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Math 280 Assignment #2 due Monday, October 19, 2015

Problem 1. CH is the statement $2^{\aleph_0} = \aleph_1$. Prove that CH implies that

$$(\aleph_n)^{\aleph_0} = \aleph_n$$

whenever $1 \leq n < \omega$. **Hint:** One way is to prove the so-called *Hausdorff formula* $\aleph_{n+1}^{\aleph_0} = \aleph_n^{\aleph_0} . \aleph_{n+1}$ for all n.

Problem 2.

(a) Which axioms of ZFC hold in V_ω? [1] Which fail? Explain your answers.

(b) Show that the only elementary substructure of V_ω is itself.

(c) Which axioms of ZFC hold in $V_{\omega+\omega}$? Which fail? Explain your answers.

Problem 3. Define a class function $\mathrm{F} : \mathrm{V} \times \omega \longrightarrow \mathrm{V}$ by the equations

$$\mathrm{F}(x, 0) = x$$

and

$$\mathrm{F}(x, n+1) = \bigcup \mathrm{F}(x, n).$$

Then define

$$\mathrm{trcl}(x) = \bigcup_{n \in \omega} \mathrm{F}(x, n).$$

Finally, put

$$\mathrm{HF} = \{x \mid |\mathrm{trcl}(x)| < \aleph_0\}.$$

Note that, from this definition, it is not obvious that HF is a set.

(a) Show that $\mathrm{HF} \supseteq \mathrm{R}(\omega)$.

(b) Show that $\mathrm{HF} \subseteq \mathrm{R}(\omega)$.

Now we know that HF is a set!

Problem 4. The countable Axiom of Choice states that: Every countable family of nonempty sets has a choice function. The principle of Dependent Choice (DC) states that: If E is a binary relation on some nonempty set A such that for every $a \in A$, there is a $b \in A$ such that bEa, then there is a sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ such that $a_{n+1} E a_n$ for all n.

Prove that the countable Axiom of Choice proves that every infinite set has a countable subset. Prove that DC implies the countable Axiom of Choice.

Problem 5. Assume ZF. Show that ω_2 is not a countable union of countable sets.

[1] Remember our convention that if we use a set A to refer to a structure, it is the structure $\langle A, E \rangle$ where $E = \{(x, y) \in A \times A \mid x \in y\}$. It should always be clear from context whether we mean A or $\langle A, E \rangle$.