MATH 150 HOMEWORK 3 SOLUTION

1. (**5 pts**) A set Σ of formulas is independent iff for every formula $\phi \in \Sigma$,

$$\Sigma \backslash \{\phi\} \not\models \phi.$$

Here $\Sigma \backslash \{\phi\}$ is the set of formulas obtained by removing the formula ϕ from Σ. (More generally, if X, Y are sets then $X \backslash Y$ is the set obtained by removing all elements of Y from X.) Of course, if $\phi \notin \Sigma$, then $\Sigma \backslash \{\phi\} = \Sigma$.

(a) (**1 pt**) Is the set of formulas $\Sigma = \{A \to C, A \vee B, \neg B, C\}$ independent? Prove or disprove.

Answer. Σ is NOT independent. For example, $\Sigma \backslash \{C\} = \{A \to C, A \vee B, \neg B\} \models C$. You can easily check this using truth table. But perhaps it is faster to observe that for any truth assignment v such that $v(A \to C) = v(A \vee B) = v(\neg B) = T$, we must have $v(B) = F$. Since $v(A\vee) = T$ and $v(B) = F$, $v(A) = T$. Now since $v(A \to C) = T$ and $v(A) = T$, $v(C) = T$.

(b) (**0.5pt+0.5pt**) Let Σ and Σ' be sets of formulas such that $\Sigma' \subseteq \Sigma$. Prove or disprove. (Try to understand what is going on.)

(i) For every formula σ we have: If $\Sigma' \models \sigma$ then $\Sigma \models \sigma$.

Answer. This is TRUE. To see this, let v be a truth assignment such that $v(\tau) = T$ for every $\tau \in \Sigma$. Since $\Sigma' \subseteq \Sigma$, $v(\tau) = T$ for every $\tau \in \Sigma'$. Since $\Sigma' \models \sigma$, $v(\sigma) = T$.

(ii) For every formula σ we have: If $\Sigma \models \sigma$ then $\Sigma' \models \sigma$.

Answer. This is FALSE. We just need a counter-example. So fix a formula τ and let $\Sigma = \{\tau, \neg\tau\}$ and $\Sigma' = \{\tau\}$. Clearly $\Sigma' \subset \Sigma$. Now, $\Sigma \models \neg\tau$ (vacuously because there are no truth assignments that satisfies Σ) but $\Sigma' \not\models \neg\tau$ (this is because for any truth assignment v such that $v(\tau) = T$, $v(\neg\tau) = F$.

(c) (**2 pts**) Given a finite set $\Sigma = \{\phi_1, \phi_2, \ldots, \phi_n\}$ of formulas, write a recipe how to pick a subset $\Sigma' \subseteq \Sigma$ such that Σ' is independent and $\Sigma' \models \Sigma$.

Answer. We define this set inductively. Let $\Sigma_1 = \{\phi_1\}$. At step $k \geq 1$ and $k < n$, assume we already have defined Σ_k such that for $l < k$, $\Sigma_l \subseteq \Sigma_k$. Now consider ϕ_{k+1}: if $\Sigma_k \models \phi_{k+1}$, then let $\Sigma_{k+1} = \Sigma_k$; otherwise, let $\Sigma_{k+1} = \Sigma_k \cup \{\phi_{k+1}\}$. Finally, let $\Sigma' = \Sigma_n$. I'll leave it to you to check Σ' is independent.

(d) (**1pt**) Let $\Sigma' \subseteq \Sigma$ be sets of formulas such that $\Sigma' \models \phi$ for all $\phi \in \Sigma$. Prove or disprove: For every formula σ, if $\Sigma \models \sigma$ then $\Sigma' \models \sigma$.

Answer. This is a true statement. To see this, fix a formula σ such that $\Sigma \models \sigma$. Let v be a truth assignment such that $v(\tau) = T$ for all $\tau \in \Sigma'$. Since $\Sigma' \models \phi$ for all $\phi \in \Sigma$, $v(\phi) = T$ for all $\phi \in \Sigma$. This means $v(\tau) = T$ as desired.

2. (**5 pts**) Let Σ be a set of formulas.

(a) (**2 pts**) Assume Σ is satisfiable. Show that if ϕ is any formula then one of the sets $\Sigma \cup \{\phi\}, \Sigma \cup \{\neg\phi\}$ is satisfiable.

Answer. Let v witnesses Σ is satisfiable. Fix ϕ. Now if ϕ only uses sentence symbols in Σ, then either $v(\phi) = T$ or $v(\neg\phi) = T$. If some sentence symbols, say $A_1, \dots, A_n$ are used in ϕ but not used in Σ, then we can extend v to a v^* that assigns truth values to $A_1, \dots, A_n$. So $v^*(\tau) = v(\tau) = T$ for every $\tau \in \Sigma$ and either $v^*(\phi) = T$ or $v^*(\neg\phi) = T$. So either $\Sigma \cup \{\phi\}$ or $\Sigma \cup \{\neg\phi\}$ is satisfiable.

(b) (**1 pt**) Prove that Σ is not satisfiable if and only if $\Sigma \models A \wedge \neg A$.

Answer. If Σ is not satisfiable, i.e. there are no truth assignment v such that $v(\tau) = T$ for all $\tau \in \Sigma$, then vacuously $\Sigma \models A \wedge \neg A$.

Conversely, if $\Sigma \models A \wedge \neg A$, and suppose for contradiction that Σ is satisfiable, say by v. Then $v(A \wedge \neg A) = T$. This cannot happen. So Σ is not satisfiable.

(c) (**2 pts**) Assume Σ is such that for every formula ϕ,

$$(\star) \quad \text{If } \Sigma \models \phi \text{ then } \Delta \models \phi \text{ for some finite } \Delta \subseteq \Sigma.$$

Prove: If every finite $\Delta \subseteq \Sigma$ is satisfiable then Σ is satisfiable. Do not use the Compactness Theorem in your proof. This gives you a proof of the Compactness Theorem from the property $(\star)$. **Hint.** Use (b).

Answer. Suppose Σ is not satisfiable. By (b), $\Sigma \models A \wedge \neg A$. Then by $(\star)$, there is some finite $\Delta \subset \Sigma$ such that $\Delta \models A \wedge \neg A$. But this Δ is satisfiable by our hypothesis, so again by (b), $\Delta \not\models A \wedge \neg A$. We get a contradiction.

3.(**5 pts**) Consider the following situation. You have a collection of infinitely many cubes $C_0, C_1, C_2, C_3, \dots$. Some of them are connected with wire. Apply the Compactness Theorem to the following tasks.

(a) (**3 pts**) Assume you have two colors, and whenever you pick a finite subcollection of cubes, it is possible to color them using just those two colors so that no two cubes connected with a wire have the same color. Prove that the entire infinite collection can be colored in the same manner, that is, you use just those two colors and no two cubes connected with a wire get the same color.

Answer. As in the hint, we view cube C_i as the i-th sentence symbol. We view the colors as the truth values; so one can color can just be T and the other can just be F. The statement "cubes C_i, C_j have distinct colors when they are connected by a wire" can be expressed by the formula $\phi_{i,j}$ = "$(C_i \wedge \neg C_j) \vee (\neg C_i \wedge C_j)$". Note that $\phi_{i,j} = \phi_{j,i}$ for all $i \neq j$. Let $\Sigma = \{\phi_{i,j} \mid \text{cubes } C_i \text{ and } C_j \text{ are connected by a wire}\}$. Now by our hypothesis, every finite $\Delta \subset \Sigma$ is satisfiable, by Compactness Theorem, Σ is satisfiable. This is exactly the requirement of the problem.

(b) (**2 pts**) Do the analogous task as in (a), but now use four colors.

Answer. Now add sentence symbols D_i for each $i \in \mathbb{N}$ (these are distinct from the symbols C_i's already available to us from part (a)). Now we have 4 colors, say $0, 1, 2, 3$, but remember our truth assignments only map to T or F so we can't code colors by truth values as before. We want to do the following: for the i-th cube, we code the fact that "cube i is colored by color 0" by the value $C_i \wedge D_i$, "cube i is colored by color 1" by the value $C_i \wedge \neg D_i$, "cube i is colored by color 2" by the value $\neg C_i \wedge D_i$, "cube i is colored by color 3" by the value $\neg C_i \wedge \neg D_i$.

Then for $i, j \in \mathbb{N}$ for $k, l \in \{0, 1, 2, 3\}$, we can express the fact that "cubes C_i and C_j are connected by a wire and cube C_i is colored by color k and C_j is colored by color l" a following formula: $\phi_{i,j,k,l}$. For example, the formula $\phi_{2,5,1,2}$ is "$(C_2 \wedge \neg D_2) \wedge (\neg C_5 \wedge D_5)$".

Let $\Sigma = \{\phi_{i,j,k,l} \mid k \neq l \wedge i \neq j \wedge$
cubes C_i and C_j are connected by a wire and C_i has color k and C_j has color $l\}$.

By the exact same argument as above, using compactness, we get Σ is satisfiable.

Hint. (a): View C_i as sentential symbols, and colors $0, 1$ as truth evaluations. Write down formulas in those sentential symbols that express that two cubes connected with a wire get different colors. Subtle point: Do not express by a formula in your formal language that two cubes are connected with a wire. Instead, use the information about which are connected to write the correct set of formulas. (b): Add sentential letters D_i and code each color by evaluations of the pair C_i, D_i (note that there are four possible evaluations, which correspond to the number of colors).