SOLUTIONS TO SELECTED PROBLEMS FROM HOMEWORK 5

Additional problem 1: (We assume the sets A, B are nonempty. Otherwise, there is nothing to do.) Given injection $f : A \to B$, one can define surjection $g : B \to A$ by $g = f^{-1} \cup h$ where $h : B \backslash rng(f) \to A$ by fixing some $a \in A$ and let $h(x) = a$ for all $x \in B \backslash rng(f)$ if $B \backslash rng(f)$ is nonempty; if $B = rng(f)$ then simply let $g = f^{-1}$.

Let $g : B \to A$ be surjective. For each $a \in A$, choose an element b_a b of $g^{-1}(a)$; recall $g^{-1}(a) = \{b \in B : g(b) = a\}$. Notice that if $a_0 \neq a_1$ then $g^{-1}(a_0) \cap g^{-1}(a_1) = \emptyset$ because g is a function. Indeed any $c \in g^{-1}(a_0) \cap g^{-1}(a_1)$ would have the property that $g(c) = a_0 \neq a_1 = g(c)$. Of course, this violates the fact that g is a function.

Now define $f : A \to B$ as: $f(a) = b_a$. Since if $a_0 \neq a_1$, then $b_{a_0} \neq b_{a_1}$, we get that f is injective.

Additional problem 2: Let $C = rng(f) = \{b \in B : \exists a \in A f(a) = b\}$. By $\text{dom}(f^{-1})$, I mean the set C. This simply means that f^{-1} is defined only on C.

Assume f is one-to-one. We want to see that f^{-1} is a function from C to A. To see that f^{-1} is a function. Suppose not. This means there is some $b \in C$ such that there are distinct $x, y \in A$ such that $f^{-1}(b) = x$ and $f^{-1}(b) = y$. In other words, $f(x) = b = f(y)$. This contradicts the fact that f is one-to-one. So f^{-1} is indeed a function.

Now assume f is onto but not one-to-one. So in this case $C = B$. But f^{-1} is NOT a function (from C to B). The reasoning is almost the same as above. Let x, y be distinct such that $f(x) = f(y) = b$; x, y exist because we assume f is not one-to-one. By definition, $(x, b) \in f$ so $(b, x) \in f^{-1}$ and similarly, $(b, y) \in f^{-1}$. This means the element b is "mapped" by f^{-1} to two distinct elements x, y. So f^{-1} is not a function.

Problem 5.1.1:
(a) The recurrence relation is: $h_{n+1} = h_n + n$. Note that $h_1 = 0$.

(b) The formula for h_n in terms of n is: for $n \geq 1$, $h_n = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$.

One way you can prove this is to observe that we know the formula for $\sum_{i=0}^{n} i$ is $\frac{n(n+1)}{2}$, so the sum

$$h_n = \sum_{i=0}^{n-1} i = \sum_{i=0}^{n} i - n = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}.$$

Problem 5.2.2:

Base case: when $n = 0$, $\sum_{j=0}^{0} 2^j = 2^0 = 1$. On the other hand, $2^{0+1} - 1 = 2 - 1 = 1$. So these values are equal. We're done with the base case.

Inductive step: We show $\forall n \geq 0$ $(\sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \Rightarrow \sum_{j=0}^{n+1} 2^j = 2^{n+1+1} - 1)$.

So we fix $n \geq 0$ and assume $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$ (this is our inductive hypothesis). We show

$$\sum_{j=0}^{n+1} 2^j = 2^{n+1+1} - 1.$$

$\sum_{j=0}^{n+1} 2^j = 2^0 + 2^1 + \dots 2^n + 2^{n+1} = \sum_{j=0}^{n} 2^j + 2^{n+1}$.
Now applying the inductive hypothesis, we have:

$$\sum_{j=0}^{n+1} 2^j = (2^{n+1} - 1) + 2^{n+1} = 2.2^{n+1} - 1 = 2^{n+2} - 1$$

as desired.

Problem 5.2.7 (a): The formula for the sum of the first n odd numbers is:

$$\sum_{i=1}^{n}(2i - 1) = 1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Proof by induction that the above formula is true for all $n \geq 1$.
Base case: $n = 1$. $\sum_{i=1}^{1}(2i - 1) = 1 = 1^2$. So the base case holds.
Inductive step: We need to show: $\forall n \geq 1(\sum_{i=1}^{n}(2i - 1) = n^2 \Rightarrow \sum_{i=1}^{n+1}(2i - 1) = (n + 1)^2)$.
So fix $n \geq 1$ and assume $\sum_{i=1}^{n}(2i - 1) = n^2$. We need to show

$$\sum_{i=1}^{n+1}(2i - 1) = (n + 1)^2.$$

Now, $\sum_{i=1}^{n+1}(2i - 1) = \sum_{i=1}^{n}(2i - 1) + [2 * (n + 1) - 1]$.
Apply the inductive hypothesis, we have:

$$\sum_{i=1}^{n+1}(2i - 1) = n^2 + 2 * (n + 1) - 1 = n^2 + 2n + 1 = (n + 1)^2$$

as desired.