MATH 150 HOMEWORK 5
Due: Nov 22, 2017

IMPORTANT INSTRUCTION: It is crucial that you clearly explain how you arrive at your conclusions.

1. (**5pts**) Let $\mathcal{L}$ be a first order language language, Σ be a set of $\mathcal{L}$-sentences, and σ, τ be $\mathcal{L}$-sentences. Prove or disprove the following statements. Recall the $\models$ symbol used below is "logical implication" as defined in lectures.

(a) **(1pt)** If $\Delta \subseteq \Sigma$ and $\Delta \models \sigma$ then $\Sigma \models \sigma$.

(b) **(2pts)** $\Sigma \cup \{\tau\} \models \sigma$ if and only if $\Sigma \models \tau \to \sigma$.

(c) **(2pts)** Σ is satisfiable if and only if there is an $\mathcal{L}$-sentence ρ such that $\Sigma \not\models \rho$.

2. (**5 pts**) Let $\mathcal{L} = \{P, Q\}$ be a first order language with P a 2-ary relation symbol and Q a 1-ary relation symbol. Show the following.

(a) **(1pt)** $\forall v_1\ Qv_1 \models \exists v_2\ Qv_2$.

(b) **(1pt)** $\exists x\ \forall y\ Pxy \models \forall y\ \exists x\ Pxy$.

(c) **(2pt)** $\forall y\ \exists x\ Pxy \not\models \exists x\ \forall y\ Pxy$.

(d) **(1pt)** $\emptyset \models \exists x(Qx \to \forall x\ Qx)$.

3. (**5pt**) Let $\mathcal{L} = \{\dot{+}, \dot{\times}, \dot{<}, \dot{0}, \dot{1}\}$ be the language of ordered rings introduced in class. Let $\mathfrak{M} = (\mathbb{R}; +, \times, <, 0, 1)$ be the $\mathcal{L}$-structure, where $\mathbb{R}$ is the set of real numbers, $+ = \dot{+}^{\mathfrak{M}}$ is the usual addition on real numbers, $\times = \dot{\times}^{\mathfrak{M}}$ is the usual multiplication on real numbers, $<= \dot{<}^{\mathfrak{M}}$ is the usual "less than" relation on real numbers, $0 = \dot{0}^{\mathfrak{M}}$, and $1 = \dot{1}^{\mathfrak{M}}$.

Express each of the following statements as a formula (or sentence) in $\mathcal{L}$ and check whether $\mathfrak{M}$ satisfies the formula with respect to the evaluation $s : V \to \mathbb{R}$ defined as: $s(x_{2n+1}) = \sqrt{2n}$ for all $n \in \mathbb{N}$.

(a) **(1.5 pt)** $\varphi_1 \equiv$ "there is no largest negative number".

(b) **(1.5 pt)** $\varphi_2(x_3) \equiv$ "x_3 is in the interval $[\frac{1}{2}, \frac{2}{3})$".

Define the following sets in $\mathfrak{M}$.

(i) **(1 pt)** The binary relation R consisting of tuples (a, b) such that $a \geq b$ and b is less than -1.

(ii) **(1 pt)** The set of squares less than $\frac{1}{2}$.