MATH 150 HOMEWORK 6 SOLUTION FOR PROBLEM 2

Problem 1.

(b) "G consists of disjoint cycles."

Answer. We start the same way Alberto's solution starts. We have the formula $\phi_n(x) :=$ "x is in a cycle of length n and the cycle is disjoint from the rest of the nodes" for $n \geq 3$ (this simply is because every cycle, by definition, has at least 3 elements).

Now, the property "G consists of disjoint cycles" should be interpreted more specifically; as it is, we cannot come up with a Σ such that for every graph G, $G \vDash \Sigma$ iff G consists of disjoint cycles. (The Σ in Alberto's solution doesn't work).

We interpret as question as follows: for each $N \geq 3$, we have the property "G consists of disjoint cycles of length at most N". For each such N, we simply let $\Sigma_N = \{\Phi_N\}$, where Φ_N is as in Alberto's solution: $\Phi_N := \forall x \ \bigvee_{i=3}^{N} \phi_n(x)$ (this just says "the graph consists of disjoint cycles of length at most N").

(c) "Any two nodes in G have the same degree."

Answer. Again, we have to interpret the property in question as: for each $N \geq 1$, "any two nodes in G have the same degree N". So for each value of N we have a Σ_N such that for all G, $G \vDash \Sigma_N$ iff "any two nodes in G have the same degree N".

Let, for $n \geq 1$, $\varphi_n(x, y)$ be as in Alberto's notes. $\varphi_n(x, y)$ just says "x, y both have degree n".

Now fix an $N \geq 1$, and let $\Phi_N := \forall x \forall y \varphi_N(x, y)$. Φ_N just says "every element in the graph G has degree N". So our $\Sigma_N = \{\Phi_N\}$.

Problem 2. (**6pts**) Let $\mathcal{L} = \{\dot{E}\}$ be the language of equivalence relations.
(**4pts**) Express each of the following statements as a formula/sentence in $\mathcal{L}$.

(a) (**1pt**) "There are infinitely many equivalence classes of size 2"
Solution: For each $n \in \mathbb{N}$, we have a sentence ϕ_n in $\mathcal{L}$ that expresses "there are at least n equivalence classes of size 2". ϕ_n is

$$\exists x_1 \exists x_2 \ldots \exists x_{2n-1} \exists x_{2n} (x_1 \dot{E} x_2 \wedge x_3 \dot{E} x_4 \cdots \wedge x_{2n-1} \dot{E} \wedge x_{2n} \wedge (\neg x_1 \dot{E} x_3) \wedge \cdots \wedge \\ \neg(x_1 \dot{E} x_{2n-1}) \neg(x_3 \dot{E} x_5) \wedge \cdots \wedge \neg(x_3 \dot{E} x_{2n-1}) \wedge \cdots \wedge \neg(x_{2n-3} \dot{E} x_{2n-1})).$$

Now let $\Sigma = \{\phi_n : n \in \mathbb{N}\}$. Then Σ expresses the statement "there are infinitely many equivalence classes of size 2". This is because any model $\mathcal{M} \vDash \Sigma$ has infinitely many equivalence classes of size 2.

(b) (**1pt**) "There are arbitrarily large finite equivalence classes"
Solution: We do the same thing as before. Let $\Sigma = \{\phi_n : n \in \mathbb{N}\}$; but now for each n, there is a $k \geq n$ such that the formula ϕ_n says "there is an equivalence class of size k". So ϕ_n is

$$\exists x_1 \exists x_2 \ldots \exists x_k (x_1 \dot{E} x_2 \wedge x_2 \dot{E} x_3 \ldots x_{k-1} \dot{E} x_k \wedge \forall y (\neg (x_1 \dot{E} y))$$

(c) (**1pt**) "All finite equivalence classes are of even size"
Solution: There are a couple of ways to interpret this question. I will deal with each case. (In general, if you have a problem that is a bit open to interpretation; you should state your interpretation and solve the problem according to that interpretation.

Interpretation 1: There is a fixed even number N such that every equivalence relation has size N.

For each $n \geq 1$ you just let ϕ_n say "there is an equivalence class of size exactly n". This is similar to part (b) (where we replace the k in the formula ϕ_n by n). Let $\Sigma = \{\phi_N\} \cup \{\neg \phi_n : n \neq N\}$. Then Σ expresses the idea that "each equivalence class has even size N". This is because if $\mathfrak{M} \vDash \Sigma$, then $\mathfrak{M} \vDash \phi_N$ so there is an equivalence class in $\mathfrak{M}$ that has size N. On the other hand, if $n \neq N$, then $\mathfrak{M} \vDash \neg \phi_n$; so there is no equivalence class in $\mathfrak{M}$ of size n. In other words, all equivalence classes have size N.

Interpretation 2: There is a subset $A \subseteq \{2n : n \in \mathbb{N}, n \geq 1\}$ such that every equivalence relation has size m for some $m \in A$ (m is even because $A \subseteq \{2n : n \in \mathbb{N}, n \geq 1\}$).

For this interpretation, for each $n \geq 1$ you just let ϕ_n say "there is an equivalence class of size exactly n". This is similar to part (b) (where we replace the k in the formula ϕ_n by n). Now let $\Sigma = \{\phi_n : n \in A\} \cup \{\neg \phi_n : n \notin A\}$. Then Σ expresses the idea that "each equivalence class has even size n and $n \in A$". This is because if $\mathfrak{M} \vDash \Sigma$, then for each $n \in A$, $\mathfrak{M} \vDash \phi_n$ so there is an equivalence class in $\mathfrak{M}$ that has size n. On the other hand, if $n \notin A$, then $\mathfrak{M} \vDash \neg \phi_n$; so there is no equivalence class in $\mathfrak{M}$ of size n.

Remark 0.1. *Technically, interpretation 1 is a special case of interpretation 2, where you just let $A = \{N\}$.*

(d) (**1pt**) "All equivalence classes are infinite".
Solution: First, let ϕ_n say "there is an equivalence class of size n". This is similar to part (b) (where we replace the k in the formula ϕ_n by n). Now let $\Sigma = \{\neg \phi_n : n \geq 1\}$. Then Σ expresses the idea that "all equivalence classes are infinite" because for any model $\mathcal{M} \vDash \Sigma$, for each n, $\mathcal{M} \vDash \neg \phi_n$ means that $\mathcal{M}$ has no equivalence class of size n.

(**2pts**) Prove that there is no set Σ (possibly infinite) of $\mathcal{L}$-sentences such that for every equivalence relation structure $\mathcal{A} = (A, E)$ the following holds:

$$\mathcal{A} \vDash \Sigma \text{ if and only if } \mathcal{A} \text{ has finitely many equivalence classes of size 1.}$$

Solution: Suppose such a Σ exists. For each n, let ϕ_n say "there are at least n equivalence classes of size 1". The formula ϕ_n is similar to part (a). Now we use compactness to check that $\Delta = \Sigma \cup \{\phi_n : n \geq 1\}$ is satisfiable. So let $\Delta_0 \subset \Delta$ be finite. Let N be the largest such that $\phi_N \in \Delta_0$ (note that there may not be such an N, in which case $\Delta_0 \subseteq \Sigma$ and it is satisfiable by our assumption; so we may as well assume N exists). Then there is a model $\mathcal{A}$ such that $\mathcal{A}$ has at least N equivalence classes of size 1 and $\mathcal{A}$ only has finitely equivalence classes of size 1 (this follows from our assumption on Σ). So $\mathcal{A} \vDash \Delta_0$.