HOMEWORK 7 ANSWER KEYS

PROBLEMS FROM THE NOTES: 6.1.3, 6.1.7, 6.2.3, 6.2.4, 6.3.2, 6.3.5, 6.3.6, 7.1.3, 7.1.5, 7.2.2, 7.2.4

Problem 6.1.7 (a,c): (a) Suppose (for contradiction) that the cardinality of $(a,b) = \{a, \{a,b\}\}$ is 1. This must mean that $a = \{a,b\}$. This implies (by the definition of two sets being equal) $\forall x(x \in \{a,b\} \to x \in a)$. Since $a \in \{a,b\}$, we get that $a \in a$. This contradicts the Axiom of Regularity.

(c) (We use the results of parts (a,b) in what follows). We want to show the second conclusion in part (b) is impossible. Suppose it holds, i.e. $a = \{c,d\}$ and $c = \{a,b\}$. The fact that $c = \{a,b\}$ implies $a \in c$. The fact that $a = \{c,d\}$ implies $c \in a$. So we have $a \in c \in a$. This again violates the Axiom of Regularity.

Remark: We did not use the full Axiom of Regularity, but two consequences of the axiom, namely: (i) $\forall a \neg(a \in a)$, and (ii) $\forall a \forall b \neg(a \in b \in a)$.

Problem 6.2.3 (a,b): (a) FALSE. Counterexample: let $A = \{7, \{7\}\}$. In this case, $\mathcal{P}(A) = \{\emptyset, \{7\}, \{\{7\}\}, A\}$. Clearly, $\{7\} \in \mathcal{P}(A)$, $7 \in A$ (by definition of A) but also $\{7\} \in A$.

(b) TRUE. We have shown in class that for finite B, if $|B| = n$, then $|\mathcal{P}(B)| = 2^n$. Since $A \subsetneq \mathcal{P}(B)$ and $|A| = 2$, $|\mathcal{P}(B)| \geq 3$. Since 3 is not 2^n for any n, $|\mathcal{P}(B)| \geq 4$. Now since $\mathcal{P}(B) \subsetneq C$, $|C| \geq 5$. This is what we want.

Problem 6.3.5: $\bigcup_{x\in\mathbb{R}} A_x = \mathbb{R}$. Since $A_x \subseteq \mathbb{R}$, $\bigcup_{x\in\mathbb{R}} A_x \subseteq \mathbb{R}$. To see the $\supseteq$-direction, let $z \in \mathbb{R}$ be arbitrary, we want to show $z \in \bigcup_{x\in\mathbb{R}} A_x$. It is enough to find some $x \in \mathbb{R}$ such that $z \in A_x$ (by the the definition of $\bigcup_{x\in\mathbb{R}} A_x$). Let $x = z - 1$. Then since $z > z - 1 = x$, $z \in A_x$ as desired.

$\bigcap_{x\in\mathbb{R}} A_x = \{3, -2\}$. To see the $\supseteq$-direction, observe that $\{3, -2\} \subseteq A_x$ for all $x \in \mathbb{R}$ by the definition of A_x. To see the $\subseteq$-direction, it is enough to show that for $z \neq 3, -2$, $z \notin \bigcap_{x\in\mathbb{R}} A_x$. Fix such a z, to see that $z \notin \bigcap_{x\in\mathbb{R}} A_x$, it is enough to show that there is some $x \in \mathbb{R}$ such that $z \notin A_x$ (this uses the de Morgan's law for sets: $\neg(\bigcap_x A_x) = \bigcup_x \neg A_x$). Let $x = z$, then $A_x = A_z = \{3, -2\} \cup \{y \in \mathbb{R} : y > z\}$. It is easy, using the fact that $x3, -2$, to see that $z \notin A_z$.

Problem 7.1.5: The answer is 6. To see this, notice that the relation R is a subset of $A \times A$. Since $|A| = 4$, $|A \times A| = 4.4 = 16$. Write $A = \{a, b, c, d\}$. Write $A \times A$ as $A_1 \cup A_2 \cup A_3$, where

- $A_1 = \{(a,a), (b,b), (c,c), (d,d)\}$
- $A_2 = \{(a,b), (a,c), (a,d), (b,c), (b,d), (c,d)\}$
- $A_3 = \{(b,a), (c,a), (d,a), (c,b), (d,b), (d,c)\}$.

$|A_1| = 4$, $|A_2| = |A_3| = 6$. Also note that in order for $R \cap R^{-1} = \emptyset$, R cannot contain any element of A_1. This implies $|R| \leq 12$. Now observe that for $x, y \in A$, $(x,y) \in A_2$ iff $(y,x) \in A_3$. Also, if

$(x, y) \in R$, then (y, x) cannot be in R. This shows that $|R| \leq 6$. $|R|$ can be 6, for example we can let $R = A_2$.

Problem 7.2.2 (a,b): (a) R is NOT a function. For example, $(0, 1)$ and $(0, -1)$ are both in R.

(b) S is a function. This is because for each $x \in [-1, 1]$, the unique element $y \in [0, 1]$ satisfying $x^2+y^2 = 1$ is $y = \sqrt{1 - x^2}$. The function S is SURJECTIVE because for any $y \in [0, 1]$, $x = \sqrt{1 - y^2}$ is such that $(x, y) \in S$. S is NOT INJECTIVE because $(-1, 0)$ and $(1, 0)$ are both in S.