Homework 6 Math 150, Fall 2017

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Problem 1. Let $\mathcal{L} = \{\dot{E}\}$ be the language of graphs (recall the definition of a graph discussed in ~~class~~ discussion section).

Express each of the following statements about graphs as a set (possibly infinite) of sentences in $\mathcal{L}$. That is, in each of the following cases find a set of $\mathcal{L}$-sentences Σ such that for every graph G,

$$G \text{ has the named property iff } G \vDash \Sigma.$$

(a) "G contains arbitrarily large finite cliques."

(b) "G consists of disjoint cycles."

(c) "Any two nodes in G have the same degree."

Prove that there is no set Σ (possibly infinite) of $\mathcal{L}$-sentences such that for every graph $G = (V, E)$ (so G is an $\mathcal{L}$-structure), the following holds:

$$G \vDash \Sigma \text{ if and only if } G \text{ has finitely many cliques of size 5}$$

Solution.

(a) Observe "G contains arbitrarily large finite cliques" is equivalent to "G contains a clique of size n for every natural number n." Define $\Sigma := \{\varphi_n : n \in \mathbb{N}_{\geq 1}\}$ by

$$\begin{aligned}\varphi_n &:= \text{"there exists a clique of size } n\text{"}\\ &:= (\exists x_1)\ldots(\exists x_n)\text{"}x_i\text{, for } 1 \leq i \leq n\text{, form a clique"}\\ &:= (\exists x_1)\ldots(\exists x_n)\left[\bigwedge_{i\neq j}(x_i \neq x_j) \wedge \bigwedge_{i\neq j}(\dot{E}x_i x_j)\right].\end{aligned}$$

(b) Define $\Sigma := \{\Phi_n : n \in \mathbb{N}_{\geq 3}\}$ by

$$\begin{aligned}\varphi_n(x) &:= \text{"}x\text{ is in a cycle of length } n \text{ and the cycle is disjoint from the rest of the nodes"}\\ &:= \text{"}x\text{ is in a cycle of length } n \text{ and the cycle is disjoint from the rest of the nodes"}\\ &:= (\exists x_1)\ldots(\exists x_n)\begin{array}{l}\text{"}\{x_1,\ldots,x_n\}\text{ is a cycle of length } n \text{ and } x = x_1\\ \text{and the cycle is disjoint from the rest of the nodes"}\end{array}\\ &:= (\exists x_1)\ldots(\exists x_n)\left[\bigwedge_{i=1}^{n-1}(\dot{E}x_i x_{i+1}) \wedge \dot{E}x_n x_1 \wedge x = x_1 \wedge \begin{array}{l}\text{"the cycle is disjoint from the rest of the}\\ \text{nodes"}\end{array}\right]\\ &:= (\exists x_1)\ldots(\exists x_n)\left[\bigwedge_{i=1}^{n-1}(\dot{E}x_i x_{i+1}) \wedge \dot{E}x_n x_1 \wedge x = x_1 \wedge\right.\\ &\qquad\left.(\forall y)\left[\bigwedge_{i=2}^{n-1}((\dot{E}yx_i \wedge y \neq x_{i-1}) \to y = x_{i+1}) \wedge ((\dot{E}yx_1 \wedge y \neq x_n) \to y = x_2)\right]\right]\end{aligned}$$

and $\Phi_n := (\forall x)[\bigvee_{i=1}^{n} \varphi_n(x)]$.

(c) Define $\Sigma := \{\Phi_n : n \in \mathbb{N}\}$ by

$$\begin{aligned}\varphi_n(x, y) &:= \text{"}x\text{ and } y \text{ have degree } n\text{"}\\ &:= \text{"}x\text{ has degree } n\text{"} \wedge \text{"}y\text{ has degree } n\text{"}\end{aligned}$$

and "z has degree 0" is defined to be "$(\forall v)[\neg\dot{E}vz]$" and for $n \geq 1$,

$$\text{"}z\text{ has degree } n\text{"} := (\exists x_1)\ldots(\exists x_n)\left[\bigwedge_{i\neq j}(x_i \neq x_j) \wedge \bigwedge_{i=1}^{n}(\dot{E}x_i z) \wedge (\forall w)\left[\dot{E}wz \to \bigvee_{i=1}^{n}(w = x_i)\right]\right]$$

and $\Phi_n := \bigvee_{i=1}^{n}(\forall x)(\forall y)[\varphi_n(x, y)]$.

By contradiction, suppose a set Σ of $\mathcal{L}$-sentences exists such that for each graph G,

$$G \vDash \Sigma \text{ if and only if } G \text{ has finitely many cliques of size 5}$$

Define Δ to be the set of sentences φ_n stating that there exist at least n distinct cliques of size 5. Observe $\Delta = \{\varphi_n\}$ is infinite. Consider the union $\Sigma \cup \Delta$. Observe the union is finitely satisfiable, because for each finite subset $\Delta_0 \subseteq \Sigma \cup \Delta$, there exists a model satisfying the sentences (or axioms) in Δ_0. By the compactness theorem, there exists a model $\mathfrak{M}$ satisfying every sentence (or axiom) in the entire set $\Sigma \cup \Delta$. Observe $\mathfrak{M} \vDash \varphi_n$ for every n. Therefore $\mathfrak{M}$ is a model satisfying every axiom in Σ but $\mathfrak{M}$ does not have finitely many cliques of size 5—contradiction. As a result, Σ cannot exist. ∎

Problem 3. Let $\mathcal{L} = \{P\}$ be a language with one 2-ary relation symbol P. Let $\mathcal{M} = \left(\mathbb{Z}, P^{\mathcal{M}}\right)$ be an $\mathcal{L}$-structure. Here $\mathbb{Z}$ is the set of integers and for all $a, b \in \mathbb{Z}$,

$$(a, b) \in P^{\mathcal{M}} \Leftrightarrow |b - a| = 1.$$

Show that there is an elementary equivalent $\mathcal{L}$-structure $\mathcal{N} = \left(N, P^{\mathcal{N}}\right)$ (i.e. $\mathcal{M} \equiv \mathcal{N}$) that is not connected. Hint: Add constant symbols c, d to the language $\mathcal{L}$. Write down the sentences which say that c and d are "far apart." Apply compactness theorem to this set of sentences.

Solution. Define $\mathscr{L} := \mathcal{L} \cup \{c, d\}$ where c and d are distinct symbols not in $\mathcal{L}$. Observe $\mathscr{L}$ is a more expressive language than $\mathcal{L}$; that is to say, formulae φ and terms t in $\mathscr{L}$ can utilize the constant symbols c and d. Define $\Sigma = \{\varphi : \varphi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \varphi\}$. Observe every $\varphi \in \Sigma$ cannot have the symbols c and d. Define Δ to be the set of sentences φ_n stating that there does not exist n distinct elements $x_1, \ldots, x_n$ such that $c = x_1$ and $d = x_n$ and for $1 \leq i \leq n$, Px_ix_{i+1}. Observe φ_n is an $\mathscr{L}$-sentence which guarantees that c and d are at least $n+1$ distance apart. Consider the union $\Sigma \cup \Delta$.

Claim. The union is finitely satisfiable.

Let $\Delta_0 \subseteq \Sigma \cup \Delta$ be finite. For each natural number n, define $\mathcal{M}_n := \left(\mathbb{Z}, P^{\mathcal{M}_n}, c^{\mathcal{M}_n}, d^{\mathcal{M}_n}\right)$ by

- $P^{\mathcal{M}_n} = P^{\mathcal{M}}$.
- $c^{\mathcal{M}_n} = -2^n$.
- $d^{\mathcal{M}_n} = 2^n$.

Because Δ_0 is finite, there exists a finite enumeration $\{\sigma_1, \ldots, \sigma_p, \delta_1, \ldots, \delta_q\}$ of Δ_0 where

- $\sigma_i \in \Sigma$ for every $i \leq p$.
- $\delta_i \in \Delta$ for every $i \leq q$.
- Δ_0 can have no elements of Σ; in which case, $p = 0$.
- Δ_0 can have no elements of Δ; in which case, $q = 0$.

Because $\delta_i \in \Delta$, there exists m_i such that δ_i and φ_{m_i} are equal. Define $\mathfrak{m} := \max\{m_1, \ldots, m_q, 0\}$. Observe $\mathcal{M}_{\mathfrak{m}}$ believes that c and d are $2^{\mathfrak{m}+1}$ distance part. Observe $2^{\mathfrak{m}+1} > \mathfrak{m}$. Therefore $\mathcal{M}_{\mathfrak{m}} \models \delta_i$ for every $i \leq q$. Observe σ_i are sentences which do not use the symbols c and d, and $\mathcal{M} \models \sigma_i$. Observe $\mathcal{M}_{\mathfrak{m}}$ has the same domain as $\mathcal{M}$ and $\mathcal{M}_{\mathfrak{m}}$ interprets P exactly the same. Therefore $\mathcal{M}_{\mathfrak{m}} \models \sigma_i$ for every $i \leq p$. Therefore $\mathcal{M}_{\mathfrak{m}} \models \Delta_0$. That is to say, Δ_0 is satisfiable. As a result, the union is finitely satisfiable.

By the compactness theorem, the union is satisfiable. That is to say, there exists an $\mathscr{L}$-structure $\mathfrak{M}$ such that $\mathfrak{M} \models \Sigma \cup \Delta$.

Claim. $\mathfrak{M}$ and $\mathcal{M}$ are elementarily equivalent. That is to say, for each $\mathcal{L}$-sentence φ, $\mathfrak{M} \models \varphi$ iff $\mathcal{M} \models \varphi$.

Let φ be an arbitrary $\mathcal{L}$-sentence.[1]

($\Rightarrow$) Assume $\mathfrak{M} \models \varphi$. Therefore $\mathfrak{M} \models \neg\neg\varphi$. Therefore $\mathfrak{M} \not\models \neg\varphi$. By contradiction, suppose $\mathcal{M} \not\models \varphi$. Therefore $\mathcal{M} \models \neg\varphi$. Therefore $\neg\varphi \in \Sigma$. Because $\mathfrak{M} \models \Sigma$, $\mathfrak{M} \models \neg\varphi$—contradiction. Therefore $\mathcal{M} \models \varphi$.

($\Leftarrow$) Assume $\mathcal{M} \models \varphi$. Therefore $\varphi \in \Sigma$. Because $\mathfrak{M} \models \Sigma$, $\mathfrak{M} \models \varphi$.

Claim. $\mathfrak{M}$ is not connected.

Because $\mathfrak{M} \models \Delta$, for each natural number n, $\mathfrak{M} \models \varphi_n$. Therefore $c^{\mathfrak{M}}$ and $d^{\mathfrak{M}}$ are not connected. Therefore $\mathfrak{M}$ is not connected. □

[1] The reason φ cannot be an $\mathscr{L}$-sentence is because $\mathcal{M}$ has no interpretation for c and d. Recall an $\mathscr{L}$-sentence may have the symbols c and d.