Math 8: Homework Questions 4

Submit answers to questions 1, 2, and 4 on Canvas by Thursday 12$^{\text{th}}$ May

1. Plutonium-238 has a *half-life* of 88 years, meaning that after 88 years half of the isotope has decayed to another element (in this case uranium-234).
   (a) If you start with 100 grams of plutonium, find a model for how much remains after t years.
   (b) How long will take for the mass to decay to 10 grams, and at what rate will it be decaying?
2. The *dumb breeding* model is an alternative model for population growth. It posits that the rate of increase of a population is proportional to the frequency of interactions between members of the population.
   (a) Explain, in words, why the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$ is appropriate for this model.
   (b) Show that $P(t) = \frac{P_0}{1-kP_0t}$ is a solution, where P_0 is the initial population. What happens to the population as t increases?
   (c) A population of 100 rabbits breeds according to the dumb model. Suppose there are 110 rabbits after one month. How long is it until doomsday?
3. You invest \$1000 in an account that pays 4% simple interest per year.
   (a) How much money will you have after 5 years?
   (b) To what *continuous* rate of interest is 4% equivalent?
   (c) If you close the account after 2 years and 3 months, the bank needs to decide how much interest to credit you with. Do this is two ways (the answers will be different!):
      i. Compute using the simple interest rate for 2.25 years.
      ii. Suppose that interest is paid at 4% for all completed years and then at 4% paid monthly for any completed months of an incomplete year. Find the balance of the account at closing.
4. Suppose (as in the notes) that a population (in thousands) of fish in a lake behaves according to the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{16}P(10-P)$$

   where t is measured in months.
   (a) Suppose that a fixed *fraction* $f \in (0,1)$ of the fish are removed each month. Show that you still have a logistic model. What happens to the population as $t \to \infty$?
   (b) Suppose that a constant number h thousand fish are harvested each month. Find constants $P_0 < P_1$ such that $Q(t) = P(t) - P_0$ satisfies a logistic equation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{16}Q(P_1 - P_0 - Q)$$

   What do you expect to happen to the population of fish in the lake, and how does this depend on the value of h? How many fish can you harvest per month if the population is to exist long-term?