MATH 13 HOMEWORK 4 ANSWER KEY

Problem 2(a): ($\Leftarrow$): We assume $d|c$. By our assumption, let (x_0, y_0) be an integer solution to $ax + by = d$. So $x_0, y_0 \in \mathbb{Z}$ and

$$ax_0 + by_0 = d. \tag{0.1}$$

Since $d|c$, there is an integer k such that $c = kd$. Multiply both sides of Equation 0.1 by k, we get

$$a(kx_0) + b(ky_0) = c. \tag{0.2}$$

Since $kx_0, ky_0 \in \mathbb{Z}$, we have shown that $ax + by = c$ has integer solutions.

($\Rightarrow$): Now assume $ax + by = c$ has an integer solution. Let (x_0, y_0) be such a solution. Since $d|a$, $d|ax$. Similarly, $d|by$. So $d|(ax + by)$. So $d|c$.

Problem 4.1.1(e): First, we list out the elements of the set $A = \{x \in \frac{1}{2}\mathbb{Z} : 0 \leq x \leq 4\}$. $A = \{0, 1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4\}$. Then we check for each $x \in A$, whether $4x^2 \in 2\mathbb{Z} + 1$. It is easy to see that this property holds for $x = 1/2, 3/2, 5/2, 7/2$. So the set

$$\{x \in \tfrac{1}{2}\mathbb{Z} : 0 \leq x \leq 4 \text{ and } 4x^2 \in 2\mathbb{Z} + 1\} = \{1/2, 3/2, 5/2, 7/2\}.$$

Problem 4.2.5(d): $[10] = \{\dots, -10, -5, 0, 5, 10, \dots\}$. In other words, $[10]$ is the set of integers which are divisible by 5. By definition, for all $x \in \mathbb{Z}$, $x \in M_{[10]}$ iff $-x \in [10]$. But we know $5|x$ iff $5|(-x)$. So $M_{[10]} = [10]$.

Problem 4.3.5: ($\Rightarrow$): Assume $B \backslash A = B$. This means $B \cap A^c = B$. This implies that $B \subseteq A^c$. To see this, let $x \in B$. Since $B = B \cap A^c$, $x \in B \cap A^c$; hence $x \in A^c$. So for each $x \in B$, $x \notin A$. In other words, $A \cap B = \emptyset$.

($\Leftarrow$): Suppose $A \cap B = \emptyset$. This implies, for each $x \in B$, $x \notin A$, so $x \in A^c$. Hence $B \subseteq A^c$. This in turn imlies $B \cap A^c = B$. This means $B \backslash A = B$.

Problem 4.4.9: Suppose $g \circ f$ is injective. We show f is injective. Let $x_1 \neq x_2 \in dom(f)$. We show $f(x_1) \neq f(x_2)$. Suppose not. Then $f(x_1) = f(x_2)$. Since g is a function, $g(f(x_1)) = g(f(x_2))$. This means $g \circ f(x_1) = g \circ f(x_2)$, while $x_1 \neq x_2$. This contradicts the fact that $g \circ f$ is injective. So it must be the case that $f(x_1) \neq f(x_2)$.