Nam Trang
Math 280 Assignment #6 due Friday, Dec 04, 2015

Problem 1. In the following, stationarity, clubness etc. are in the sense of Woodin (as defined in class). Fix sets $\emptyset \neq A \subseteq B$. Prove that:

(i) If S is stationary in $\wp(B)$ then $S \restriction A = \{x \cap A \mid x \in S\}$ is stationary in $\wp(A)$.

(ii) If S is stationary in $\wp(A)$ then $S^B = \{x \subseteq B \mid x \cap A \in S\}$ is stationary in $\wp(B)$.

Problem 2. Let $\mathcal{L}, \mathcal{L}'$ be languages such that $\mathcal{L} \subseteq \mathcal{L}'$. (So $\mathcal{L}'$ may have properly more function, relation and constant symbols.) Let Σ be a set of $\mathcal{L}$-sentences and σ be an $\mathcal{L}$-sentence. Prove:

$$\Sigma \vdash_{\mathcal{L}} \sigma \Leftrightarrow \Sigma \vdash_{\mathcal{L}'} \sigma.$$

Hint. For the non-trivial direction use the completeness theorem.

Problem 3. Compactness Theorem can be proved directly, without going over Completeness Theorem, by mimicking the proof of Completeness Theorem in the semantical setting. This direct proof of Compactness Theorem is even simpler than the proof of Completeness Theorem, because it avoids subtleties related to formal proofs. Of course, the information one gets from this proof is restricted: It does not seem to yield a proof of Completeness Theorem. Obviously it is not clear how to obtain a proof of Completeness Theorem from a semantical proof of Compactness Theorem. The following exercise is devoted to the semantical proof of Compactness Theorem. For this, we have to reformulate several notions that occurred in the proof of Completeness Theorem into semantical terms.

Recall that a set Σ of $\mathcal{L}$-sentences is **satisfiable** iff it has a model, that is, iff there is an $\mathcal{L}$ structure $\mathcal{M}$ such that $\mathcal{M} \models \Sigma$. We say that Σ is **finitely satisfiable** iff every finite $\Delta \subseteq \Sigma$ is satisfiable. We consider the following statement of Compactness Theorem, which highlights the more interesting direction.

Theorem 0.1 (Compactness Theorem)**.** *Let Σ be a finitely satisfiable set of $\mathcal{L}$-sentences. Then Σ has a model of cardinality $\leq$ card$(\mathcal{L}) + \aleph_0$.*

As in the case of Completeness Theorem, we split the proof into two logical parts, namely the Model Existence Theorem and Henkin Extensions Theorem. We now reformulate the notions of completeness and Henkin constants into semantical terms (but by Completeness Theorem, we already know that these reformulations are equivalent to the syntactical ones! Only for the purpose of this exercise we pretend we do not know this!)

Definition 0.2 (Semantical completeness)**.** *A set Σ of $\mathcal{L}$-sentences is **semantically complete** iff for every $\mathcal{L}$-sentence σ there is a finite $\Delta \subseteq \Sigma$ such that*

$$\Delta \models \sigma \text{ or } \Delta \models \neg\sigma.$$

Definition 0.3 (Henkin constants in the semantical sense)**.** *We say that a set Σ of $\mathcal{L}$-sentences has **Henkin constants in the semantical sense** iff for every $\mathcal{L}$-formula ϕ and every variable v,*

$$\text{if there is a finite } \Delta \subseteq \Sigma \text{ such that } \Delta \models (\exists v)\phi$$

then there is a constant symbol c and a finite $\Delta' \subseteq \Sigma$ such that $\Delta' \models \phi(v/c)$.

By mimicking the proofs of the corresponding theorems we proved above, prove the following.

Theorem 0.4 (Model Existence Theorem, ZF)**.** *Let Σ be a set of $\mathcal{L}$-sentences satisfying the following.*

(a) Σ is finitely satisfiable.

(b) Σ is semantically complete.

(c) Σ has Henkin constants in the semantical sense.

Then Σ has a model of cardinality $\leq card(C^{\mathcal{L}})$ where $C^{\mathcal{L}}$ is the set of all constant symbols of $\mathcal{L}$.

Theorem 0.5 (Henkin Extension Theorem, ZFC)**.** *Let Σ be a finitely satisfiable set of $\mathcal{L}$-sentences. Then there is a language $\mathcal{L}^*$ of cardinality $\leq card(\mathcal{L}) + \aleph_0$ obtained by adding only constant symbols to $\mathcal{L}$ and a set $\Sigma^* \supseteq \Sigma$ of $\mathcal{L}^*$-sentences such that clauses (a) – (c) in the Model existence theorem hold.*

Now derive Compactness Theorem from these theorems.