.7	2.75	3	5
$\frac{a^{0.1}-1}{0.1}$	0.718	0.960	1.044	1.065	1.161	1.746
$\frac{a^{0.01}-1}{0.01}$	0.696	0.921	0.998	1.017	1.105	1.622
$\frac{a^{0.001}-1}{0.001}$	0.693	0.917	0.994	1.012	1.099	1.611
$\frac{a^{0.0001}-1}{0.0001}$	0.693	0.916	0.993	1.012	1.099	1.610

What is happening to the proportionality constant as a increases? As h decreases?

It appears as if there is a special number somewhere between 2.7 and 2.75 for which the proportionality constant is precisely $k = 1$.

Definition 3.5. The value $e = 2.71828\ldots$ is the unique real number such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.
The *natural*[10] *exponential function* $\exp(x) = e^x$ has derivative $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$.

[10] *Natural* here means *unavoidable*: an old cliche suggests that if aliens were to land on Earth, they'd have to understand e given the technology they'd require to get here. Of course they'd likely use a different symbol; ours comes from Leonhard Euler around 1728. Like π and $\sqrt{2}$, the constant e is an *irrational number*: its decimal representation contains no repeating pattern. There isn't the same geeky fascination with memorizing the digits of e as there is with π, neither is there an 'e-day' (Feb 7th at 6:28 p.m. anyone?).

The function $f(x) = \frac{1}{2}e^x$ is plotted in Example 3.1. Of course there are many other solutions to the natural growth equation $\frac{dy}{dx} = y$: for any constants c, k,

$$y = ce^{kx} \implies \frac{dy}{dx} = \frac{d}{dx}ce^{kx} = cke^{kx} = ky$$

In fact the converse also holds; for the details, take a differential equations course!

Theorem 3.6. *The solutions to the natural growth equation* $\frac{dy}{dx} = ky$ *are precisely the functions*

$$y(x) = y_0 e^{kx} = y_0 \exp(kx)$$

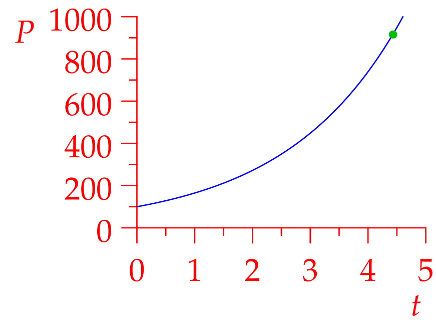
where $y_0 = y(0)$ *is the initial value.*

Example 3.7. A Petri dish contains a population $P(t)$ of bacteria satisfying the natural growth equation $\frac{dP}{dt} = 0.5P$ where time is measured in weeks from the start of the year.

If $P(0) = 100$ bacteria, then $P(t) = 100e^{0.5t}$. Specifically, at the end of January ($4\frac{3}{7}$ weeks) one expects there to be

$$P(\tfrac{31}{7}) = 100 \exp \tfrac{31}{14} = 915 \text{ bacteria}$$

Note that the exponential doesn't return 915 exactly; this is only an approximation. Models like this work best for large populations where integer rounding errors are of minimal concern.



Compound Interest and the Discovery of e

The first description of e came in 1683 when Jacob Bernoulli tried to model the growth of money in a hypothetical bank account. We give a modernized version of his approach.

Example 3.8. \$1 is deposited in an account paying 100% interest per year (nice!). Bernoulli observed that the money in the account at the end of the year depends on *when* the interest is paid.

- If the interest is paid once at the end of the year (this is called *simple* interest), you'll have \$2.
- If half the interest (50¢) is paid at six months, then the balance (\$1.50) earns $\frac{1}{2} \cdot 1.50 = 75$¢ interest for the rest of the year; you'll finish the year with \$2.25 in the account.
- If the interest is paid in four installments, we have the following table of transactions (data is rounded to the nearest cent)

Date	Interest Paid	Balance
1st Jan	—	\$1
1st Apr	25¢	\$1.25
1st July	$\frac{1}{4} \cdot 1.25 = 31$¢	\$1.56
1st Oct	$\frac{1}{4} \cdot 1.56 = 39$¢	\$1.95
New Year	$\frac{1}{4} \cdot 1.95 = 49$¢	\$2.44

More succinctly, the year-end balance is $\left(1 + \frac{1}{4}\right)^4 = \$2.44$.

- More generally, if the interest is paid over n equally spaced intervals, the account balance at the end of the year would be $\$(1+\frac{1}{n})^n$. Here are a few examples rounded things to 5 d.p.

Frequency	Balance after 1 year ($)
Every month	$(1+\frac{1}{12})^{12} = 2.61304$
Every day	$(1+\frac{1}{365})^{365} = 2.71457$
Every hour	$(1+\frac{1}{8760})^{8760} = 2.71813$
Every second	$(1+\frac{1}{31536000})^{31536000} = 2.71828$

As the frequency of payment increases, it appears as if the balance is increasing to $\$e\ldots$

In fact this is a theorem, though it requires significant work (beyond this class) to prove it:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \text{and more generally} \quad e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

This again shows that e arises very naturally.

Simple, Monthly & Continuous Interest In finance, interest is typically computed in one of three ways. In each case we describe the result of investing $1 at an interest rate of $r\% = \frac{r}{100}$ per year.

Simple interest You are paid $\frac{r}{100}$ dollars at the end of the year. Your invested dollar becomes $1+\frac{r}{100}$ dollars.

Monthly interest Each month you are paid $\frac{r}{12}\%$ of your current balance. This amounts to a balance of $(1+\frac{r}{1200})^{12}$ dollars at year's end. The period need not be monthly: if interest is paid in n installments, the balance would be $(1+\frac{r}{100n})^n$.

Continuous interest After t years (can be any fraction of a year!) your dollar-balance is

$$e^{\frac{rt}{100}} = \exp\frac{rt}{100} = \lim_{n\to\infty}\left(1+\frac{rt}{100n}\right)^n$$

Example 3.9. A bank account earns 6% annual interest paid monthly. To what simple interest rate does this correspond? Would you perfer an account paying 6% continuously?

At the end of the year, $1 becomes

$$(1+0.0612)^{12} = 1.005^{12} \approx 1.06168\ldots$$

corresponding to a simple interest rate of 6.17%. By cobntrast, 6% continuous interest would result in your dollar becoming $e^{6/100} \approx 1.06184$, corresponding to a (marginally) higher simple interest rate of 6.18%. You should prefer this, particularly if you have a lot of money to invest! The difference is more noticable with an investment of $1000 over ten years:

$$1000 \times 1.005^{120} = \$1819.40 \quad \text{versus} \quad 1000e^{0.6} = \$1822.12$$

There are several reasons for these varying approaches, not all of them consumer-friendly:

1. Simple interest is simple! It is easy to understand and compute, but hard to decide how or even whether to compute interest for parts of a year.
2. Monthly interest fits with most paychecks, so is sensible for loans, particularly mortgages.
3. Continuous interest allows the balance of an account to be found easily at any time, even between interest payment dates. It is also much easier to apply mathematical analysis (calculus).
4. A company can make an interest rate appear *higher* (if a savings account) or *lower* (if a loan) by choosing which way to quote an interest rate.

Example 3.10. A bank quotes you a loan with a continuously compounded interest rate of 7%. If you borrow \$100,000, then at the end of the year you'll owe

$$100000e^{0.07} = \$107,250.82$$

not the \$107,000 you might have expected! This corresponds to a simple interest rate (one payment at the end of the year) of 7.25%.[11]

Exercises 3.1. 1. Draw a slope field for the natural decay equation $\frac{dy}{dx} = -\frac{1}{3}y$ and use it to sketch the solution curve with initial condition $y(0) = 6$. What is the *function* $y(x)$ in this case?

2. Which of the following would you prefer for a savings account, and why?
   - 5% interest paid continuously.
   - 5.05% compounded monthly.
   - 5.1% paid at the end of the year.

3. You invest \$1000 in an account that pays 4% simple interest per year.
   (a) How much money will you have after 5 years?
   (b) If you close the account after 2 years and 3 months, the bank needs to decide how much interest to credit you with. Do this is two ways (the answers will be different!):
      i. Compute using the simple interest rate for 2.25 years.
      ii. Suppose that interest is paid at 4% for all completed years and then at 4% paid monthly for any completed months of an incomplete year. Find the balance of the account at closing.

4. See if you can explain why the proportionality constant for $\left(\frac{1}{a}\right)^x$ is *negative* that for a^x: that is,

$$\lim_{h\to 0} \frac{(\frac{1}{a})^h - 1}{h} = -\lim_{h\to 0} \frac{a^h - 1}{h}$$

   Try to find both an *algebraic* reason and a *pictorial* one.

5. Sketch the function $f(x) = e^{-x^2}$. Where have you seen this before, and what uses does this function have?

[11]In the US, mortgage companies typically quote an interest rate which they use to compound *monthly*. For example, if the quoted rate is 7%, then the effective annual (simple) interest rate is $\left(1 + \frac{0.07}{12}\right)^{12} - 1 = 7.229\%$. By law, this higher *effective APR* must be quoted somewhere, though it is unlikely to be as prominently posted...