$AD^+$, Derived Models, and $\Sigma_1$-Reflection

John Steel, Nam Trang

Let $AD^+$ be the theory $AD + DC_{\omega} + \text{"Every set of reals is } \infty\text{-Borel"} + \text{"Ordinal Determinacy"}$. For any $\Gamma \subseteq P(\mathbb{R})$, let $M_\Gamma = \cup \{ m \mid m \text{ is transitive and } \exists E, F \subseteq \mathbb{R} \times \mathbb{R} \ (E, F \in \Gamma \text{ and } (\mathbb{R}/E, F) \cong (m, \in)) \}$. We’ll prove the following theorems:

**Theorem 1.** (Woodin) Assume $ZF + AD + V = L(P(\mathbb{R}))$. Then the following are equivalent:

1. $AD^+$
2. Letting $S = \{ B \subseteq \mathbb{R} \mid B \text{ is Suslin co-Suslin} \}$, $M_S \prec \Sigma_1 V$.

Let us call the statement in (2) above “$\Sigma_1$-reflection” to Suslin co-Suslin.

**Theorem 2.** (Woodin) Assume $ZF + AD^+ + V = L(P(\mathbb{R}))$, then

1. $\Sigma_2^1$ has the scale property.
2. $M_{\Delta^2_1} \prec \Sigma_1 V$.

**Proof.** The theorem follows immediately from Theorem 1 and lemma 7.2 in [?], whose proof is essentially due to Woodin.

In the course of proving Theorem ??, we shall prove part of the determinacy-to-large-cardinals direction of the Derived Model Theorem. Let $\lambda$ be a limit of Woodin cardinals, and $G$ be $V$-generic over $Col(\omega, < \lambda)$. We set

$$R^*_G = \cup_{\alpha < \lambda} V[G|\alpha],$$

$$\text{Hom}^*_G = \{ p[T] \cap R^*_G \mid \exists \alpha < \lambda (T \in V[G|\alpha], V[G|\alpha] \models T \text{ is } \lambda\text{-absolutely complemented}) \},$$

$$\mathcal{A}_G = \{ A \subseteq R^*_G \mid A \in V(R^*_G) \text{ and } L(A, R^*_G) \models AD^+ \}, \text{where } V(R^*_G) = \text{HOD}^{V[G]}_{\text{VU}R^*_G \cup \{ R^*_G \}}.$$

**Theorem 3.** (Woodin) Assume $ZF + AD^+ + V = L(P(\mathbb{R}))$. Suppose also that if $AD_R$ holds, then $\Theta$ is singular. Then there is a set $X$ in some generic extension of $V$ such that setting $M = L[X]$, then

1. for some $\lambda$, $M \models ZFC + \lambda$ is a limit of Woodins;
2. for some $M$-generic $G$ over $Col(\omega, < \lambda)$:
   
   - $V = L(A_G, R^*_G)$, and
• $\text{Hom}_G^* = \{ B \subseteq \mathbb{R}_G^* \mid B \text{ is Suslin co-Suslin in } V \}$.

**Remark 4.**

- The model $L(\mathcal{A}_G, \mathbb{R}_G^*)$ as in 2 of the previous theorem is called the “new” derived model to distinguish it from the “old” derived model which is $L(\text{Hom}_G^*, \mathbb{R}_G^*)$.

- [?] shows that if $V \models AD^+ + \text{"there is a largest Suslin cardinal"}$, then we have the same conclusions as those of Theorem ?? . What we handle here is the case that $AD_{\mathbb{R}}^+ \text{"\(\Theta\) is singular" holds in } V$.

- Characterization of derived models is one of the main themes in this paper. We want to answer the question: Is every model of $AD^+$ a derived model? Theorem ? and the previous remark answer this question positively for the “no largest Suslin cardinal + \(\Theta\) singular” and the “largest Suslin cardinal” cases. Woodin has shown that if $V \models AD_{\mathbb{R}}^+ \text{ + \(\Theta\) is regular, then V is elementarily embeddable into a derived model of HOD}$. A proof of this fact can be found in [?]. It’s not known whether $V$ is actually a derived model in this case.

The proof of Theorem 3 is implicit in that of the direction (1) $\Rightarrow$ (2) of theorem 1. Before giving the proof of theorem 1, we’ll state a couple of corollaries of the above theorems, and a key definition.

**Corollary 5.** Let $M \models ZFC + \lambda$ is a limit of Woodins, and let $D$ be a derived model of $M$ below $\lambda$; then $D$ satisfies: $\Sigma_1$-reflection (to Suslin co-Suslin), $\Sigma_2$ has the scale property, and every non-empty $\Sigma_1$ set $A \subseteq P(\mathbb{R})$ has a $\Delta_2^1$ member.

**Proof.** Woodin has shown that $D \models AD^+$ (see [?] for a proof). Applying theorems 1 and 2 gives us the conclusions.

**Corollary 6.** Assume $AD^+$. Then $\text{Ult}(V, \mu)$ is well-founded where $\mu$ is the Martin measure on Turing degrees.

**Proof.** If not, then by Theorem 1, there is $\alpha, \beta < \Theta$ such that $L_\alpha(\mathcal{P}_\beta(\mathbb{R})) \models \text{"Ult}(V, \mu)$ is ill-founded." Since $DC_{\mathbb{R}}$ holds and there is a surjection from $\mathbb{R}$ onto $L_\alpha(\mathcal{P}_\beta(\mathbb{R}))$, $L_\alpha(\mathcal{P}_\beta(\mathbb{R})) \models DC$ and this is a contradiction.

**Definition 7.** $(ZF + AD + DC_{\mathbb{R}})$ Suppose $X$ is a set. The **Solovay sequence** defined relative to $X$ is the sequence $\langle \Theta^X_\alpha : \alpha \leq \Upsilon_X \rangle$ where

1. $\Theta^X_0$ is the supremum of the ordinals $\xi$ such that there is a surjection $\phi : \mathbb{R} \rightarrow \xi$ such that $\phi$ is OD from $X$.
2. $\Theta^X_\alpha = \sup\{ \Theta^X_\beta \mid \beta < \alpha \}$ if $\alpha > 0$ is limit.
3. If $\Theta^X_\alpha < \Theta$ then $\Theta^X_{\alpha+1}$ is the supremum of the ordinals $\xi$ such that there is a surjection $\phi : \mathbb{R} \rightarrow \xi$ such that $\phi$ is OD($X, A$) where $A$ is a set of reals of Wadge rank $\Theta^X_\alpha$.

**Remark 8.** Suppose $AD^+$ holds. Let $\Theta^X_\alpha < \Theta$ be a member of the Solovay sequence and $A$ be a set of reals with Wadge rank $\Theta^X_\alpha$. Let $\kappa = \sup\{ \delta^A_n(\kappa) \mid n < \omega \}$. Clearly $\kappa < \Theta^X_{\alpha+1}$. It’s an $AD^+$ theorem that any $B$ with Wadge rank $\Theta^X_\alpha$ has an $\infty$-Borel code $C_B \subseteq \kappa$. Let $\xi < \Theta^X_{\alpha+1}$. We can define an $OD_X$ surjection $\pi : P(\mathbb{R}) \rightarrow \xi$ as follows. Given $C \subseteq \kappa$, if $C$ codes a tuple $\langle C_B, x, y \rangle$ where $x, y \in \mathbb{R}$, $C_B$ is an $\infty$ – Borel code for a set $B$ of Wadge
rank $\Theta^X_\alpha$, and if there is a pre-wellordering of the reals of order type $\xi$ that is $OD_X(B,x)$, then we let $\pi(C) = \pi_B(y)$ where $\pi_B : \mathbb{R} \to \xi$ is the surjection associated with the least such pre-wellordering; otherwise, $\pi(C) = 0$. So in fact, under $AD^+$, $\Theta^X_{\alpha+1}$ is the supremum of ordinals $\xi$ such that there is an $OD_X$ surjection from $P(\kappa)$ onto $\xi$.

**Remark 9.** It’s worth pointing out that the Solovay sequence defined in Definition ?? is “globally defined” i.e. defined in $V$. On the other hand, one can define the notion of “locally defined” Solovay sequences, i.e. Solovay sequences defined in some $L(A,\mathbb{R})$, for $A \subseteq \mathbb{R}$. If $\Theta_{\alpha+1} < \Theta^{L(A,\mathbb{R})}$ then $\Theta_{\alpha+1}$ is a member of the “locally defined” Solovay sequence in $L(A,\mathbb{R})$. $\Theta_{\alpha+1}$ cannot be a limit of Suslin cardinals in $L(A,\mathbb{R})$ as otherwise, it would have an $OD^V(A)$ uniformization. Thus $\Theta_{\alpha+1} = (\Theta^{L(A,\mathbb{R})}_{\gamma+1})$, for some $\gamma$. Another key point is the following. Suppose $A \subseteq \Theta_{\alpha+1}$ is $OD^V(B)$ for some $B \subseteq \mathbb{R}$ such that $w(B) < \Theta_{\alpha+1}$. Let $\mathcal{C} = \langle C_\beta \mid \beta < \Theta_{\alpha+1} \rangle$, where $\mathcal{C}$ is an $OD^V(D)$ sequence such that each $C_\beta$ is a pre-wellordering of $\mathbb{R}$ of length $\beta$, where $w(D) = \Theta_\alpha$. Then $\Theta_{\alpha+1}$ is regular in $L(\mathbb{R})[A,C]$. This is important because it makes the Woodins’ techniques for constructing measures under $AD$ described in [?] relevant. We state here a theorem which will be used heavily.

**Theorem 10.** (Woodin, see Theorem 5.6 of [?]) Assume $ZF + DC + AD$. Suppose $X$ and $Y$ are sets and let $\Theta_{X,Y} = \sup \{ \alpha \mid \text{there is an } OD_{X,Y} \text{ surjection } \pi : \mathbb{R} \to \alpha \}$.

Then

$$HOD_X \models ZFC + \Theta_{X,Y} \text{ is a Woodin cardinal.}$$

**Proof of Theorem 1:**

We deal with the easy direction $(2) \Rightarrow (1)$ first. Suppose there is a set of reals in $V$ that has no $\omega$-Borel codes. One can show that $A$ has an $\omega$-Borel code if and only if $A$ has an $\omega$-Borel code which is coded by a set of reals projective in $A$. So our supposition is $\Sigma^2_2$. By (2), there is a Suslin co-Suslin set $B$ that has no $\omega$-Borel codes; but this is absurd since any tree $T$ such that $p[T] = B$ is an $\omega$-Borel code of $B$.

For Ordinal Determinacy, again suppose there is a set $B$ in $V$ such that Ordinal Determinacy fails for $B$. The ordinal game associated to $B$ and pre-wellordering $\leq$ of $\mathbb{R}$ has a winning strategy if and only if it has a winning strategy projective in $\leq$, by the Coding Lemma. So our supposition is $\Sigma^2_1$. By (2), there is a Suslin co-Suslin set $B$ such that Ordinal Determinacy fails for $B$. This contradicts a theorem of Moschovakis and Woodin which states that Ordinal Determinacy holds for any Suslin co-Suslin set.

Finally, to see $DC_{\mathbb{R}}$ holds. Suppose not. Again, by our hypothesis, there is a Suslin co-Suslin relation $E \subseteq \mathbb{R} \times \mathbb{R}$ witnessing the failure of $DC_{\mathbb{R}}$. However, we can uniformize $E$ using the scale associated with a tree $T$ such that $p[T] = E$. This gives us an infinite $E$-chain, which is a contradiction. This completes the proof of $(2) \Rightarrow (1)$.

**Remark 11.** Our proof used that $\Sigma^2_1$ reflects to Suslin co-Suslin, rather than the full $\Sigma^1_1$-reflection in (2). Derived models satisfy $\Sigma^1_1$-reflection, hence they satisfy $AD^+$; see [?] and [?].
The rest of the paper is dedicated to the proof of (1) \( \Rightarrow \) (2). First, assume there is a largest Suslin cardinal. This is the easier case.

**Lemma 12.** If \( \Theta \) is regular and \( V = L(P(\mathbb{R})) \models \phi[x] \) where \( x \in \mathbb{R} \) and \( \phi \) is \( \Sigma_1 \), then there is a transitive \( M \) such that \( M \) is a surjective image of \( \mathbb{R} \) and \( (M, \in) \models \phi[x] \).

*Proof.* By relativization, \( L_\alpha(P(\mathbb{R})) \models \phi[x] \) for some ordinal \( \alpha \). We’ll form a Skolem hull \( H \) of \( L_\alpha(P(\mathbb{R})) \). First, fix a surjection \( h : \alpha \times P(\mathbb{R}) \to L_\alpha(P(\mathbb{R})) \). Let \( H_0 = \mathbb{R} \). Suppose we already have \( H_n \) and a surjection \( \pi_n : \mathbb{R} \to H_n \). To build \( H_{n+1} \), for any \( a \in H_n \) and any formula \( \varphi \) such that \( L_\alpha(P(\mathbb{R})) \models \exists y \varphi[y, a] \), pick the least \( \beta \) such that there is an \( A \subseteq \mathbb{R} \) such that \( L_\alpha(P(\mathbb{R})) \models \varphi[h(\beta, A), a] \). Then let \( \gamma \) be the least such that there is an \( A \subseteq \mathbb{R} \) such that \( w(A) = \gamma \) and \( L_\alpha(P(\mathbb{R})) \models \varphi[h(\beta, A), a] \). Denote the \( (\beta, \gamma) \) above \( (\beta_0, \gamma_0) \). Now, let \( H_{n+1} = H_n \cup \{ h(\beta_0, A) \mid a \in H_n, w(A) = \gamma_0 \} \). By regularity of \( \Theta \) and the fact that \( \pi_n : \mathbb{R} \to H_n \) is surjective, \( \sup \{ \gamma_n \mid a \in H_n \} \in \Theta \). Hence, there is a surjection \( \pi_{n+1} : \mathbb{R} \to H_{n+1} \). Finally, let \( H = \cup_n H_n \). Hence \( H < L_\alpha(P(\mathbb{R})) \) by construction. Since \( \Theta \) is regular, \( \mathbb{R} \subseteq H \), and \( V = L(P(\mathbb{R})) \), it is easy to see that \( H \) collapses to some \( L_\alpha(P_\gamma(\mathbb{R})) \) for some \( \delta, \gamma < \Theta \). Since \( L_\delta(P_\gamma(\mathbb{R})) \models \phi[x] \), \( L_\delta(P_\gamma(\mathbb{R})) \) is the desired \( M \). \( \Box \)

**Lemma 13.** Suppose there is a largest Suslin cardinal, then \( \Theta \) is regular.

*Proof.* Let \( \kappa \) be the largest Suslin cardinal and \( T \) be a tree on \( \omega^2 \times \kappa \) such that \( p[T] \) is a universal \( \Gamma \)-set (where \( \Gamma \) is the boldface pointclass of \( \kappa \)-Suslin sets of reals).

For each \( A \subseteq \mathbb{R} \), we have \( L(T, A, \mathbb{R}) \models DC \) because \( \mathbb{V} \models DC_{\mathbb{R}} \). Let \( T_A \) be the image of \( T \) under the Martin measure ultrapower map where the ultrapower is computed with respect to functions in \( L(T, A, \mathbb{R}) \). Because \( L(T, A, \mathbb{R}) \models DC \), \( Ult(L(T, A, \mathbb{R}), \mu_T) \) is wellfounded. By relativizing the proof that \( P(\mathbb{R}) \subseteq L(T^*, \mathbb{R}) \) to the universe \( L(T, A, \mathbb{R}) \) (see [?]), we get that \( A \in L(T_A, \mathbb{R}) \). Notice that \( T_A \) only depends on \( w(A) \) but not \( A \) itself. So we in fact have an enumeration \( \langle T_\alpha \mid \alpha < \Theta \rangle \) where for each \( \alpha < \Theta \), \( T_\alpha = T_A \) for any \( A \) with Wadge rank \( \alpha \). Now let \( \gamma = sup \{ sup T_\alpha \mid \alpha < \Theta \} \) and \( C \subseteq \Theta \times \gamma \) is such that \( (\alpha, \beta) \in C \Leftrightarrow \beta < T_\alpha \). Then \( T_A \in L[C] \) for any \( A \subseteq \mathbb{R} \). So \( P(\mathbb{R}) \subseteq L(C, \mathbb{R}) \). So \( V = L(C, \mathbb{R}) \). The following claim supplies an important step toward proving \( \Theta \) is regular.

**Claim 14.** \( \Theta \) is regular if and only if \( Collection \) holds, where \( Collection \) is the following statement:

\[
(\forall x \in \mathbb{R}) (\exists A \subseteq \mathbb{R}) (x, A) \in U \to (\exists B \subseteq \mathbb{R}) (\forall x \in \mathbb{R}) (\exists y \in \mathbb{R}) (x, B_{x,y}) \in U, \text{ where } B_{x,y} = \{z \mid (x, y, z) \in B\}.
\]

*Proof.* \( (\Leftarrow) \) Suppose \( \Theta \) is singular. Let \( f : \mathbb{R} \to \Theta \) be cofinal. So \( (\forall x \in \mathbb{R}) (\exists A \subseteq \mathbb{R}) (A \text{ is a pre-wellordering of } \mathbb{R} \text{ of length } f(x)) \). By \( Collection \), \( (\exists B \subseteq \mathbb{R}) (\forall x \in \mathbb{R}) (\exists y \in \mathbb{R}) (B_{x,y} \text{ is a pre-wellordering of } \mathbb{R} \text{ of length } f(x)) \). Define \( g : \mathbb{R} \to \Theta \) as follows: for any \( x \in \mathbb{R} \), if \( x = (x_0, x_1, x_2) \) and \( B_{x_0,x_1} \) is a pre-wellordering of \( \mathbb{R} \) of order type \( f(x_0) \), then let \( g(x) = \text{rank of } x_2 \text{ in the pre-wellordering } B_{x_0,x_1} \); otherwise, let \( g(x) = 0 \). Clearly, \( g \) is onto. This is a contradiction.

\( (\Rightarrow) \) Suppose \( \Theta \) is regular. Let \( U \) be as in the hypothesis of \( Collection \). For \( x \in \mathbb{R} \), let \( f(x) \) be the least \( \xi \) such that there is \( A \subseteq \mathbb{R} \) with Wadge rank \( \xi \) and \( (x, A) \in U \). Since \( \Theta \) is regular, \( f \) is bounded in \( \Theta \). Fix an \( \alpha < \Theta \) such that \( \alpha \geq sup(rng(f)) \). Let \( B = \{(x, y, B_{x,y}) \mid x, y \in \mathbb{R}, y \text{ Wadge reduces } B_{x,y} \text{ to } A\} \). Clearly, \( B \) satisfies the conclusion of \( Collection \). \( \Box \)
By claim ??, it suffices to prove that $L(C, \mathbb{R}) \models Collection$. So let $U$ be as in the hypothesis of Collection. Let $B = \{(x, y, B_{x,y}) \mid B_{x,y}$ is the least $OD_C(y)$ set such that $(x, B_{x,y}) \in U\}$. This $B$ clearly works because every set of reals in $V = L(C, \mathbb{R})$ is $OD_C(y)$ for some real $y$.

The proof of Lemma ?? also implies that DC holds, hence the Martin measure ultrapower is well-founded. This fact is used to show that there is an $M$ in a generic extension of $V$ such that $V$ is a derived model of $M$. See [?] for a proof of this. The conclusion 2 of Theorem ?? then follows from Lemma ?? above and Lemma 7 of [?].

Now, we’re on to the “no largest Suslin cardinal” case. So we have $AD_\mathbb{R}$. First, assume $\Theta$ is regular. By Lemma ??, $M_\mathbb{P}(\mathbb{R}) \prec \Sigma_1 V$. Since all sets of reals are Suslin co-Suslin, we’re done.

From now on, we may assume that $\Theta$ is singular. We have that every set of reals is Suslin co-Suslin. Our strategy is to Prikry-force a universe $M$ such that $V$ is a derived model of $M$. This guarantees that $\Sigma_2$-reflection holds in $V$, but with a little more argument, we’ll be able to show $\Sigma_1$-reflection holds in $V$. Most of what we are doing here, then, is proving Theorem ?? in the case $AD_\mathbb{R} + \Theta$ is singular.

Case 1: $\text{cof}(\Theta) = \omega$.

Let $\langle \Theta_\alpha \mid \alpha < \gamma \rangle$ be the Solovay sequence of $V$. Notice that $\text{cof}(\gamma) = \omega$. Hence, there is a sequence $\langle \alpha_i \mid i < \omega \rangle$ cofinal in $\gamma$. We can and do take the sequence $\langle \alpha_i \mid i < \omega \rangle$ to be definable from a set of reals and from no ordinal parameters. The hypothesis implies that every set of reals is Suslin, so given an $\alpha < \gamma$, let $\kappa$ be the largest Suslin cardinal below $\Theta_{\alpha+1}$. Set $HOD_{P(\kappa)} = \{A \mid \forall C \in TC(A \cup \{A\}) C \text{ is } OD \text{ from some } B \in P(\kappa)\}$, then the following hold:

$(1.1)$ $\Theta_{\alpha+1}$ is the supremum of the ordinals $\xi$ for which there is a surjection $\phi : P(\kappa) \to \xi$ such that $\phi$ is OD.

$(1.2)$ $\Theta_{\alpha+1} = \Theta^{HOD_{P(\kappa)}}$.

$(1.3)$ $HOD_{P(\kappa)} = HOD_X$, where $X = \{B \subseteq \mathbb{R} \mid w(B) < \Theta_{\alpha+1}\}$.

$(1.1)$ follows from Remark 8. Both $(1.2)$ and $(1.3)$ are immediate consequences of $(1.1)$. By $(1.3)$, every bounded subset of $\Theta_{\alpha+1}$ belongs to $HOD_{P(\kappa)}$. Now, for each $i < \omega$, let $\kappa_i$ be the largest Suslin cardinal below $\Theta_{\alpha_i+1}$ and $\mu_i$ be the supercompact (nonprincipal, fine, and normal) measure on $P_{\omega_1}(P(\kappa_i))$. Notice here that by $AD_\mathbb{R}$, Solovay’s super-compactness measure on $P_{\omega_1}(\mathbb{R})$ exists and is unique. Since $P(\kappa_i)$ is the surjective image of $\mathbb{R}$, $\mu_i$ exists and is unique. Because it is unique, $\mu_i$ is OD. Also, let $X_i$ be the set of all $\sigma \in P_{\omega_1}(P(\kappa_i))$ such that

$(2.1)$ $HOD_{\sigma \cup \{\sigma\}} \models AD^+$

$(2.2)$ $HOD_{\sigma \cup \{\sigma\}} \not\models AD_\mathbb{R}$

$(2.3)$ the transitive collapse of $\sigma$ is $P(\kappa_{\sigma}) \cap HOD_{\sigma \cup \{\sigma\}}$ where $\kappa_{\sigma}^\sigma$ is the largest Suslin
Lemma 15. $\mu_i(X_i) = 1$

Proof. Let

$$\prod_{\sigma} HOD_{\sigma \cup \{\sigma\}} / \mu_i = M,$$

where the ultraproduct is formed in the universe $HOD_{P(\kappa_i)}$. The reason we do this is that we do not have DC in $V$, and thus the ultraproduct formed in $V$ might be illfounded. On the other hand, $HOD_{P(\kappa_i)} \models DC$, so $M$ is well-founded, and we take it to be transitive. Let $\sigma^\infty$ be the element of $M$ represented by the identity function. By Los, for all formulas $\phi$,

$$M \models \phi[\sigma^\infty] \iff \mu_i(\{\sigma \in P_{\omega_1}(P(\kappa_i)) \mid HOD_{\sigma \cup \{\sigma\}} \models \phi[\sigma]\}) = 1.$$

We should remark here that even though we don’t have AC, Los theorem still goes through because of normality (closure under diagonal intersections) of $\mu_i$. The following claim will complete the proof of the lemma.

Claim 16. The following hold:

1. The transitive collapse of $\sigma^\infty$ is $P(\kappa_i)$.
2. $\mathbb{R} \cap M = \mathbb{R}$.
3. $P(\mathbb{R}) \cap M = \{B \mid w(B) < \Theta_{\alpha_i + 1}\} = P(\mathbb{R}) \cap HOD_{P(\kappa_i)}$.

Proof. (1) and (2) are easy consequences of normality, so we leave them to the reader. To prove (3), suppose first that $w(B) < \Theta_{\alpha_i + 1}$. So $B \in HOD_{P(\kappa_i)}$. Let $f(\sigma) = B \cap \sigma$ for $\sigma \in P_{\omega_1}(P(\kappa_i))$. Then $f \in HOD_{P(\kappa_i)}$ and $[f]_{\mu_i} = B$. On the other hand, $M \subseteq HOD_{P(\kappa_i)}$ as the ultraproduct is formed in $HOD_{P(\kappa_i)}$.

Let

$$T_0 = \{\langle \sigma_0, \ldots, \sigma_n \rangle \mid \sigma_i \in P_{\omega_1}(P(\kappa_i)) \text{ for all } i\}.$$

Let $T$ be the set of all $s = \langle \sigma_0, \ldots, \sigma_n \rangle \in T_0$ such that for all $i \leq n$

1. $P(\mathbb{R})^{HOD(s)} = P(\mathbb{R})^{HOD}$,
2. $\sigma_i \in X_i$,
3. $\sigma_k \subset \sigma_i$ and $\sigma_k \in HOD_{\sigma_{i \cup \{\sigma_i\}}}$ for all $k \leq i$,
4. $\sigma_k$ is countable in $HOD_{\sigma_{i \cup \{\sigma_i\}}}$ for all $k < i$,
5. $\theta^{\sigma_i}$ is Woodin in $HOD_{\sigma_{i \cup (i+1)}}$ and $P(\theta^{\sigma_i}) \cap HOD_{\sigma_{i \cup (i+1)}} = P(\theta^{\sigma_i}) \cap HOD_{\{s\}}$, where $\theta^{\sigma_i} = \Theta_{HOD_{\sigma_{i \cup \{\sigma_i\}}}}$. Note here that $\theta^{\sigma_i}$ is a successor in the Solovay sequence of $HOD_{\sigma_{i \cup \{\sigma_i\}}}$.
Remark 17. For any \( s = \langle \sigma_0, \ldots, \sigma_n \rangle \in T_0 \), \( \text{HOD}_{\{s\}} = \text{HOD}_s \). From now on, we’ll write \( \text{HOD}_s \) for \( \text{HOD}_{\{s\}} \).

Lemma 18. Let \( t = \langle \sigma_0, \ldots, \sigma_n \rangle \) be such that (3.1)-(3.4) hold. Let \( \sigma = \sigma_n \), and set \( H = \text{HOD}_t \). Then

\[
H = \text{HOD}_H^{\text{HOD}_{\sigma \cup \{\sigma\}}}.
\]

Proof. Here \( \text{HOD}_H \) consists of all sets \( \text{HOD} \) from members of \( H \). Notice here that \( H \subseteq H \cap \text{HOD}_{\sigma \cup \{\sigma\}} \); hence the right hand side of the equation makes sense and \( H \subseteq H \cap \text{HOD}_H^{\text{HOD}_{\sigma \cup \{\sigma\}}} \).

The \( \supseteq \) direction follows from the fact that \( \sigma \) is OD from \( t \). 

Lemma 19. Let \( s \in T \) and \( \text{dom}(s) = i \); then \( \forall \mu \sigma \ (s + \langle \sigma \rangle \in T) \).

Proof. Fix \( s \in T \) with \( \text{dom}(s) = i \). It is easy to see that \( \forall \mu \sigma, s + \langle \sigma \rangle \) satisfies (3.1)-(3.4), so we address (3.5). We want to show \( \forall \mu \sigma, H \cap \text{HOD}_{s + \langle \sigma \rangle} \models \theta^\sigma \) is Woodin. Let \( H = H \cap \text{HOD}_{s + \langle \sigma \rangle} \). Let us work now in \( \text{HOD}_{\sigma \cup \{\sigma\}} \), where \( \text{AD}^+ \) holds and \( \text{AD}_\kappa \) fails. This implies that \( \Theta = \Theta_Y \) for some \( Y \). Also, \( \Theta \) is regular and DC holds. We have then from Theorem 5.6 of [?] that

\[
H \models \Theta \text{ is Woodin.}
\]

By the previous theorem, \( H = H \cap \text{HOD}_H \), hence we’re done.

Let \( s = \langle \sigma_0, \ldots, \sigma_{i-1} \rangle \). Without loss of generality, it is enough to see that \( (\forall \mu \sigma, \theta \models \text{HOD}_s = \text{HOD}_s + P(\theta^{\sigma-1}) \cap \text{HOD}_s = P(\theta^{\sigma-1}) \cap \text{HOD}_{s + \langle \sigma \rangle}) \). It is clearly enough to show \( (\forall \mu \sigma, \theta \models \text{HOD}_s \supseteq P(\theta^{\sigma-1}) \cap \text{HOD}_{s + \langle \sigma \rangle}) \).

Suppose not. We have \( (\forall \mu \sigma, (\exists A_\sigma \subseteq \theta^{\sigma-1}) (A_\sigma \in \text{HOD}_{s + \langle \sigma \rangle} \setminus \text{HOD}_s) \).

Here we take \( A_\sigma \) to be the least such set. Since \( \theta^{\sigma-1} \) is a fixed countable ordinal, we have \( (\exists A \subseteq \theta^{\sigma-1})(\forall \mu \sigma) (A = A_\sigma) \). But this A is in fact \( \text{HOD}_s \) since the supercompactness measures are OD. Contradiction.

Lemma 20. Let \( s \in T \) with \( \text{dom}(s) = i \). Let \( \sigma = s(\text{dom}(s) - 1) \). Then there is a partial order \( \mathbb{P} \) such that

1. \( \text{HOD}_s \models \mathbb{P} \) is a \( \theta^\sigma \)-c.c. complete boolean algebra of cardinality \( \theta^\sigma \), and

2. for any \( A \subseteq \kappa^\sigma \) such that \( A \in \text{HOD}_{\sigma \cup \{\sigma\}} \), there is a filter \( G_A \) on \( \mathbb{P} \) such that

   \[
   \bullet \ G_A \text{ is } \text{HOD}_s \text{-generic over } \mathbb{P}, \text{ and}
   \]

   \[
   \bullet \ \text{HOD}_{\{s,A\}} = \text{HOD}_s[G_A].
   \]

Proof. Let \( H = \text{HOD}_s \). Working in \( \text{HOD}_{\sigma \cup \{\sigma\}} \), where \( H = H \cap \text{HOD}_H \) by Lemma ??, let \( \mathbb{P} \) be the Vopenka algebra for adding subsets of \( \kappa^\sigma \) to \( \text{HOD}_H \). So \( \mathbb{P} \) is isomorphic to \( (O, \subseteq) \), where \( O \) is the collection of all \( \text{OD}_H \) subsets of \( P(\kappa^\sigma) \). Then (1) and (2) are standard properties of the Vopenka algebra, where the filter \( G_A \) in (2) is the filter generated by \( A \).

Now we’re ready to define our Prikry forcing \( \mathbb{P} \). Conditions in \( \mathbb{P} \) are pairs \( (s,F) \) such that \( s \in T \) and \( F : T \to V, F(\emptyset) \in \mu_0 \), and for all \( \langle \sigma_0, \ldots, \sigma_n \rangle \in T \), \( F(\langle \sigma_0, \ldots, \sigma_n \rangle) \in \mu_{n+1} \). The ordering is defined by

\[
(s_0, F_0) \preceq (s_1, F_1) \iff s_1 \subseteq s_0, (\forall s \in T)(F_0(t) \subseteq F_1(t)), (\forall i \in \text{dom}(s_0) - \text{dom}(s_1))(s_0(i) \in F_1(s_0[i])).
\]
Lemma 21. Suppose $Z \subset V^\mathbb{P}$ is countable, $\phi$ is a formula, and $(s_0, F_0) \in \mathbb{P}$. Then there is a condition $(s_0, G) \in \mathbb{P}$ deciding $\phi[\tau]$ for all $\tau \in Z$.

Proof. Since the usual proof requires DC, which we don’t have, we’ll give here a DC-free proof. Fix $\tau \in Z$. We’ll show that there is an $(s_0, G)$ deciding $\phi[\tau]$ such that $G$ is OD from $s_0$, $F$, and $\tau$. Let us say that $u \in T$ is positive if and only if $(\exists G) ((u, G) \Vdash \phi[\tau])$, negative if and only if $(\exists G) ((u, G) \Vdash -\phi[\tau])$, and ambiguous if and only if it is neither positive nor negative. Notice that $u$ cannot be both positive and negative.

For notational convenience, for $u \in T$ with $\text{dom}(u) = n + 1$, we write $\forall^*_u P(\sigma)$ to mean $\{\sigma \mid P(\sigma)\} \in \mu_{n+1}$. Now define $G = G_\tau$ by: for $v \in T$, $G(v) = \{\sigma \mid v + \langle \sigma \rangle$ is positive\} $\cap F_0(v)$ if $(\forall^*_u \sigma) (v + \langle \sigma \rangle$ is positive); $G(v) = \{\sigma \mid v + \langle \sigma \rangle$ is negative\} $\cap F_0(v)$ if $(\forall^*_u \sigma) (v + \langle \sigma \rangle$ is negative); $G(v) = \{\sigma \mid v + \langle \sigma \rangle$ is ambiguous\} $\cap F_0(v)$ if $(\forall^*_u \sigma) (v + \langle \sigma \rangle$ is ambiguous). Clearly $G$ is OD from $s_0$, $\tau$, $F_0$ and $(s_0, G) \preceq (s_0, F_0)$. If remains to see that $(s_0, G)$ decides $\phi[\tau]$.\hfill\Box

Claim 22. Let $u \in T$ with $\text{dom}(u) = n + 1$. Then

1. $u$ is positive $\Rightarrow \forall^*_u \sigma (u + \langle \sigma \rangle$ is positive);  
2. $u$ is negative $\Rightarrow \forall^*_u \sigma (u + \langle \sigma \rangle$ is negative);  
3. $u$ is ambiguous $\Rightarrow \forall^*_u \sigma (u + \langle \sigma \rangle$ is ambiguous).

Proof. If $u$ is positive, then there is an $H$ such that $(u, H) \Vdash \phi[\tau]$. But then whenever $\sigma \in H(u)$, $(u + \langle \sigma \rangle, H) \Vdash \phi[\tau]$. Since $H(u) \in \mu_{n+1}$, we’re done. The proof is the same for $u$ being negative.

Suppose $u$ is ambiguous and the conclusion of (3) is false. Without loss of generality, we may assume $\forall^*_u \sigma (u + \langle \sigma \rangle$ is positive). Let $G = G_\tau$ be as above. Then $(u, G) \Vdash \phi[\tau]$ since if $(v, H) \preceq (u, G)$, then $v$ is positive by and easy induction using part (1), and thus $(v, H) \not\preceq -\phi[\tau]$. Hence $u$ is in fact positive. Contradiction.\hfill\Box

Claim 23. No $u \in T$ is ambiguous.

Proof. Suppose $u$ is ambiguous. Let $G = G_\tau$ be as in the previous claim. Let $(v, H) \preceq (u, G)$ and $(v, H)$ decide $\phi[\tau]$. Then $v$ is not ambiguous. On the other hand, by induction using Claim 18 part (3), $v$ is ambiguous. Contradiction.\hfill\Box

By the previous claim, we may assume without loss of generality that $s_0$ is positive. But then $(s_0, G_\tau) \Vdash \phi[\tau]$, for otherwise, we have $(v, H) \preceq (s_0, G_\tau)$ forcing $-\phi[\tau]$. This implies that $v$ is negative. However, an induction using Claim 18 part (1) shows that $v$ is positive.

Finally, let $H(v) = \cap_{\tau \in Z} G_\tau(v)$. We get that $(s_0, H)$ decides $\phi[\tau]$ for all $\tau \in Z$.\hfill\Box

Let $G \subset \mathbb{P}$ is $V$-generic and $s_G = \bigcup\{s \mid (s, F) \in G\}$. Now we use Lemma ?? to prove the following:

Lemma 24. For all $i < \omega$, $P(\theta_i) \cap \text{HOD}^V_{s_G[i+1]} = P(\theta_i) \cap \text{HOD}^V_{\{s_G\}}$, where $\theta_i = \Theta^{\text{HOD}^V_{s_G[i]}}$.\hfill\Box
Proof. The \( \subseteq \) direction is evident because we use \( V \) as a predicate in the definition of \( HOD_{\{s_G^i(V,G)\}} \). Suppose the converse direction fails for some \( i \). Then there is a formula \( \varphi(x_0,x_1,x_2) \), an ordinal \( \xi \), an \( n > i \), an \( F \) such that \( (s_G|n,F) \in G \), and

\[
(s_G|n,F) \models \{ \beta < \theta_i \mid (V[G],V) \models \varphi[\beta,\xi,s_G]\} \notin HOD_{\{s_G^i(i+1)\}}.
\]

By Lemma ??, given any \((s_G^i,n,F)\) as above, there is \((s_G^i|n,F^*) \subseteq (s_G^i|n,F)\) such that for all \( \beta < \theta_i \), either \((s_G^i|n,F^*) \models (V[G],V) \models \varphi[\beta,\xi,s_G]\) or \((s_G^i|n,F^*) \models (V[G],V) \models -\varphi[\beta,\xi,s_G]\). Hence we can find such a \((s_G^i|n,F^*)\) in \( G \). So \( \{ \beta < \theta_i \mid (s_G^i|n,F^*) \models (V[G],V) \models \varphi[\beta,\xi,s_G]\}\) = \( \{ \beta < \theta_i \mid \exists F^*(s_G^i|n,F^*) \models (V[G],V) \models \varphi[\beta,\xi,s_G]\}\) \( \in HOD_{\{s_G^i|n\}} \). But \( s_G^i|n \in T \) and \( n > i \), so by (3.5) \( \{ \beta < \theta_i \mid (s_G^i|n,F^*) \models (V[G],V) \models \varphi[\beta,\xi,s_G]\}\) \( \in HOD_{\{s_G^i(i+1)\}} \). This is a contradiction.

Fix a \( G \subseteq \mathbb{P} \) such that \( G \) is \( V \)-generic. Let

\[
N = HOD_{\{s_G^i(V,G)\}}.
\]

It’s easy to see that \( \omega^V_1 \) is a limit of Woodin cardinals in \( N \), \( N \models ZFC \). Here is the key lemma.

**Lemma 25.** \( V \) is a derived model of \( N \).

**Proof.** To simplify the notation, let \( N_i = HOD_{s_G^i(i+1)} \) and \( \theta_i = \Theta^{HOD_{s_G^i(i+1)}} \) for each \( i < n \). Then \( \theta_i \) is Woodin in \( N_i \) and \( P(\theta_i) \cap N_i = P(\theta_i) \cap N_j = P(\theta_i) \cap N \) for all \( j \geq i \). As mentioned above, \( \omega^V_1 = \sup \{ \theta_i \mid i < \omega \} \).

Now, let \( K \) be a \( Col(\omega,\omega^V_1) \)-generic over \( N \) such that \( \mathbb{R}^K = \mathbb{R}^V \). To see that there is such a \( K \), it suffices to show that any \( x \in \mathbb{R}^V \) is generic over \( N \) for some poset \( \mathbb{P} \in N[sup(\theta_i)] \). Fix such an \( x \) and pick \( i \) such that \( x \in s_G^i(i) \). By Lemma ??, \( x \) is \( \mathbb{P} \)-generic over \( N_i \), where \( \mathbb{P} \) is the Vopenka algebra of \( HOD_{s_G^i(i+1)} \) for adding a subset of \( \kappa_{s_G^i(i)} \) to \( HOD_{s_G^i(i+1)} = N_i \). But \( P(\theta_i)^N = P(\theta_i)^N \), so \( x \) is \( \mathbb{P} \)-generic over \( N \).

To finish the proof, we need to see that \( P(\mathbb{R})^V \subseteq Hom^*_K \). It suffices to show that \( P(\mathbb{R})^V \subseteq Hom^*_K \). Because then if \( P(\mathbb{R})^V \not\subseteq Hom^*_K \), we get a sharp for \( V \) in a generic extension of \( V \). This is impossible.

So let \( B \in P(\mathbb{R})^V \). \( B \) is Suslin co-Suslin. By Martin’s theorem, \( B \) and \( \mathbb{R} \setminus B \) are homogeneousy Suslin as witnessed by homogeneous trees on \( \omega \times \kappa \) for some \( \kappa < \Theta \). So we can find a countable sequence of ordinals \( f \) such that \( \sup(\text{range}(f)) < \Theta \) from which we can define a pair of trees \( (T,U) \) over \( V \) such that \( p[T] = B = \mathbb{R} \setminus p[U] \). The sequence \( f \) comes from the measures of the homogeneity systems from which \( T \) and \( U \) are defined. Pick \( k \) large enough so that \( \text{ran}(f) \subseteq s_G^i(k) \). Also \( s_G^i(k) \cap \text{Ord} \in N \). \( s_G(k) \) is made countable in \( N(\mathbb{R}^V) \) and some real coding \( \text{ran}(f) \) is added. Hence, for some \( i < \omega \) and \( g \in V \) generic over \( N_i \) for the collapse of an ordinal \( \theta_i \), we have \( f \in N_i[g] \). So, for any \( j \geq i \), \( N_j[g] \) can decode \( f \) to get the pair \( (T,U) \). Moreover, \( p[T]^N_{j[g]} = B \cap \mathbb{R}_{N_j[g]} = \mathbb{R}_{N_j[g]} - p[U]^N_{j[g]} \). Hence, \( B \in Hom^*_K \) as desired.

Now let \( \phi \) be a \( \Sigma_1 \) formula such that \( V \models \phi[\mathbb{R}] \). We want to show that there are \( \alpha, \beta < \Theta \) such that \( L_\alpha(P_{\beta}(\mathbb{R})) \models \phi[\mathbb{R}] \).

9
Lemma 26. There is an $A \in (\text{Hom}_{<\omega_1^Y})^N$ such that $L(A, \mathbb{R}^N) \models \phi[\mathbb{R}^N]$.

Proof. Let $\gamma$ be the least such that $L_\gamma(P(\mathbb{R})) \models \phi[\mathbb{R}]$ and $\langle \alpha_i \mid i < \omega \rangle$ is definable in $L_\gamma(P(\mathbb{R}))$. Since $V$ is the derived model of $N$ at $\omega_1^Y$, the station of (Q version of) stationary tower forcing gives an elementary embedding $j: N \to (M, E)$ such that

(10.1) $\text{crt}(j) = \omega_1^N$ and $j(\omega_1^N) = \omega_1^V$;

(10.2) $\mathbb{R}^{(M,E)} = \mathbb{R}^V$;

(10.3) $P(\mathbb{R})^V = (\text{Hom}_{<\omega_1^V})^* \subseteq j((\text{Hom}_{<\omega_1^Y})^N)$

(10.4) $j(A) = A^*$ for each $A \in (\text{Hom}_{<\omega_1^Y})^N$, where $A^* = p[T] \cap \mathbb{R}^V$ for $T$ a homogeneous tree in $N$ such that $p[T] \cap \mathbb{R}^N = A$;

(10.5) $\gamma$ is in the well-founded part of $(M,E)$.

If $(P(\mathbb{R}))^V \neq j((\text{Hom}_{<\omega_1^Y})^N)$, then there is an $A \in j((\text{Hom}_{<\omega_1^Y})^N) \setminus (P(\mathbb{R}))^V$. Since $\phi$ is $\Sigma_1$ and by (10.2), $(M,E) \models L(A, \mathbb{R}^{(M,E)}) \models \phi[\mathbb{R}^{(M,E)}]$. By elementarity, there is an $A \in (\text{Hom}_{<\omega_1^Y})^N$ such that $L(A, \mathbb{R}^N) \models \phi[\mathbb{R}^N]$. Hence, we may assume $(P(\mathbb{R}))^V = j((\text{Hom}_{<\omega_1^Y})^N)$. Since $\langle \alpha_i \mid i < \omega \rangle$ is definable in $L_\gamma(P(\mathbb{R}))$, from some $B \in P(\mathbb{R})^V = (\text{Hom}_{<\omega_1^Y})^*$, let $\beta < \omega_1^V$ such that there is a $D \in N[K][\beta]$ such that $B = D^*$. Replacing $N$ by $N[K][\beta]$ if necessary where $K$ is as in the previous lemma, we can assume $\langle \alpha_i \mid i < \omega \rangle$ is in the range of $j$, say $j(\langle \alpha_i^* \mid i < \omega \rangle) = \langle \alpha_i \mid i < \omega \rangle$. Since $N$ is a model of choice, we can choose (using $\langle \alpha_i^* \mid i < \omega \rangle$) a sequence $\langle A_i \mid i < \omega \rangle \in N$ cofinal in $(\text{Hom}_{<\omega_1^Y})^N$. Let $A \in (\text{Hom}_{<\omega_1^Y})^N$ code the $A_i$’s, say $A = \{\langle i, x(0), x(1)\rangle \mid x = \langle x(0), x(1)\rangle \in A_i\}$. Then $A$ is in $\text{Hom}_{<\omega_1^Y}$ but not Wadge reducible to any $A_i$. Contradiction.

Lemma ?? and the elementarity of the map $j$ defined there finish the proof of the theorem in the case $\text{cof}(\Theta) = \omega$.

Case 2: $\text{cof}(\Theta) > \omega$

By a result of Solovay, DC holds in this case (see [?]). Let $\mu$ be a measure on $\{\alpha \mid \text{cof}(\alpha) = \omega\}$ induced by the measure on $\text{cof}(\Theta) < \Theta$ which in turn is induced by the Martin measure on Turing degrees.

For each $\alpha < \Theta$ such that $\text{cof}(\alpha) = \omega$, let $I_{\alpha} = \{A \subset \Theta_{\alpha} \mid \text{sup}(A) < \Theta_{\alpha}\}$. Therefore,

(11.1) $\text{HOD}_{I_{\alpha}} \models AD^+ + AD_{\mathbb{R}}$

(11.2) $\Theta^{\text{HOD}_{I_{\alpha}}} = \Theta_{\alpha}^V$

(11.3) for each $X \in \text{HOD}_{I_{\alpha}}$, $\Theta^{\text{HOD}_{I_{\alpha}}}$ is a limit of Woodin cardinals in $\text{HOD}(X)$.

We’ll use a slightly different Prikry forcing to add an inner model $N$ like before. The only difference in this case is that we want $\omega_1^V$ to be a limit of limits of Woodin cardinals in $N$. 10
For each $\alpha < \Upsilon$ such that $\text{cof}(\alpha) = \omega$, let $\mu_\alpha$ be the supercompact measure on $P_{\omega_1}(I_\alpha)$ induced by the Solovay measure on $P_{\omega_1}(\mathbb{R})$.

**Lemma 27.** For each $\alpha < \Upsilon$ such that $\text{cof}(\alpha) = \omega$, there are $\mu_\alpha$-measure 1 many $\sigma$ such that

\[
(12.1) \ HOD_{\sigma \cup \{\sigma\}} \models AD_{\mathbb{R}}
\]

\[
(12.2) \ \text{The transitive collapse of } \sigma \text{ is the set } \{A \subset \Theta \mid \sup(A) < \Theta\} \text{ as computed in } HOD_{\sigma \cup \{\sigma\}}.
\]

**Proof.** Notice that because of DC, the ultraproduct $\prod_\sigma HOD_{\sigma \cup \{\sigma\}}/\mu_\alpha$ is wellfounded. So identifying it with its transitive collapse, we get $I_\alpha \subset \prod_\sigma HOD_{\sigma \cup \{\sigma\}}/\mu_\alpha \subset HOD_{I_\alpha}$. Also $\Theta_\alpha = \Theta^{HOD_{I_\alpha}} = \Theta^{\prod_\sigma HOD_{\sigma \cup \{\sigma\}}/\mu_\alpha}$. This proves the claim. \[\square\]

Now like before, let $T_0$ be the set of all finite sequences $\langle \sigma_i \mid i \leq n \rangle$ such that for all $i \leq n$, there is an $\alpha < \Upsilon$ such that

\[
(13.1) \ \text{cof}(\alpha) = \omega
\]

\[
(13.2) \ \Theta_\alpha = \sup \{\gamma \mid \gamma \in \sigma_i\}
\]

\[
(13.3) \ \sigma_i \in P_{\omega_1}(I_\alpha)
\]

\[
(13.4) \ HOD_{\sigma_i \cup \{\sigma_i\}} \models AD_{\mathbb{R}}
\]

\[
(13.5) \ \text{The transitive collapse of } \sigma_i \text{ is } \{A \subset \Theta \mid \sup(A) < \Theta\} \text{ as computed in } HOD_{\sigma_i \cup \{\sigma_i\}}
\]

For each $\langle \sigma_i \mid i \leq n \rangle \in T_0$, let $\alpha_{\sigma_i} = \sup \{\gamma \mid \gamma \in \sigma_i\}$. Now let $T$ be the set of all $s = \langle \sigma_i \mid i \leq n \rangle \in T_0$ such that for all $i \leq n$,

\[
(14.1) \ P(\mathbb{R})^{HOD(s)} = P(\mathbb{R})^{HOD}
\]

\[
(14.2) \ \alpha_{\sigma_i} < \alpha_{\sigma_{i+1}}
\]

\[
(14.3) \ \sigma_k \subset \sigma_i, \ \sigma_k \in HOD_{\sigma_i \cup \{\sigma_{i+1}\}} \text{ for all } k \leq i, \text{ and } \sigma_k \text{ is countable in } HOD_{\sigma_i \cup \{\sigma_i\}} \text{ for all } k < i,
\]

\[
(14.4) \ P(\theta^\sigma) \cap HOD_{\{s(i+1)\}} = P(\theta^\sigma) \cap HOD_{\{s_i\}}, \text{ where } \theta^\sigma = \Theta^{HOD_{\sigma_i \cup \{\sigma_i\}}}.
\]

From the definition of $T$ and a similar proof to that of Lemma 27, if $s \in T$ then for $\mu$-almost all $\alpha < \Upsilon$, for $\mu_\alpha$-almost all $\sigma \in P_{\omega_1}(I_\alpha)$, $s + \langle \sigma \rangle \in T$. Now we’re ready to define the Prikry forcing $\mathbb{P}$. Conditions in $\mathbb{P}$ are pairs $(s,F)$ such that $s \in T$ and $F : T \to V$ such that for all $t \in T$, $t + \langle \sigma \rangle \in T$ for all $\sigma \in F(t)$ and for $\mu$-almost all $\alpha < \Upsilon$, for $\mu_\alpha$-almost all $\sigma \in P_{\omega_1}(I_\alpha)$, $\sigma \in F(t)$. The ordering on $\mathbb{P}$ is defined by:

$(s_1,F_1) \leq (s_0,F_0) \iff s_0 \subset s_1, \forall i \in \text{dom}(s_1) - \text{dom}(s_0), s_1(i) \in F_0(s_1[i]),$ and $F_1 \subset F_0$ pointwise.
Lemma 28. Suppose $Z \subset V^\mathbb{P}$ is countable, $\phi$ is a formula, and $(s_0, F_0) \in \mathbb{P}$. Then there is a condition $(s_0, F_1) \in \mathbb{P}$ that decides $\phi[\tau]$ for every $\tau \in Z$.

Proof. Same as that of Lemma ??.

Let $G \subset \mathbb{P}$ be $V$-generic and let $s_G = \{ s \mid \exists F(s, F) \in G \} = \{ \sigma_i \mid i < \omega \}$.

Lemma 29. (a) For all $i < \omega$, $P(\theta^{s_i}) \cap \text{HOD}^V_{s_G(i+1)} = P(\theta^{s_i}) \cap \text{HOD}^{(V[G], V)}_{\{s_G\}}$, where $\theta^{s_i} = \Theta^\text{HOD}_{s_G(i+1)}$.

(b) For all $i < \omega$, for all $A$ bounded subset of $\theta^{s_i}$ and $A \in \text{HOD}_{\sigma_i \cup \{s_i\}}$, there is a partial order $\mathbb{P}$ such that $|\mathbb{P}| < \theta^{s_i}$ and $\mathbb{P}$ is $\theta^{s_i}$-c.c. as computed in $\text{HOD}^V_{s_G(i+1)}$, and $\text{HOD}^V_{\{s_G(i+1), A\}} = \text{HOD}^V_{s_G(i+1)+1}[G_A]$ for some $\text{HOD}^V_{s_G(i+1)}$-generic filter $G_A \subset \mathbb{P}$ in $V$.

(c) $\theta^{s_i}$ is a limit of Woodin cardinals in $\text{HOD}^{(V[G], V)}_{\{s_G\}}$.

Proof. (a),(b) have the same proofs as those of Lemma ?? and ??, It remains to prove (c). By (a), it suffices to prove

$$\text{HOD}^V_{s_G(i+1)} \models \theta^{s_i} \text{ is Woodin.}$$

We know $\text{HOD}^V_{\sigma_i \cup \{s_i\}} \models \text{AD}_\mathbb{R}$, and in $\text{HOD}^V_{\sigma_i \cup \{s_i\}}$, $\text{HOD}^V_{s_G(i+1)} = \text{HOD}^V_{\text{HOD}_{s_G(i+1)}}$, so by Theorem 5.6 of [?], $\theta^{s_i}$ is a limit of Woodin cardinals in $\text{HOD}^V_{s_G(i+1)}$. Hence we’re done. 

Now, fix some $G \subset \mathbb{P}$ such that $G$ is $V$-generic, and let

$$N = \text{HOD}^{(V[G], V)}_{\{s_G\}}.$$ 

As before, for any $x \in \mathbb{R}^V$, $N[x] \equiv ZFC$, and $V$ is the derived model of $N[x]$. By part (c) of the previous lemma, $\omega_1^V$ is a limit of limits of Woodin cardinals in $N[x]$. Before stating the next lemma, we need the following:

**Definition 30.** Suppose $\delta$ is a limit of Woodin cardinals, then $\text{Hom}_{<\delta}$ is weakly sealed if the following hold.

1. Suppose $\kappa < \delta$ is a Woodin cardinal and $G \subset Q_{<\kappa}$ is $V$-generic. Let $j : V \to M \subset V[G]$ be the associated generic embedding. Then $j(\text{Hom}_{<\delta}) = (\text{Hom}_{<\delta})^{(V[G], V)}$.

2. Suppose that $G \subset \mathbb{P}$ is $V$-generic and $\mathbb{P} \in V_\delta$. Then (1) holds in $V[G]$.

**Lemma 31.** One of the following must hold.

(a) There is an $x \in \mathbb{R}^V$ and $A \in (\text{Hom}_{<\omega_1^V})^{N[x]}$ such that $L(A, \mathbb{R}^{N[x]}) \equiv \phi[\mathbb{R}^{N[x]}]$.

(b) $\text{Hom}^N_{<\omega_1^V}$ is weakly sealed in $N$.

Proof. Let $\gamma$ be large enough that $L_\gamma(P(\mathbb{R}^V)) \equiv \phi[\mathbb{R}^V]$. For any $x \in \mathbb{R}^V$, there is a generic elementary embedding $j_x : N[x] \to (M_x, E_x)$ induced by a $Q_{<\omega_1^V}$-generic such that

15.1 $\text{crt}(j_x) = \omega_1^{N[x]}$ and $j_x(\omega_1^{N[x]}) = \omega_1^V$,

15.2 $\mathbb{R}^{(M_x, E_x)} = \mathbb{R}^V$,

15.3 $(P(\mathbb{R}))^V \subseteq j_x(\text{Hom}^N_{<\omega_1^V})$. 

12
(15.4) \( \forall A \in \text{Hom}^{N[x]}_{<\omega_1^c}, j_x(A) = A^* \),

(15.5) for all successor Woodin cardinals \( \kappa < \omega_1^V \) in \( N[x] \), there is an \( N[x] \)-generic \( H \subset \mathcal{Q}^{N[x]}_{<\kappa} \) inducing a generic elementary embedding \( j_H : N[x] \to \text{Ult}(N[x], E_H) \), and an elementary embedding \( k_H : \text{Ult}(N[x], E_H) \to (M_x, E_x) \) such that \( j_x = k_H \circ j_H \).

(15.6) \( \gamma \) is in the well-founded part of \((M_x, E_x)\).

If overspill occurs, i.e. if there is some \( x \in \mathbb{R}^V \) such that \( P(\mathbb{R})^V \neq j_x(\text{Hom}^{N[x]}_{<\omega_1^V}) \) then (a) holds by the same argument as in Lemma ???. So suppose \( P(\mathbb{R})^V = j_x(\text{Hom}^{N[x]}_{<\omega_1^V}) \) for all \( x \in \mathbb{R}^V \). Then \( j_H(\text{Hom}^{N[x]}_{<\omega_1^V}) = \text{Hom}^{N[x][H]}_{<\omega_1^V} \) for all \( H \) in (15.5) because \( k_H(\text{Hom}^{N[x]}_{<\omega_1^V}(H)) \supset P(\mathbb{R})^V \) and \( j_H(\text{Hom}^{N[x]}_{<\omega_1^V}) \supset \text{Hom}^{N[x][H]}_{<\omega_1^V} \). By varying \( j_x \) and \((M_x, E_x)\) to ensure the filters \( H \) contain any specified condition, we get (b).

If (a) holds in the previous lemma, we’re done with the proof of case 2. So suppose (b) holds.

**Lemma 32.** \( \text{Hom}^N_{<\omega_1^V} = L(\text{Hom}^N_{<\omega_1^V}) \cap P(\mathbb{R}^N) \)

**Proof.** We first show:

(16.1) If \( \mathbb{P} \in V^N_{\omega_1^V} \) and \( G \subset \mathbb{P} \) is \( N \)-generic then in \( N[G] \), there is an elementary embedding \( j_G : L(\text{Hom}^N_{<\omega_1^V}) \to (L(\text{Hom}^N_{<\omega_1^V}))^N[G] \) such that \( j_G(\text{Hom}^N_{<\omega_1^V}(G)) = (\text{Hom}^N_{<\omega_1^V})^N[G] \).

To show (16.1), fix \( \mathbb{P} \in V^N_{\omega_1^V} \) and an \( N \)-generic \( G \subset \mathbb{P} \). Fix an increasing sequence \( \langle \delta_i \mid i < \omega \rangle \) of Woodin cardinals in \( N \) bounded below \( \omega_1^V \) and let \( \kappa = \sup \{ \delta_i \mid i < \omega \} > |\mathbb{P}|^N \). Let \( \delta_\omega < \omega_1^V \) be a Woodin cardinal in \( N \) larger than \( \kappa \).

Let \( \sigma \) be the symmetric reals for a \( \text{Col}(\omega, < \kappa) \)-generic over \( N \). Let \( G_\omega \subset \mathcal{Q}_{<\delta_\omega} \) be \( N \)-generic such that for all \( i \), \( G_i = G_\omega \cap \mathcal{Q}_{<\delta_i} \) is \( N \)-generic and \( \sigma = \bigcup \{ \mathcal{R}^N[G_i] \mid i < \omega \} \).

Let, for each \( i < \omega \), \( j_i : N \to M_i \subset N[G_i] \) be the generic elementary embedding given by \( G_i \). Let \( j_{i_1,i_2} : M_{i_1} \to M_{i_2} \) be the induced embeddings for pairs \( i_1 < i_2 \) and \( M^* \) be the corresponding direct limit with associated embedding \( j^* : N \to M^* \). \( M^* \) can be embedded into \( M_\omega \), hence it is well-founded. Also, since \( \text{Hom}^N_{<\omega_1^V} \) is weakly-sealed, \( j_i(\text{Hom}^N_{<\omega_1^V}) = \text{Hom}^N_{<\omega_1^V} \), hence \( j^*(\text{Hom}^N_{<\omega_1^V}) = \text{Hom}^{N(\sigma)}_{<\omega_1^V} \). Using this, we’ll show (16.1).

Using the notation of (16.1), let \( N[G](\tau) \) be a symmetric extension of \( N[G] \) for \( \text{Col}(\omega, < \kappa) \) such that \( N(\sigma) = N[G](\tau) \). Now, \( j^* \) induces an elementary embedding \( j_\sigma : L(\text{Hom}^N_{<\omega_1^V}) \to L(\text{Hom}^N_{<\omega_1^V})^{N(\sigma)} \) such that \( j_\sigma(\text{Hom}^N_{<\omega_1^V}) = \text{Hom}^{N(\sigma)}_{<\omega_1^V} \). Similarly, there is an elementary embedding \( j_\tau : (L(\text{Hom}^N_{<\omega_1^V})^{N[G]} \to (L(\text{Hom}^N_{<\omega_1^V}))^{N[G]}(\tau)) \) such that \( j_\tau(\text{Hom}^N_{<\omega_1^V}) = \text{Hom}^{N[G]}_{<\omega_1^V} \).

But \( N[G](\tau) = N(\sigma) \) so this induces an elementary embedding \( j_G : L(\text{Hom}^N_{<\omega_1^V}) \to (L(\text{Hom}^N_{<\omega_1^V}))^{N[G]} \) such that \( j_G(\text{Hom}^N_{<\omega_1^V}) = \text{Hom}^{N[G]}_{<\omega_1^V} \). This proves (16.1)

Now to see that (16.1) implies the lemma, we need to use Woodin’s tree production
lemma. Suppose for contradiction that $Hom^N_{\omega_1^Y} \neq L(Hom^N_{\omega_1^Y}) \cap P(\mathbb{R}^N)$. Let $\alpha$ be least such that $Hom^N_{\omega_1^Y} \neq L_\alpha(Hom^N_{\omega_1^Y}) \cap P(\mathbb{R}^N) \backslash Hom^N_{\omega_1^Y}$. Let $A \in L_\alpha(Hom^N_{\omega_1^Y}) \cap P(\mathbb{R}^N) \backslash Hom^N_{\omega_1^Y}$ such that $N$ can define $A$ by a formula $\phi$ with parameters a pair of trees $(T, S)$ representing a $Hom^N_{\omega_1^Y}$ set. It is then easy to check the hypotheses of the tree production lemma hold true for $N$ and $\phi$, i.e.

(a) (Generic Absoluteness) Let $\delta < \omega_1^Y$ be Woodin in $N$, $G$ be $< \delta$-generic over $N$, and $H$ be $< \delta^+$-generic over $N[G]$. For all $x \in \mathbb{R} \cap N[G]$, $N[G] \models \phi[x, T, S] \iff N[G][H] \models \phi[x, T, S]$.

(b) (Stationary Tower Correctness) Let $\delta < \omega_1^Y$ be Woodin in $N$, $G$ be $Q_{< \delta}$-generic over $N$, and $j : N \rightarrow M \subseteq N[G]$ be the induced embedding. Then for all $x \in \mathbb{R} \cap N[G]$, $N[G] \models \phi[x, T, S] \iff M \models \phi[x, j(T), j(S)]$

The tree production lemma then implies that $A \in Hom^N_{\omega_1^Y}$. This is a contradiction. □

This implies that $L(Hom^N_{\omega_1^Y})$ is a counterexample to the theorem in the sense that $L(Hom^N_{\omega_1^Y}) \models AD^+$ but no $A \in (P(\mathbb{R}))^{L(Hom^N_{\omega_1^Y})}$ satisfies that $L(A, \mathbb{R}^N) \models \phi[\mathbb{R}^N]$. By induction on $\Theta$ of $AD^+$ models and the fact that $\Theta^{L(Hom^N_{\omega_1^Y})} < \Theta^V$, we have a contradiction. So (b) of Lemma ?? can’t hold; hence, (a) is the only possibility. (Theorem 1)