HOMEWORK 8 ANSWER KEYS

PROBLEMS FROM THE NOTES: 7.3.1, 7.3.4, 7.3.5, 7.4.3, 7.5.3, 7.6.1, 7.6.2, 7.6.5

Problem 7.3.1 (a,c): (a) $R = \{(1,1),(2,2),(3,3)\}$ is both symmetric and antisymmetric.

(c) $R = \{(1,2),(2,3),(1,3)\}$ is transitive. Now $R^{-1} = \{(2,1),(3,2),(3,1)\}$. $R \cup R^{-1} = \{(1,2),(2,3),(1,3),(2,1),(3,2),(3,1)\}$ is NOT transitive because: $(3,1)$ and $(1,3)$ are in $R \cup R^{-1}$ but $(3,3) \notin R \cup R^{-1}$.

Problem 7.3.5(a): $aRb \Leftrightarrow a \equiv b \pmod 3$ or $a \equiv b \pmod 4$

Is R reflexive: YES. Because for all $a \in \mathbb{Z}$, $a \equiv a$ (mod n) for all n.

Is R symmetric: YES. Suppose aRb. This means either $a \equiv b \pmod 3$ or $a \equiv b \pmod 4$. If the former holds, then $b \equiv a \pmod 3$. So $b \equiv a \pmod 3$ or $b \equiv a \pmod 4$. So bRa. If the latter holds, then similarly, we get $b \equiv a \pmod 4$, hence $b \equiv a \pmod 4$ or $b \equiv a \pmod 3$. Hence bRa.

Is R transitive: NO. Because $1 \equiv 4 \pmod 3$ and $4 \equiv 8 \pmod 4$ but $\neg(1 \equiv 8) \pmod 3$ and $\neg(1 \equiv 8) \pmod 4$.

Problem 7.4.3(a,b): (a) R is reflexive, symmetric, but is NOT transitive. To see why R is not transitive, just note that, for example, $(2,1) \in R$ and $(1,3) \in R$ but $(2,3) \notin R$.

(b) $A_1 = \{1,2,3\}$.

$A_2 = \{1,2\}$.

$A_3 = \{1,3\}$.

$\{A_1, A_2, A_3\}$ do not form a partition of X because (even though $\bigcup_i A_i = X$) they are not pairwise disjoint, e.g. $A_1 \cap A_2 = \{1,2\} \neq \emptyset$.

Problem 7.5.3(c): I just do this for $\oplus$ (the $\otimes$ is similar and easier; grader: please just grade on the work on $\oplus$).

Suppose $(a,b) \sim (c,d)$ and $(x,y) \sim (z,t)$. We want to see that $[(a,b)] \oplus [(x,y)] = [(c,d)] \oplus [(z,t)]$.

By the definition of $\sim$:

- $(a,b) \sim (c,d)$ means $ad = bc$
- $(x,y) \sim (z,t)$ means $xt = yz$.

Now, by the definition of $\oplus$:

- $[(a,b)] \oplus [(x,y)] = [(ay+bx, by)]$.
- $[(c,d)] \oplus [(z,t)] = [(ct+dz, dt)]$.

To see that $[(ay+bx, by)] = (ct+dz, dt)]$, we need to see that

$$(ay+bx)dt = (ct+dz)by.$$

Now $(ay+bx)dt = (ad)(yt)+(bd)(xt) = (bc)(yt)+(bd)(xt) = (bc)(yt)+(bd)(yz) = (ct)(by)+(by)(dz) = (ct+dz)by$. As desired.

Problem 7.6.1: $f : \mathbb{Z}_3 \to \mathbb{Z}_5$: $x \mapsto x^3 \pmod 5$ is NOT well-defined.

Counter-example: $[2]_3 = [5]_3$ (i.e. $2 \equiv 5 \pmod 3$). f(2) $= 2^3 = 8 \equiv 3 \pmod 5$, but $f(5) = 5^3 = 125 \equiv 0 \pmod 5$. Clearly $f(2) \neq f(5)$ even though $[2]_3 = [5]_3$.