**SOLUTIONS TO SELECTED PROBLEMS FROM HOMEWORK 6**

**Additional problem 2:** (We assume the sets $A, B$ are nonempty. Otherwise, there is nothing to do.) Given injection $f : A \to B$, one can define surjection $g : B \to A$ by fixing some $a \in A$ and let $h(x) = a$ for all $x \in B \setminus \text{rng}(f)$ if $B \setminus \text{rng}(f)$ is nonempty; if $B = \text{rng}(f)$ then simply let $g = f^{-1}$.

Let $g : B \to A$ be surjective. For each $a \in A$, choose an element $b_a$ of $g^{-1}(a)$; recall $g^{-1}(a) = \{b \in B : g(b) = a\}$. Notice that if $a_0 \neq a_1$ then $g^{-1}(a_0) \cap g^{-1}(a_1) = \emptyset$ because $g$ is a function. Indeed any $c \in g^{-1}(a_0) \cap g^{-1}(a_1)$ would have the property that $g(c) = a_0 \neq a_1 = g(c)$. Of course, this violates the fact that $g$ is a function.

Now define $f : A \to B$ as: $f(a) = b_a$. Since $a_0 \neq a_1$, then $b_{a_0} \neq b_{a_1}$, we get that $f$ is injective.

**Problem 6.1.7:** (a) The key point is that to show $(a, b)$ has cardinality 2, we just need to see that $a \neq \{a, b\}$. But if $a = \{a, b\}$, then $a \in a$. This contradicts the regularity axiom.

(b) Assume $(a, b) = (c, d)$. By definition, we get that $(a = c \ \text{AND} \ \{a, b\} = \{c, d\}) \ \text{OR} \ (a = \{c, d\} \ \text{AND} \ \{a, b\} = c)$. This is as desired.

(c) The main point is that we want to rule out the case $a = \{c, d\}$ and $c = \{a, b\}$. If this holds, then $c \in a$ and $a \in c$. So $a \in c \in a$. This again contradicts the axiom of regularity. So the other case has to hold, namely $a = c \ \text{AND} \ b = d$.

**Problem 6.2.7:** $f$ is one-to-one: if $(X, i), (Y, j)$ are in $\mathcal{P}(A) \times \{1, 2\}$ and are distinct, then:

(a) $i = j = 1$ and $X \neq Y$: in this case, $f(X, i) = X \neq f(Y, j) = Y$, or

(b) $i = j = 2$ and $X \neq Y$: in this case, $f(X, i) = X \cup \{b\}$ and $f(Y, j) = Y \cup \{b\}$. Since $b$ is not in $A$ and $X, Y$ are subsets of $A$ and $X, Y$ are distinct, $X \cup \{b\} \neq Y \cup \{b\}$, or

(c) $i \neq j$: say $i = 1$ and $j = 2$. Then $f(X, i) = X$ and $f(Y, j) = Y \cup \{b\}$. Since $b$ is not in $X$, $Y \cup \{b\} \neq X$.

$f$ is onto: every $Z \in \mathcal{P}(B)$ is either in $\mathcal{P}(A)$ or else $b \in Z$. In the first case, $Z = f(Z, 1)$. In the second case, write $Z = Y \cup \{b\}$, then $Z = f(Y, 2)$.

**Problem 8.1.9:** $g$ is a bijection because:

$g$ is surjective: let $(q, r)$ in $\mathbb{Q} \times \mathbb{Q}$, since $q \in \mathbb{Q}$, there is some $m \in \mathbb{N}$ such that $f(m) = q$ because $f$ is surjective. Similarly, there is some $n \in \mathbb{N}$ such that $f(n) = r$. So $g(m, n) = (f(m), f(n)) = (q, r)$.

$g$ is injective: suppose $g(m, n) = g(l, k)$. Then $(f(m), f(n)) = (f(l), f(k))$. So $f(m) = f(l)$ and $f(n) = f(k)$. So $m = l$ and $n = k$ because $f$ is injective. So $(m, n) = (l, k)$. We’re done.

**Additional problem 2:** We define a bijection $f : (-1, 1) \to \mathbb{R}$. There are many choices for $f$, but one of them is: $f(x) = \frac{1}{1-x}$.

$f$ is one-to-one: Suppose $f(a) = f(b)$. So $a = b$. If $\frac{a}{1-|a|} = \frac{b}{1-|a|}$, then it’s easy to see that $a = b = 0$. If $\frac{a}{1-|a|} > 0$, then it’s easy to see that $a, b > 0$ (this is because
a, b \in (-1, 1)). So |a| = a and |b| = b and hence \( \frac{a}{1-a} = \frac{b}{1-b} \). This gives, \( b(1 - a) = a(1 - b) \). So \( b - ab = a - ab \). So \( a = b \). Finally, assume \( \frac{a}{1-|a|} = \frac{b}{1-|b|} < 0 \). We now conclude \( a, b < 0 \). So |a| = −a and |b| = −b. Hence, \( \frac{a}{1+a} = \frac{b}{1+b} \). So \( a(1 + b) = b(1 + a) \). So \( a = b \). Therefore, \( f \) is one-to-one.

**f is onto:** Let \( r \in \mathbb{R} \). Since \( f(0) = 0 \), we may assume \( r \neq 0 \). If \( r > 0 \), then \( r \frac{1}{1+r} \in (0, 1) \). It’s easy to see that \( f(r \frac{1}{1+r}) = \frac{r}{1+r} \). This is because \( f(r \frac{1}{1+r}) = r \frac{1}{1+r} \). Similarly, for \( r < 0 \), \( r \frac{1}{1+r} \in (-1, 0) \) and \( f\left(\frac{r}{1+r}\right) = r \).

So we just verified that \( f \) is a bijection.

**Problem 6.3.4:** (You should draw pictures of the sets involved.)

(b) The formula for the set \( A_n \) is: \( A_n = (-\frac{2n+1}{n}, 2n) \) for \( n \geq 1 \). So \( A_1 = (-1, 2) \), \( A_2 = (-\frac{3}{2}, 4) \), \( A_3 = (-\frac{5}{3}, 6) \) etc.

\[ \bigcup_{n \geq 1} A_n = (-2, +\infty) \] This is because \( \lim_{n \to +\infty} 2n = +\infty \) and \( \frac{-2n+1}{n} = -2 + \frac{1}{n} \). and \( \lim_{n \to +\infty} (-2 + \frac{1}{n}) = -2 \).

\[ \bigcap_{n \geq 1} A_n = (-1, 2) \] This is because \((-1, 2) \subset (-\frac{3}{2}, 4) \subset (-\frac{5}{3}, 6) \subset \ldots \).

**Problem 6.3.7:** (a) Suppose \( A_1 \supseteq A_2 \supseteq A_3 \ldots \). We show \( \bigcup_{n \geq 1} A_n = A_1 \).

Clearly, \( A_1 \subseteq \bigcup_{n \geq 1} A_n \). For the converse, it suffices to prove:

(†): for all \( n \geq 1 \), \( A_n \subseteq A_1 \).

Assuming (†), we show \( \bigcup_{n \geq 1} A_n \subseteq A_1 \). Let \( x \in \bigcup_{n \geq 1} A_n \). By definition, there is some \( n \geq 1 \) such that \( x \in A_n \). By (†), \( A_n \subseteq A_1 \), so \( x \in A_1 \).

Now we prove (†) by induction on \( n \geq 1 \). The base case when \( n = 1 \) is obvious because \( A_1 \subseteq A_1 \).

Now suppose \( n \geq 1 \) and \( A_n \subseteq A_1 \). We want to show \( A_{n+1} \subseteq A_1 \). By the hypothesis, \( A_{n+1} \subseteq A_n \).

But \( A_n \subseteq A_1 \). So \( A_{n+1} \subseteq A_n \subseteq A_1 \). So \( A_{n+1} \subseteq A_1 \). We’re done.