4 Sequences as Functions

We've seen many different types of function in this course and used them to model various situations. In practice, one is often faced with the opposite problem: given experimental data, what type of function should you try?

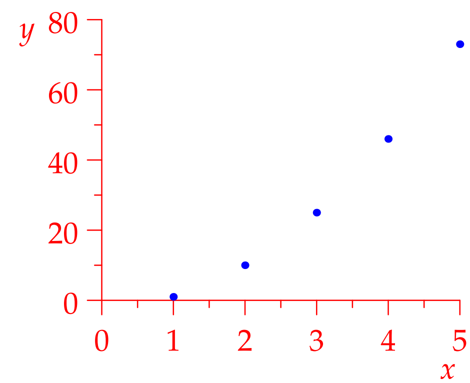
4.1 Polynomial Sequences: First, Second, and Higher Differences

To begin to answer this, first ask yourself, "What is a sequence?" Hopefully you have a decent intuitive idea already. More formally, a sequence a function whose domain is a set like the natural numbers, for example

$$f : \mathbb{N} \to \mathbb{R} : n \mapsto 3n^2 - 2$$

defines the sequence

$$\big(f(1), f(2), f(3), \dots\big) = (1, 10, 25, 46, 73, \dots)$$



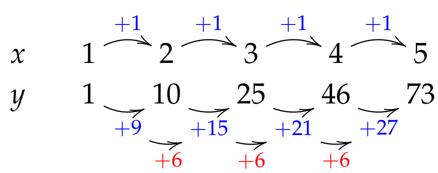
This is indeed the intuitive idea of a function to many grade-school students: continuity and domains including fractions or even irrational numbers are more advanced concepts.

Suppose instead that all you have is a data set

x	1	2	3	4	5
y	1	10	25	46	73

perhaps arising from an experiment. Could you recover the original function $y = f(x)$ directly from this data? You could try plotting data points as we've done, though it is hard to decide directly from the plot whether we should try a quadratic model, some other power function/polynomial, or perhaps an exponential. Of course, the physical source of real-world data might also provide clues.

A more mathematical approach involves considering how data values *change*:



The first-differences in the x-values are constant whereas those for the y-values are increasing

$$(y_{n+1} - y_n) = (9, 15, 21, 27, \dots)$$

You likely already notice the pattern: the sequence of first-differences is increasing *linearly* as the *arithmetic sequence*

$$y_{n+1} - y_n = 3 + 6n$$

To make this even clearer, note that the sequence of second-differences in the y-values is *constant* (+6). These facts are huge clues that we expect a quadratic function.

But why? Well we can certainly check the following directly:

Linear Model If $f(n) = an + b$, then the sequence of first-differences is constant

$$f(n+1) - f(n) = a$$

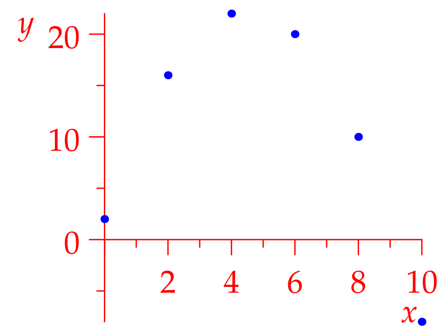
Quadratic Model If $f(n) = an^2 + bn + c$, then the sequence of first-differences is linear and the second-differences are constant:

$$g(n) := f(n+1) - f(n) = 2an + a + b, \qquad g(n+1) - g(n) = 2a$$

The relationship between these results and the *derivative(s)* of the original function $f(x)$ should feel intuitive: what happens if you differentiate a quadratic twice?

Example 4.1. You are given the following data set

x	0	2	4	6	8	10
y	2	16	22	20	10	−8



The x-values have constant first-differences while the y-values have constant second-differences

First-differences: $14, 6, -2, -10, -18$

Second-differences: $-8, -8, -8, -8$

We therefore suspect a quadratic model $y = f(n) = an^2 + bn + c$. Rather than using the above formulae, particularly since the x-differences are not 1, it is easier just to substitute:

$$2 = y(0) = c, \quad \begin{cases} 16 = f(2) = 4a + 2b + 2 \\ 22 = f(4) = 16a + 4b + 2 \end{cases} \implies \begin{cases} 2a + b = 7 \\ 8a + 2b = 10 \end{cases} \implies 4a = -4$$

whence $a = -1$, $b = 9$ and $c = 2$. A quadratic model is therefore

$$y = f(n) = -n^2 + 9n + c = -n^2 + 9n + 2$$

It is easily verified that the remaining data values satisfy this relationship.

There are at least two issues with our method:

1. The question we're answering is, "Find a quadratic model satisfying given data." Constant second-differences don't guarantee that only a quadratic model is suitable. For example,

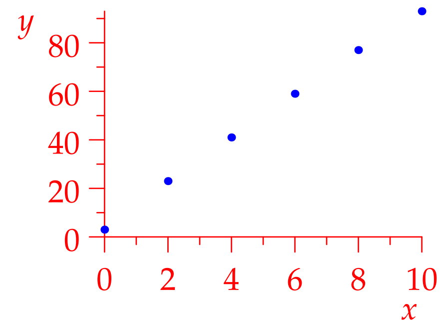
   $$y = -n^2 + 9n + 2 + 297n(n-2)(n-4)(n-6)(n-8)(n-10)$$

   is a very complicated model satisfying the same data set!

2. It is very unlikely that experimental data will fit such precise patterns (why not?). However, if the differences are *close* to satisfying such patterns, then you should feel confident that a linear/quadratic model is a good choice.

Example 4.2. Given the data set

x	0	2	4	6	8	10
y	3	23	41	59	77	93



with sequences of first- and second-differences

First-differences: 20, 18, 18, 18, 16

Second-differences: $-2, 0, 0, -2$

do you think a linear or quadratic model would be superior?
If you wanted a linear model, you'd likely be inclined to try $f(x) = 9x + b$ for some constant b. Here are two options:

1. $f(x) = 9x + 5$ fits the middle four data values perfectly, but as a predictor is too large at the endpoints: $f(0) = 5 > 3$ and $f(10) = 95 > 93$.

2. $f(x) = 9x + 5 - \frac{2}{3}$ doesn't pass through any of the data values but seems to reduce the net error to zero:

x	0	2	4	6	8	10
$f(x) - y$	$-\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{4}{3}$

$$\implies \sum_x f(x) - y = 0$$

Neither model is perfect, but then this is what you expect with real-world data!

Exercises 4.1. 1. For each data set, find a function $y = f(x)$ modelling the data.

(a)

x	2	4	6	8
y	-1	2	7	14

(b)

x	2	5	8	11	14
y	-6	-15	-6	21	66

(c)

x	0	6	9	15
y	3	15	21	33

(*Be careful with (c): the x-differences aren't constant!*)

2. Suppose a table of data values containing (x_0, y_0) has constant first-differences in both variables

$$\Delta x = x_{n+1} - x_n = a, \quad \Delta y = b$$

Find the equation of the linear function $y = f(x)$ through the data.

3. What relationship do you expect to find with the sequential differences of a cubic function $f(n) = an^3 + bn^2 + cn + d$? What about a degree-$m$ polynomial $f(n) = an^m + bn^{m-1} + \cdots$?

4. If $f(n) = an^2 + bn + c$ is a quadratic model for the data in Example 4.2 with constant second-differences -1, show that $a = -\frac{1}{8}$. What might be reasonable values for b, c?

5. (Hard) Suppose $f(x)$ is a twice-differentiable function and $h > 0$ is constant. Use the mean value theorem from calculus to explain the following.

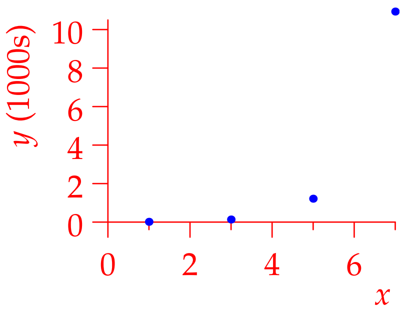
   (a) First-differences $f(x + h) - f(x)$ are proportional to $f'(\xi)$ for some $\xi \in (x, x + h)$.

   (b) Second-differences satisfy $\big(f(x + 2h) - f(x + h)\big) - \big(f(x + h) - f(x)\big) = f''(\xi)h\alpha$ for some ξ between x and $x + h$ and some α. Why is it unlikely that α is constant?

4.2 Exponential, Logarithmic & Power Sequences

To observe relationships between data values, you might also have to consider *ratios* between successive terms or skip values.

Example 4.3. From a first glance at the given data, it is hard to decide whether an exponential or a quadratic (or higher degree polynomial) model is more suitable. If we try to apply the constant-difference method, we don't seem to get anything helpful:



		$+2$		$+2$		$+2$	
x	1		3		5		7
y	15		135		1215		10935
		$+120$		$+1080$		$+9720$	
			$+960$		$+8640$		

By the time we're looking at second-differences, any conclusion would be very weak since we only have two data values!

If instead we think about *ratios* of y-values, then a different pattern emerges:

		$+2$		$+2$		$+2$	
x	1		3		5		7
y	15		135		1215		10935
		$\times 9$		$\times 9$		$\times 9$	

The question remains: what type of function scales its output by 9 when 2 is added to its input: $f(x+2) = 9f(x)$? This is a function that converts *addition* to *multiplication*: an exponential! If we try $y = f(x) = ba^x$ for some constants a, b, then

$$f(x+2) = ba^{x+2} = ba^2b^x = a^2 f(x)$$

from which a suitable model is $y = 5 \cdot 3^x$.

We can see the pattern in the example more generally:

Exponential Model If $f(x) = ba^x$, then adding a constant to x results in

$$f(x+k) = ba^{x+k} = a^k f(x)$$

If x-values have constant differences $(+k)$, then y-values will be related by a constant *ratio* $(\times a^k)$. You might remember this as 'addition–product' or 'arithmetic–geometric.'

Such a simple pattern is often disguised:

- Complete data might not be given so you might have to skip some data values to see a pattern. For example, if our original data was

x	1	3	4	5	7
y	15	135	405	1215	10935

  then the x-values are not in a strictly arithmetic sequence.

- As in Example 4.2, real-world/experimental data will only *approximately* exhibit such patterns.

Example 4.4. A population of rabbits is measured every two months resulting in the data set

t	0	2	4	6	8	10
P	5	7	10	14	19	28

The data seems very close to being quadratic; consider the first and second sequences of P-differences

$$\Delta P = (2,3,4,5,9), \qquad \Delta\Delta P = (1,1,1,4)$$

However, the last difference doesn't fit the pattern. Instead, the fact that we expect an exponential model is buried in the experiment: the data is measuring population growth! We therefore instead consider the ratios of P-values:

		$+2$		$+2$		$+2$		$+2$		$+2$	
t	0		2		4		6		8		10
P	5		7		10		14		19		28
		$\times 1.4$		$\times 1.43$		$\times 1.4$		$\times 1.36$		$\times 1.47$	

The ratios are very close to being constant, whence an exponential model is suggested! To exactly match the first and last data values, we could take the model

$$P(t) \approx 5\left(\frac{28}{5}\right)^{\frac{t}{10}}$$

t	0	2	4	6	8	10
P	5	7.057	9.960	14.057	19.839	28

Only $P(8)$ doesn't match when we take rounding to the nearest integer into account.

We've seen that addition-addition corresponds to a linear model and that addition-multiplication to an exponential. There are two other natural combinations.

Logarithms These operate exactly as exponentials but in reverse. If $f(n) = \log_a x + b$, then *multiplying* x by a constant results in a constant *addition/subtraction* to y:

$$f(kx) = \log_a(kx) + b = \log_a k + \log_a x + b = \log_a k + f(x)$$

This could be summarized as 'product–addition.'

Power Functions If $f(x) = ax^m$, then multiplying x by a constant will do the same to y

$$f(kx) = a(kx)^m = ak^m x^m = k^m f(x)$$

We have a 'product–product' relationship between successive terms.

Examples 4.5. Find the patterns in the following data and suggest a model $y = f(x)$ in each case.

x	6	18	54	162
y	1	2	3	4

x	3	6	9	12
y	135	1080	3645	8640

The sequential approach in this chapter is a form of *discrete calculus*: using a pattern of *differences* to predict the original function is similar to how we use knowledge of a derivative $f'(x)$ to find $f(x)$.

Example 4.6. Suppose $g(2) = 3$ and $g(4) = 9$. What do you think should be the value of $g(8)$?

It depends on the type of model you try.

1. For a linear (addition-addition) model we know that $\Delta x = 2$ corresponds to $\Delta y = 6$, so

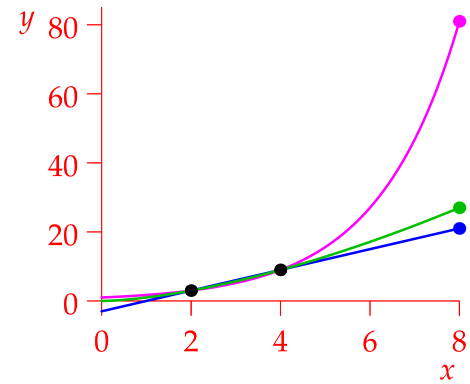
$$g(8) = g(4 + 2\Delta x) = g(4) + 2\Delta y = 9 + 12 = 21$$

2. For an exponential (addition-product) model, $\Delta x = 2$ corresponds to a *y-ratio* $r_y = \frac{9}{3} = 3$, so

$$g(8) = r_y g(6) = r_y^2 g(4) = 9 \cdot 9 = 81$$

3. For a power (product-product) model, $r_x = 2$ corresponds to a $r_y = 3$, so

$$g(8) = g(2 \cdot 4) = g(4r_x) = r_y g(4) = 3 \cdot 9 = 27$$



1. $g(x) = 3x - 3$
2. $g(x) = 3^{x/2}$
3. $g(x) = x^{\log_2 3}$

We do not need to calculate the models explicitly(!), though they are stated below the graph for convenience.

Exercises 4.2. 1. Find the patterns in the following data sets and use them to find a model $y = f(x)$.

(a)

x	0	1	2	3	4
y	80	120	180	270	405

(b)

x	2	4	8	10
y	1	16	256	625

(c)

x	1	3	5	7	9
y	15	5	19	57	119

(d)

x	1	3	4	6
y	1	36	216	7776

(e)

x	20	60	180	540
y	2	4	6	8

(f)

x	2	6	54	486	4374
y	2	4	8	12	16

2. Take logarithms of the power relationship $y = ax^m$. What is the relationship between $\ln y$ and $\ln x$? Use this to give another reason why the inputs and outputs of power functions satisfy a 'product–product' relationship.

3. How does our analysis of exponential functions change if we add a constant to the model? That is, how might you recognize a sequence arising from a function $f(x) = ba^x + c$?

4. Suppose $f(5) = 12$ and $f(10) = 18$. Find the value of $f(20)$ supposing $f(x)$ is a:

   (a) Linear function;

   (b) Exponential function;

   (c) Power function.

   If $f(20) = 39$, which of the three *models* do you think would be more appropriate?

4.3 Newton's Method

To finish our discussion of sequences we revisit a (hopefully) familiar technique for approximating solutions to equations. Variations of this approach have been in use for thousands of years.

Example 4.7. We motivate the method by considering an ancient method for approximating $\sqrt{2}$, known to the Babylonians 2500 years ago!
Suppose $x_n > \sqrt{2}$. Then $\frac{2}{x_n} < \frac{2}{\sqrt{2}} = \sqrt{2}$. It seems reasonable to guess that their average

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

should be a more accurate approximation to $\sqrt{2}$. If start with an initial guess $x_0 = 2$, then we obtain the sequence

$$x_1 = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{3}{2}, \quad x_2 = \frac{17}{12} = 1.4166\ldots, \quad x_3 = \frac{577}{408} = 1.4142\ldots, \quad \ldots$$

This sequence certainly *appears* to be converging to $\sqrt{2}$...

Since it makes use of the average, this approach is sometimes called the *method of the mean*. It may be applied to any square-root $\sqrt{a}$ where $a > 0$: let $x_0 > 0$ and define,

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{$*$}$$

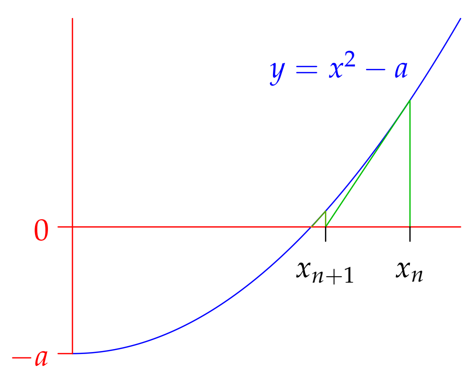
A rigorous proof that the sequence converges requires more detail than is appropriate for us (though see Exercise 3), but two observations should make it seem more believable:

1. *If* the sequence $(*)$ has a limit L, then the limit must satisfy

   $$L = \frac{1}{2}\left(L + \frac{a}{L}\right) \implies 2L^2 = L^2 + a \implies L^2 = a \implies L = \sqrt{a}$$

   where we take the positive root since all terms x_n are plainly positive.

2. The iterations have a convincing *geometric* interpretation. The sequence of iterates can be found by repeatedly taking the tangent line to the curve $y = f(x) = x^2 - a$ and intersecting it with the x-axis. To see why, observe that the tangent line at x_n has equation

   $$\begin{aligned} y &= f(x_n) + f'(x_n)(x - x_n) \\ &= x_n^2 - a + 2x_n(x - x_n) \\ &= 2x_n x - x_n^2 - a \end{aligned}$$

   which intersects the x-axis ($y = 0$) when

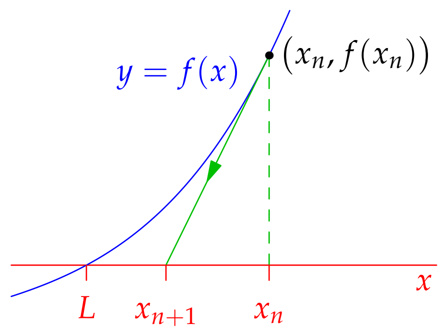
   $$x = \frac{x_n^2 + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) = x_{n+1}$$

This geometric idea generalizes...

Definition 4.8. Given a differentiable function $f(x)$ with non-zero derivative, the *Newton–Raphson iterates* of an initial value x_0 are defined by the recurrence formula

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

Our two previous observations still hold:



1. If $L = \lim_{n\to\infty} x_n$ exists and $f'(L) \neq 0$, then

$$L = L - \frac{f(L)}{f'(L)} \implies f(L) = 0$$

   That is, the limit L is a root of the function $f(x)$.

2. The tangent line at $(x_n, f(x_n))$ forms a right-triangle with base $x_n - x_{n+1}$ and height $f(x_n)$, from which its slope is

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

   Rearranging this gives the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Newton's method is particularly nice for polynomials with integer coefficients, since the iterates form a sequence of *rational numbers*. This approach was often used obtain rational approximations to irrational numbers before the advent of calculators.

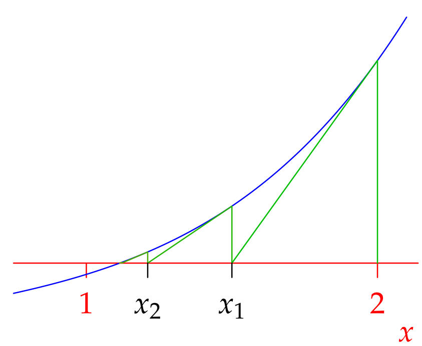
Examples 4.9. 1. To find a root of $f(x) = x^4 + 4x - 6$, start with $x_0 = 2$ and iterate

$$x_{n+1} = x_n - \frac{x_n^4 + 4x_n - 6}{4x_n^3 + 4} = \frac{3(x_n^4 + 2)}{4(x_n^3 + 1)}$$

which yields the sequence (to 3 d.p.)

$$\left(2, \frac{3}{2}, \frac{339}{280}, \ldots\right) = (2, 1.5, 1.211, 1.121, 1.114, 1.114, \ldots)$$

You can check with a calculator that 1.114 is approximately a root.



2. The irrational number $x = \sqrt{2} + \sqrt{3}$ is a root of the polynomial

$$f(x) = x^4 - 10x^2 + 1$$

By applying Newton's method with $x_0 = 3$, we obtain the sequence (to 3 d.p.)

$$x_{n+1} = x_n - \frac{x_n^4 - 10x_n^2 + 1}{4x_n^3 - 20x_n} = \frac{3x_n^4 - 10x_n^2 - 1}{4x_n(x_n^2 - 5)} \implies (x_n) = \left(3,\ \frac{19}{6} = 3.167,\ 3.147,\ \ldots\right)$$

Newton's method can be attempted for any differentiable function, though the sequence isn't guaranteed to converge: see for instance Exercise 5. You can find graphical interfaces online for this (for instance with Geogebra).

Exercises 4.3. 1. Use Newton's method to find a root of the given function to 4 decimal places.

(*Use a calculator, but explain what you are doing!*)

(a) $f(x) = x^3 - 4$ (b) $f(x) = 2x^3 + x - 1$ (c) $f(x) = e^x - \sqrt{x} - 2$

2. Use Newton's method to find a rational number approximation to $\sqrt[3]{2}$ in lowest terms $\frac{p}{q}$ where $10 < q < 100$.

3. Suppose you perform Newton's method for the function $f(x) = x^2 - 2$ starting with some positive $x_0 > 0$.

   (a) If $x_n > 0$, show that $x_{n+1} - \sqrt{2} = \frac{1}{2x_n}(x_n - \sqrt{2})^2 = \left(\frac{1}{2} - \frac{1}{2\sqrt{2}x_n}\right)(x_n - \sqrt{2})$.

   (b) Explain why $\left|x_n - \sqrt{2}\right| < \frac{1}{2^n}\left|x_0 - \sqrt{2}\right|$. Hence conclude that the sequence of iterates (x_n) converges to $\sqrt{2}$.

4. We might consider a *method of the mean* for approximating $\sqrt[3]{2}$: given x_0, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n^2}\right)$$

   (a) If the sequence (x_n) converges, show that its limit is $\sqrt[3]{2}$.

   (b) If $x_n > \sqrt[3]{2}$, show that $\frac{2}{x_n^2} < \sqrt[3]{2}$.

   (c) Let $x_0 = 1$. Compute x_1 and x_2. Compare these with the values obtained using Newton's method for the function $f(x) = x^3 - 2$ with the same initial condition $x_0 = 1$.

5. Let $f(x) = x^3 - 5x$.

   (a) What happens if you apply Newton's method to this function with initial condition $x_0 = 1$? Draw a picture to illustrate.

   (b) (Just for fun!) Investigate what happens for other values of x_0. Can you make any conjectures? Is is possible for x_0 to be *positive* and yet for $x_n \to -\sqrt{5}$? Can you make any sense of what happens if $1 < x_0 < \sqrt{\frac{5}{3}}$?