Math 150 Homework 5 Solutions

The following is written by Alberto Takase 11/22/2017. I am reachable at atakase@uci.edu.

Problem 1

Let $\mathscr{L}$ be a first-order language. Let Σ be a set of $\mathscr{L}$-sentences. Let σ and τ be $\mathscr{L}$-sentences. Prove

(a) If $\Delta \subseteq \Sigma$ and $\Delta \vDash \sigma$, then $\Sigma \vDash \sigma$.

(b) $\Sigma \cup \{\tau\} \vDash \sigma$ if and only if $\Sigma \vDash \tau \to \sigma$.

(c) Σ is satisfiable if and only if there exists an $\mathscr{L}$-sentence ρ such that $\Sigma \nvDash \rho$.

Proof. We say a structure $\mathfrak{M}$ *believes* a sentence φ if $\mathfrak{M} \vDash \varphi$.

(a) Assume $\Delta \subseteq \Sigma$ and $\Delta \vDash \sigma$. To prove $\Sigma \vDash \sigma$, let $\mathfrak{M}$ be an arbitrary structure that believes every sentence in Σ. Because $\Delta \subseteq \Sigma$, $\mathfrak{M}$ believes every sentence in Δ. Because $\Delta \vDash \sigma$, $\mathfrak{M}$ believes σ. Therefore $\Sigma \vDash \sigma$.

(b) ($\Rightarrow$) Assume $\Sigma \cup \{\tau\} \vDash \sigma$. To prove $\Sigma \vDash \tau \to \sigma$, let $\mathfrak{M}$ be an arbitrary structure that believes every sentence in Σ. Because $\Sigma \cup \{\tau\} \vDash \sigma$ and because $\mathfrak{M}$ either believes τ or $\neg\tau$, $\mathfrak{M}$ believes $\tau \to \sigma$. Therefore $\Sigma \vDash \tau \to \sigma$.

($\Leftarrow$) Assume $\Sigma \vDash \tau \to \sigma$. To prove $\Sigma \cup \{\tau\} \vDash \sigma$, let $\mathfrak{M}$ be an arbitrary structure that believes every sentence in $\Sigma \cup \{\tau\}$. Because $\mathfrak{M}$ believes every sentence in Σ, $\mathfrak{M}$ believes $\tau \to \sigma$. Because $\mathfrak{M}$ believes τ, $\mathfrak{M}$ believes σ. Therefore $\Sigma \cup \{\tau\} \vDash \sigma$.

(c) ($\Rightarrow$) Assume Σ is satisfiable. Therefore there exists a structure $\mathfrak{M}$ such that $\mathfrak{M}$ believes every sentence in Σ. Define $\rho :\!= (\exists x)[x \neq x]$. Observe $\mathfrak{M}$ does not believe ρ. Therefore $\Sigma \nvDash \rho$.

($\Leftarrow$) Assume there exists an $\mathscr{L}$-sentence ρ such that $\Sigma \nvDash \rho$. Therefore there exists a structure $\mathfrak{M}$ such that $\mathfrak{M}$ believes every sentence in Σ and $\mathfrak{M}$ does not believe ρ. Therefore Σ is satisfiable. $\square$

Problem 2

Let $\mathscr{L} = \{P, Q\}$ be a first-order language with a 2-ary relation symbol P and a 1-ary relation symbol Q. Prove

(a) $\forall x Qx \vDash \exists x Qx$.

(b) $\exists x \forall y Pxy \vDash \forall y \exists x Pxy$.

(c) $\forall y \exists x Pxy \nvDash \exists x \forall y Pxy$.

(d) $\varnothing \vDash \exists x(Qx \to \forall y Qy)$.

Proof. We say a structure $\mathfrak{M}$ *believes* a sentence φ if $\mathfrak{M} \vDash \varphi$.

(a) Let $\mathfrak{M}$ be a structure that believes $\forall x Qx$. Fix m in the domain of $\mathfrak{M}$. Therefore $\mathfrak{M}$ believes $Qx[x \mapsto m]$. Therefore $\mathfrak{M}$ believes $\exists x Qx$. Therefore $\forall x Qx \vDash \exists x Qx$.

(b) Let $\mathfrak{M}$ be a structure that believes $\exists x \forall y Pxy$. Fix m in the domain of $\mathfrak{M}$ such that $\mathfrak{M}$ believes $\forall y Pxy[x \mapsto m]$. Therefore $\mathfrak{M}$ believes $\forall y \exists x Pxy$. Therefore $\exists x \forall y Pxy \vDash \forall y \exists x Pxy$.

(c) Let $\mathfrak{M} = \left(M, P^{\mathfrak{M}}, Q^{\mathfrak{M}}\right)$ be the structure defined by

$$\begin{aligned} M &= \{x : x \text{ is a human}\} \\ P^{\mathfrak{M}} &= \{(x, y) : x \text{ is the biological parent of } y\} \\ Q^{\mathfrak{M}} &= \varnothing. \end{aligned}$$

Therefore $\mathfrak{M}$ believes $\forall y \exists x Pxy$ and $\mathfrak{M}$ does not believe $\exists x \forall y Pxy$. Therefore $\forall y \exists x Pxy \nvDash \exists x \forall y Pxy$.

(d) Let $\mathfrak{M}$ be a structure that believes every sentence in $\varnothing$. We write $\varphi \equiv \psi$ instead of $\varphi \vDash\dashv \psi$.

$$\begin{aligned} \exists x(Qx \to \forall y Qy) &\equiv \exists x(\neg Qx \vee \forall y Qy) \\ &\equiv \exists x \neg Qx \vee \forall y Qy \\ &\equiv \neg \forall x Qx \vee \forall y Qy \\ &\equiv \top \end{aligned}$$

$\therefore$ $\mathfrak{M}$ believes $\exists x(Qx \to \forall y Qy)$. Therefore $\varnothing \vDash \exists x(Qx \to \forall y Qy)$. □

Problem 3

Let $\mathscr{L} = \{\dot{+}, \dot{\times}, \dot{<}, \dot{0}, \dot{1}\}$ be the first-order language of ordered rings. Let $\mathfrak{M} = (\mathbb{R}, +, \times, <, 0, 1)$ be the standard $\mathscr{L}$-structure. Translate the following statements in $\mathscr{L}$, and check whether $\mathfrak{M}$ satisfies the formula with respect to the variable assignment $s : V \to \mathbb{R} : x_{2n+1} \mapsto \sqrt{2n}$:

(a) "there is no largest negative number."

(b) "x_3 is in the interval $[\frac{1}{2}, \frac{2}{3})$."

Define the following sets in $\mathfrak{M}$:

(i) $\{(a, b) : a \geq b \text{ and } b < -1\}$.

(ii) $\left\{x : x \text{ is a square and } x < \frac{1}{2}\right\}$.

Solution.

(a) $\varphi \coloneqq (\forall x)[x < 0 \to (\exists y)[y < 0 \wedge x < y]]$ and $\mathfrak{M} \models \varphi$.

(b) $\varphi \coloneqq (\exists x_1)(\exists x_2)[2x_1 = 1 \wedge 3x_2 = 2 \wedge (x_1 = x_3 \vee x_1 < x_3) \wedge x_3 < x_2]$ and $2 \coloneqq 1 + 1$ and $3 \coloneqq 1 + 1 + 1$ and $\mathfrak{M} \not\models \varphi(x_3)[s]$.

(i) $\{(a, b) : a \geq b \text{ and } b < -1\} = \{(a, b) : \mathfrak{M} \models (b = a \vee b < a) \wedge b + 1 < 0\}$.

(ii) $\left\{x : x \text{ is a square and } x < \frac{1}{2}\right\} = \{x : \mathfrak{M} \models (\exists y)[x = y \cdot y] \wedge (1 + 1)x < 1\}$. $\square$