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Math 280 Assignment #1 due Friday, Oct 9, 2015

For this homework, in your written solutions to the following problems, you should say which axioms of ZFC are being used.
Do not merely cite other textbooks in your answers; I want to see how you write. Understanding what needs to be written down and what can be omitted from your proof requires practice. If your write-up of a problem exceeds one full page, you might be writing too much.

Problem 1. The notation "∃!" stands for "there exists a unique". This can be expressed by a formula in the language of set theory. Let $\phi(u, v, w)$ be a formula. Show that the following principle

$$\forall p \forall A (\forall x \in A \exists! y \; \varphi(x, y, p) \rightarrow (\exists B \forall x \in A \exists y \in B \varphi(x, y, p))$$

(the Replacement Schema) follows from ZF. Show also that the Collection Schema follows from the Replacement Schema and the rest of the ZF axioms.

Problem 2. Show that if $\prec$ is a well-ordering on a set W then there exists no sequence $\langle a_n \mid n < \omega \rangle$ such that for all $n < \omega$ $a_{n+1} \prec a_n$.

Problem 3. Show that for all ordinal $\alpha > 0$, α can be uniquely represented as

$$\alpha = \omega^{\beta_1} k_1 + \cdots + \omega^{\beta_n} k_n$$

for some $n \geq 1$, $k_1, \ldots, k_1 \in \mathbb{N} \backslash \{0\}$, $\beta_n < \cdots < \beta_1 \leq \alpha$.

Problem 4, part A (Kőnig's Infinity Lemma). Define the following.

- ${}^A X$ = the set of functions from A to X.
- For any ordinal β,
$$^{<\beta}X = \bigcup_{\alpha < \beta} {}^{\alpha}X.$$
- T is a *tree on* X iff $T \subseteq {}^{<\omega}X$ and, for every $s \in T$, if $n = \text{dom}(s)$ and $m < n$, then $s \restriction m \in T$.
- If T is a tree on X, then T has *infinite height* iff for every $n < \omega$,
$$T \cap {}^n X \neq \emptyset.$$
- If T is a tree on X, then T has an *infinite branch* iff there exists $f \in {}^\omega X$ such that, for every $n < \omega$,
$$f \restriction n \in T.$$

Prove that every tree on 2 with infinite height has an infinite branch.

Problem 4, part B. Does every tree on ω with infinite height have an infinite branch? Why or why not?