MATH 150 HOMEWORK 6
Due: Wednesday Dec 6

IMPORTANT INSTRUCTION: It is crucial that you clearly explain how you arrive at your conclusions.

1. (**6pts**) Let $\mathcal{L} = \{\dot{E}\}$ be the language of graphs (recall the definition of a graph discussed in class).

(**3pts**)Express each of the following statements about graphs as a a set (possibly infinite) of sentences in $\mathcal{L}$. That is, in each of the following cases find a set of $\mathcal{L}$-sentences Σ such that for every graph G,

$$G \text{ has the named property iff } G \vDash \Sigma.$$

(a) (**1pt**) "G contains arbitrarily large finite cliques"

(b) (**1pt**) "G consists of disjoint cycles"

(c) (**1pt**) "Any two nodes in G have the same degree."

(**3pts**) Prove that there is no set Σ (possibly infinite) of $\mathcal{L}$-sentences such that for every graph $G = (V, E)$ (so G is an $\mathcal{L}$-structure), the following holds:

$$G \vDash \Sigma \text{ if and only if } G \text{ has finitely many cliques of size } 5$$

A clique of size n in a graph is a set of n nodes $\{x_1, \ldots, x_n\}$ such that for every $i \neq j$, the nodes x_i and x_j are connected with an edge.
For $k \geq 3$ we define the notion of a cycle of length k as follows. A cycle of length k in a graph is a sequence $\{x_1, \ldots, x_k\}$ of length k of nodes in the graph such that x_i is connected with x_{i+1} via an edge for every $i < k$ and also x_k is connected with x_1 via an edge. (Draw pictures!)
Two cycles in a graph are disjoint iff they do not have any common vertex.
The degree of a vertex x in a graph is the number of vertices connected with x via an edge.

2. (**6pts**) Let $\mathcal{L} = \{\dot{E}\}$ be the language of equivalence relations.

(**4pts**) Express each of the following statements as a formula/sentence in $\mathcal{L}$.

(a) (**1pt**) "There are infinitely many equivalence classes of size 2"

(b) (**1pt**) "There are arbitrarily large finite equivalence classes"

(c) (**1pt**) "All finite equivalence classes are of even size"

(d) (**1pt**) "All equivalence classes are infinite".

(**2pts**) Prove that there is no set Σ (possibly infinite) of $\mathcal{L}$-sentences such that for every equivalence relation structure $\mathcal{A} = (A, E)$ the following holds:

$$\mathcal{A} \vDash \Sigma \text{ if and only if } \mathcal{A} \text{ has finitely many equivalence classes of size 1.}$$

3. (**3pts**) Let $\mathcal{L} = \{P\}$ be a language with one 2-ary relation symbol P. Let $\mathcal{M} = (\mathbb{Z}, P^{\mathcal{M}})$ be an $\mathcal{L}$-structure. Here $\mathbb{Z}$ is the set of integers and for all $a, b \in \mathbb{Z}$,

$$(a, b) \in P^{\mathcal{M}} \Leftrightarrow |b - a| = 1.$$

(Intuitively, $(a, b) \in P^{\mathcal{M}}$ if and only if a and b are adjacent numbers).

Show that there is an elementarily equivalent $\mathcal{L}$-structure $\mathcal{N} = (N, P^{\mathcal{N}})$ (i.e. $\mathcal{M} \equiv \mathcal{N}$) that is not connected.

Here being *connected* means for every two members a, b of N, there is a path between them. A *path* of length n, from a to b is a sequence $(a_0, a_1, \ldots, a_n)$ such that $a_0 = a$, $a_n = b$ and $(a_i, a_{i+1}) \in P^{\mathcal{N}}$ for all $i < n$.

Hint: Add constant symbols c, d to the language $\mathcal{L}$. Write down the sentences which say that c and d are "far apart" (use the intuition of what "far" means in the model $\mathcal{M}$). Apply compactness theorem to this set of sentences.