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Math 280 Assignment #3 due Wednesday, October 28, 2015

Problem 1. Recall Zorn's lemma is the statement: If $(\mathbb{P}, \leq)$ is a partially ordered set and any $\leq$-chain has a $\leq$-upperbound, then $(\mathbb{P}, \leq)$ has a $\leq$-maximal element.

Prove the Axiom of Choice from Zorn's Lemma. Prove Zermelo's well-ordering theorem from Zorn's lemma. Prove the Axiom of Choice from the Zermelo's well-ordering theorem. Prove Zorn's lemma from the Zermelo's well-ordering theorem.

Problem 2. Show that $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$ by explicitly defining a surjection from $\wp(\aleph_\alpha)$ onto $\aleph_{\alpha+1}$. Using this, show also that $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.

Problem 3. Show that the definition of $\text{cof}(\alpha)$ defined in class can be weakened by omitting the phrase "strictly increasing". In other words, show that $\text{cof}(\alpha)$ is the least β such that there is a cofinal β-sequence in α.

Problem 4. (a) Show that $\text{cof}(2^\kappa) > \kappa$ for every cardinal κ.
(b) Show that if κ is singular and not strong limit then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$. Show that if κ is singular and strong limit then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\text{cof}(\kappa)}$.

Problem 5. Prove that for any infinite cardinal κ, any $\kappa \leq \alpha < \kappa^+$, there is a sequence $(X_n \subset \alpha : n < \omega)$ such that $\bigcup_n X_n = \alpha$ and for each $n < \omega$, $\text{o.t.}(X_\alpha) \leq \kappa^n$. (Hint: prove by induction on α. At each α, choose a cofinal sequence in α of length $\text{cof}(\alpha)$.)