MATH 13 HOMEWORK 2 ANSWER KEY

Problem 2.2.11: The statement you have to prove (in if ... then ... form) is:

If $x^2 + y^2$ is even, then x, y have the same parity (i.e. both even or both odd).

We assume x^2+y^2 is even. We prove this by contradiction. Suppose the conclusion fails, namely x, y have different parity.

Without loss of generality, we assume x is odd and y is even (the other case: x is even and y is odd is proved in the same manner, with obvious modifications).

So there is some integer k such that $x = 2k + 1$ and some integer l such that $y = 2l$. Hence $x^2 + y^2 = (2k + 1)^2 + (2l)^2 = 4k^2 + 4k + 1 + 4l^2 = 2(2k^2 + 2k + 2l^2) + 1$. This is clearly an odd number. Contradiction.

In conclusion, x, y have the same parity.

You could prove this by contrapositive also.

Problem 2.3.3: The statement in question is: $\exists m \in \mathbb{Z} \exists n \in \mathbb{Z}\ 2m - 3n = 15$.

To prove the statement above, you just have to find a specific pair (m, n) such that $2m-3n = 15$. Indeed, $m = 9, n = 1$ would work. There are other examples, like $m = 0, n = -5$.

The negation of the above statement is: $\forall m \in \mathbb{Z} \forall n \in \mathbb{Z}\ 2m - 3n \neq 15$.

Suppose you wanted to disprove the above statement, this means you wanted. to prove its NEGATION is true. You have to then start with arbitrary m, n and proceed to show $2m-3n \neq 15$.

Problem 2.3.15(c): We express the statement "x_n does not converge" in mathematical form. First, in English, this statement is equivalent to "for any L, x_n does not converge to L". Now let's write the definition of "x_n converges to L" given in the problem in its full mathematical form (notice the hidden quantifier):

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \in \mathbb{N} (n > N \Rightarrow |x_n - L| < \epsilon).$$

Finally, we can write the original statement as:

$$\forall L \exists \epsilon > 0 \forall N \in \mathbb{N} \exists n \in \mathbb{N} (n > N \wedge |x_n - L| \geq \epsilon).$$

Problem 2.3.15(d): We prove $x_n = n$ diverges to ∞. By the definition given in the problem, this means we need to verify the statement:

$$\forall M > 0 \exists N \in \mathbb{N} (n > N \Rightarrow x_n = n > M).$$

We start by fixing an arbitrary $M > 0$. Our job is to choose N (typically dependent on M) such that whenever $n > N$, we have x_n (which is defined to be n) $> M$.

In this case, we simply choose $N = M$, and verify that with this choice of N, the implication above is true. For any $n > N$, i.e. for any $n > M$, we know $x_n = n$, hence $x_n > M$. As desired.

Problem 4(a,b): (a) We can write this statement in the if-then form as follows: "for all a, if $a^3 - 10a + 1 = 0$, then a is not in the interval $[3, 5]$".

(b) The contrapositive of the statement in (a) is: "for all a, if a is in the interval $[3, 5]$ (equiv: $3 \leq a \leq 5$), then $a^3 - 10a + 1 \neq 0$."