MATH 150 PRACTICE PROBLEMS FOR FINAL

1. Determine if the following are tautologies:

(a) $(R \to (S \vee Q)) \vee (R \vee (S \to Q))$

(b) $(R \leftrightarrow P) \vee (P \to \neg R)$.

Answer: Both are tautologies. You should be able to check this using truth tables (or any of the shortcut methods we discussed). You should work these out.

2. The soundness theorem says that:

(a) If $\Gamma \vdash$ then $\Gamma \models \varphi$.

(b) If Γ is satisfiable (i.e. there is some model $\mathfrak{M} \models \Gamma$), then Γ is consistent.

Show that the two statements are equivalent.

Answer:
$(a) \Rightarrow (b)$: Suppose (a) holds and Γ is satisfiable. Fix some model $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$. We have to show that Γ is consistent. Suppose otherwise i.e. there is some formula β such that $\Gamma \vdash \beta \wedge \neg\beta$. Then by (a) it follows that $\Gamma \models \beta \wedge \neg\beta$. Then since $\mathfrak{A} \models \Gamma$, we get that $\mathfrak{A} \models \beta \wedge \neg\beta$, but that is a contradiction. Therefore Γ is consistent.

$(b) \Rightarrow (a)$: Suppose (b) holds and $\Gamma \vdash \phi$. We have to show that $\Gamma \models \phi$. Suppose otherwise i.e. $\Gamma \not\models \phi$. Then there is some model $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma \cup \{\neg\phi\}$ i.e. $\Gamma \cup \{\neg\phi\}$ is satisfiable (you can use this fact on the final; you can also try to prove it!!!). Then by (b) is follows that $\Gamma \cup \{\neg\phi\}$ is consistent. Therefore $\Gamma \nvdash \phi$, which is a contradiction with the assumptions. It follows that $\Gamma \models \phi$.

3. The completeness theorem says that:

(a) If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$ for all φ.

(b) If Γ is consistent, then Γ is satisfiable.

Show that the two statements are equivalent.

Answer: $(a) \Rightarrow (b)$: Suppose (a) holds and Γ is consistent. We have to show that Γ is satisfiable. Suppose otherwise i.e. for any model $\mathfrak{A}, \mathfrak{A} \not\models \Gamma$. Then we have (vacuously) that $\Gamma \models \beta \wedge \neg\beta$. Then by (a) it follows that $\Gamma \vdash \beta \wedge \neg\beta$. But that is a contradiction, since Γ was assumed to be consistent. Therefore Γ is satisfiable.

$(b) \Rightarrow (a)$: Suppose (b) holds and $\Gamma \models \phi$. We have to show that $\Gamma \vdash \phi$. Suppose otherwise i.e. $\Gamma \nvdash \phi$. Then $\Gamma \cup \{\neg\phi\}$ is consistent by RAA (Cor 24 E on pg 119, again this is a useful fact to keep in mind). Then by (b) it follows that $\Gamma \cup \{\neg\phi\}$ is satisfiable i.e. there is some model $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$ and $\mathfrak{A} \models \neg\phi$. Therefore $\Gamma \not\models \phi$, which is a contradiction with the assumptions. It follows that

$\Gamma \vdash \phi$.

4. The compactness theorem says that:

(a) If $\Gamma \models$ then $\Gamma_0 \vdash \varphi$ for some finite $\Gamma_0\Gamma$.

(b) If every finite subset of Γ is satisfiable, then so is Γ.

Show that the two statements are equivalent.

Answer: $(a) \Rightarrow (b)$: Suppose (a) holds and every finite subset of Γ is satisfiable. We have to show that is satisfiable. Suppose otherwise i.e. for any model $\mathfrak{A}$, $\mathfrak{A} \not\models \Gamma$. Then we have (vacuously) that $\Gamma \models \beta \wedge \neg\beta$. Then by (a) it follows that for some finite $\Delta \subset \Gamma$, $\Delta \models \beta \wedge \neg\beta$. But that is a contradiction, since Δ is assumed to be satisfiable. Therefore Γ is satisfiable.

$(b) \Rightarrow (a)$: Suppose (b) holds and $\Gamma \models \phi$. We have to show that for some finite $\Delta \subset \Gamma$, $\Delta \models \phi$. Suppose otherwise. Then for every finite subset Δ of Γ, $\Delta \cup \{\neg\phi\}$ is satisfiable. Then by (b) it follows that $\Gamma \cup \{\neg\phi\}$ is satisfiable i.e. there is some model $\mathfrak{A}$ such that $\mathfrak{A} \models \Gamma$ and $\mathfrak{A} \models \neg\phi$. Therefore $\Gamma \not\models \phi$ which is a contradiction with the assumptions. It follows that $\Gamma \vdash \phi$.

5. Consider the following extension of the language of rings.

- $\mathcal{L}_f = \{\dot{+}, \dot{\times}, \dot{0}, \dot{1}, \dot{<}, \dot{f}\}$ is the language of rings with an additional binary relation symbol $\dot{<}$ and a unary function symbol $\dot{f}$.

Consider the structure

$$\mathfrak{R} = (\mathbb{R}, +, \cdot, 0, 1, <, f)$$

where $\mathbb{R}$ is the set of all real numbers, and the interpretations of symbols $\dot{+}, \dot{\times}, \dot{0}, \dot{1}, \dot{<}$ in these structures are natural: $0, 1$ are numbers "zero" and "one", $+$ and $\cdot$ are usual addition and multiplication, and $<$ is the usual ordering of real numbers. Additionally, $\dot{f}$ is interpreted in $\mathfrak{R}$ as a unary function $f : \mathbb{R} \to \mathbb{R}$.

Express the following statements about numbers and function f in the structure $\mathfrak{R}$ as instructed below.

(a) Find an $\mathcal{L}$-formula $\varphi(u, v)$ which expresses:

"v is a local minimum of f at u".

(b) Find an $\mathcal{L}$-sentence τ which expresses:

"The set of arguments (i.e. points) at which f has local minimum is unbounded".

Now suppose $f(x) = e^x$ for all $x \in \mathbb{R}$; let $s : V \to \mathbb{R}$ be the evaluation of variables such that $s(x_{2k+1}) = 1$ for all $k \in \mathbb{N}$. Let t be the term

"$\dot{\times}(\dot{f}(\dot{1}), \dot{+}(\dot{1}, \dot{f}(v_1)))$"

and $\varphi(x)$ be the formula:

$$\exists v_2(v_2 = \dot{f}(x)).$$

(c) Evaluate $t^{\mathfrak{R}}[s]$.

(d) Is t substitutable for x in φ? If so, determine whether $\mathfrak{R} \vDash \varphi(x/t)[s]$.

Explanations. Number b is a local minimum of a function f at a iff $f(a) = b$, and there is an open interval (x, y) containing a such that $f(z) \geq b$ for all $z \in (x, y)$. A set $X \subseteq \mathbb{R}$ is unbounded iff X has some element outside of any open interval (x, y) where $x, y \in \mathbb{R}$.

Answer:

(a) $\varphi_{min}(u, v) =$ "$f(u) = v \wedge \exists x \exists y(x < u \wedge u < y \wedge \forall z(x < z \wedge z < y \rightarrow \neg(f(z) < v))$".

(b) $\varphi_{unbounded} =$ "$\forall z \exists u \exists v(\varphi_{min}(u, v) \wedge z < u)$"

(c) The only variable that shows up in t is v_1 and $s(v_1) = 1$. Now $f(1) = e^1 = e$. So the value of the term $t^{\mathfrak{R}}[s]$ is: $e^1 \times (1 + e^1) = e(1 + e) = e + e^2$.

(d) t is substitutable for x in φ. This is because there is only free occurrence of x in φ and when we replace x by t, we get the formula $\exists v_2(v_2 = \dot{f}(\dot{\times}(\dot{f}(\dot{1}), \dot{+}(\dot{1}, \dot{f}(v_1)))))$. Note that the variable v_1 is still free in $\varphi(x/t)$. So we did verify t is substitutable for x in φ.

To check $\mathfrak{R} \vDash \varphi(x/t)[s]$, we simply need to find a value $d \in \mathbb{R}$ such that when we extend s by s' such that $s'(v_2) = d$, $\mathfrak{R} \vDash (v_2 = \dot{f}(t))[s']$. Since by part (c), $t^{\mathfrak{R}}[s] = e + e^2$, hence $(\dot{f}(t))^{\mathfrak{R}} = e^{e+e^2}$. Let $s'(v_2) = e^{e+e^2}$, then we found the value desired value d.

6. Consider a language $\mathcal{L}$ with a 2-ary predicate symbol $\dot{<}$. Let $\mathfrak{N} = (\mathbb{N}; <)$ be the structure of $\mathcal{L}$ consisting of the natural numbers with the usual ordering. Show that one cannot express the following statement in English

"There is no infinite descending chain."

by a sentence in the language $\mathcal{L}$. **Hint.** You may want to use the Compactness Theorem here. Think about what would happen if you could express the statement by a sentence τ in $\mathcal{L}$. Does $\mathfrak{N} \vDash \tau$? Can you find a model of $\mathcal{L}$ that satisfies τ?

Answer: This is a typical application of the compactness theorem (though this one is a bit tricky). Suppose τ is a sentence in $\mathcal{L}$ that expresses the above statement. Note that $\mathfrak{N} \vDash \tau$.

Now let $\Sigma = \{\tau\} \cup \{\phi_n : n \geq 2\}$, where

$$\phi_n = \text{"}\exists x_1 \ldots \exists x_n(x_n < x_{n-1} \wedge x_{n-1} < x_{n-2} \wedge \cdots \wedge x_2 < x_1)\text{"}.$$

Σ is finitely satisfiable. This is because if $\Delta \subset \Sigma$ is finite, then $\mathfrak{N} \vDash \Delta$ (if $\tau \in \Delta$, then we already assume $\mathfrak{N} \vDash \tau$; if $\phi_n \in \Delta$ for some n, then letting N be the largest k such that $\phi_k \in \Delta$, then $\mathfrak{N} \vDash \phi_N[s]$ where $s(x_N) = 1, s(x_{N-1}) = 3, \ldots, s(x_1) = N$).

By compactness, Σ has a model $\mathfrak{M}$ and some evaluation s of the variables $\{x_n : n \in \mathbb{N}\}$ such that

(i) $\mathfrak{M} \vDash \tau$ (so there is NO infinite descending sequence in $\mathfrak{M}$); and

(ii) for all $n \geq 1$, $\mathfrak{M} \vDash \phi_n[s]$. This implies then there IS an infinite descending sequence in $\mathfrak{M}$.

Clearly, the two items above give us a contradiction. So it must be the case that the statement "there is no infinite descending chain" cannot be expressed by a sentence in $\mathcal{L}$.

7. Show that $\{\forall x(\alpha \to \beta), \exists x\alpha\} \vDash \exists x\beta$.

Answer: Suppose that $\mathfrak{A} \vDash \{\forall x(\alpha \to \beta), \exists \alpha)\}[s]$, where $\mathfrak{A}$ is a model and s is an evaluation of variables. We have to show that $\mathfrak{A} \vDash \exists x\beta[s]$.

$\mathfrak{A} \vDash \exists x\alpha[s]$, so fix an element d in the universe of $\mathfrak{A}$, such that $\mathfrak{A} \vDash \alpha[x/d]$ (in other words, we extend s to s' such that $s'(x) = d$ and such that $\mathfrak{A} \vDash \alpha[s']$. The existence of d, or equivalently s', of course follows from the fact that $\mathfrak{A} \vDash \exists x\alpha[s]$. Then since $\mathfrak{A} \vDash \forall x(\alpha \to \beta)[s]$, we have that $\mathfrak{A} \vDash (\alpha \to \beta)[x/d]$, and so $\mathfrak{A} \vDash \beta[x/d]$. It follows that $\mathfrak{A} \vDash \exists x\beta[s]$.

8. Let $\mathfrak{A} = (\mathbb{R}; +, \times)$ be an $\mathcal{L}$-structure, here $\mathcal{L}$'s nonlogical symbols are $\{\dot{+}, \dot{\times}\}$. Define the following sets in the structure $\mathcal{A}$.

(a) $\{0\}$.

(b) $\{1\}$.

(c) $\{3\}$.

(d) The interval $(0, \infty)$.

(e) $\{\langle r, s\rangle \mid r \leq s\}$ (here r, s are reals, of course).

Answer:

(a) $\phi_0(x) =$ "$x + x = x$".

(b) $\phi_1(x) =$ "$x \times x = x \wedge \neg(x + x = x)$".

(c) $\phi_3(x) =$ "$\exists y(\phi_1(y) \wedge x = y + y + y$".

(d) $\phi_{(0,\infty)}(x) =$ "$\exists y(y \times y = x)?\neg(x + x = x)$".

(e) $\psi(r, s) =$ "$\exists y \exists x(y = x \times x \wedge s = r + y)$". (the clause "$y = x \times x$" just says "$y$ is nonnegative").

9. Let $\mathfrak{A} = (\mathbb{N}; 0, 1, +, \times)$. Give a formula in the language of $\mathfrak{A}$ which defines the following. (Notice here that the language of $\mathfrak{A}$ only consists of the following non-logical symbols: $\dot{0}, \dot{1}, \dot{+}, \dot{\times}$).

(a) $\{2\}$.

(b) $\{n \mid n \text{ is even}\}$.

(c) $\{\langle m, n\rangle \mid m \text{ divides } n\}$.

(d) $\{n \mid n \text{ is a prime}\}$.

Answer: Note here our language does not have symbol $\dot{S}$ for the successor function. There are, of course, more than one way of expressing these statements above. Below, I just give you one solution.

(a) $\phi_2(x) =$ "$x = 1 + 1$".

(b) $\phi_{even}(x) =$ "$\exists y(x = y + y)$".

(c) $\phi_{divides}(x, y) =$ "$\exists z(y = x \times z)$".

(d) $\phi_{prime}(x) =$ "$\exists y \exists z(\phi_2(y) \wedge x = y + z \wedge \forall t(\phi_{divides}(t, x) \to t = 1 \vee t = x))$".

10. Assume that the language has a unary function symbol f. Find a sentence σ such that:

(a) for any model $\mathfrak{A}$, $\mathfrak{A} \models \sigma$ iff the universe of $\mathfrak{A}$ has at least two elements.

(b) for any model $\mathfrak{A}$, $\mathfrak{A} \models \sigma$ iff the universe of $\mathfrak{A}$ has exactly two elements.

(c) for any model $\mathfrak{A}$, $\mathfrak{A} \models \sigma$ iff $f^{\mathfrak{A}}$ is onto.

Answer:

(a) σ_0 is something like this: "$\exists x \exists y(\neg(x = y))$".

(b) σ_1 is: "$\phi_0 \wedge \neg(\exists x \exists y \exists z(\neg(x = y) \wedge \neg(x = z) \wedge \neg(y = z)))$".

(c) σ_2 is: "$\forall x \exists y(f(x) = y)$".

11. Consider the model ${}^*\mathfrak{R}$ discussed in class (and defined in Section 2.8). We also have standard structure $\mathfrak{R}$, where $|\mathfrak{R}| = \mathbb{R}$, $P_R^{\mathfrak{R}} = R, c_r^{\mathfrak{R}} = r, f_F^{\mathfrak{R}} = F$ for each relation symbol P_R, constant symbol c_r, and function symbol f_F. By the construction of ${}^*\mathfrak{R}$, $\mathfrak{R} \subset |{}^*\mathfrak{R}| =_{def} {}^*\mathbb{R}$. Let $<^* = P_<^{{}^*\mathfrak{R}}$.

(a) Show that for any $r, s \in \mathbb{R}$, there is some $t \in {}^*\mathbb{R} \backslash \mathbb{R}$ such that $r <^* t <^* s$.

(b) Show that there is $\epsilon \in {}^*\mathbb{R}$, $0 <^* \epsilon$ such that for positive $r \in \mathbb{R}$, $\epsilon <^* r$.

(c) Show that the set $\mathbb{R}$ is a bounded subset of ${}^*\mathbb{R}$. And there is no least upper bound for $\mathbb{R}$ in ${}^*\mathbb{R}$.

Answer: If you read section 2.8 carefully, you'll see that these are already answered there. Here I just provide you with the main points. I start with (b).

(b) By the construction of ${}^*\mathfrak{R}$, there is some element $c \in {}^*\mathbb{R}$ such that for all $r \in \mathbb{R}$, $r^*{<}c$. Now recall our language has the symbol "/" for "division"; so the interpretation ${}^*/ = f_/^{\mathfrak{R}}$ is defined and is the "division" operation in ${}^*\mathfrak{R}$. So it makes sense to let $\epsilon = 1^*/c$. We have:

  (i) $0^* < \epsilon$. This is because $0^*{<}c$ and the theory $Th(\mathfrak{R})$ of $\mathfrak{R}$ is satisfied by ${}^*\mathfrak{R}$. Part of $Th(\mathfrak{R})$ has the sentence $\forall x(0 < x \to 0 < 1/x)$.

  (ii) for any $r \in \mathbb{R}$, $\epsilon\ {}^*{<}r$. This is because $r\ {}^*{<}\ c$ and the theory $Th(\mathfrak{R})$ has the sentence "$\forall x \forall y(0 < x < y \to 0 < 1/y < /x)$".

(a) Let $w = s - r$. So $w \in \mathbb{R}$ and $w > 0$. Now use the result in part (b) to get some $\epsilon \in {}^*\mathbb{R}$ such that $0\ {}^*{<}\ \epsilon\ {}^*{<}\ w$. So $0\ {}^*{+}\ r\ {}^*{<}\ \epsilon\ {}^*{+}\ r\ {}^*{<}\ w\ {}^*{+}r$. So $r\ {}^*{<}\ \epsilon\ {}^*{+}\ r\ {}^*{<}\ s$. Notice two things here: first, $\hat{*}{+} \restriction \mathbb{R} = +$; and second, $\epsilon\ {}^*{+}\ r$ is the desired t because it is in ${}^*\mathbb{R} \backslash \mathbb{R}$ (why? if $t = \epsilon\ {}^*{+}\ r \in \mathbb{R}$, then $\epsilon = t\ {}^*{-}\ r = t - r \in \mathbb{R}$. This is a contradiction to the choice of ϵ in (a)).

(c) The first clause is clear because by construction, there is some $c \in^* \mathbb{R}$ such that for all $x \in \mathbb{R}(x\ {}^*{<}c)$. For the second clause, let c be the least upper bound for $\mathbb{R}$. Let $\epsilon = 1\ {}^*/\ c$. So ϵ has the following properties listed in part (b). Further:

  - $\epsilon \in \mathcal{F}$, where $\mathcal{F}$ is defined in Section 2.8. and is the set of all "finite elements" of ${}^*\mathbb{R}$.
  - ϵ is the largest element satisfying the conditions in (b).

Now let $0 \ {}^*\!< \delta \ {}^*\!< \epsilon$. $\delta = 1^*/d$ for some $c \ {}^*\!< d$. Let $\gamma = \delta \ {}^*\!+ \epsilon$. So $\epsilon \ {}^*\!<\gamma$. It remains to show γ satisfies (b). This would give us a contradiction to the fact that ϵ is the largest element satisfying (b).

To see γ satisfies (b). We use Corollary 28E and Theorem 28F. Note that 0 is the unique element such that $\epsilon \backsimeq 0$ and the same for δ. In other words, $0 = st(\epsilon)$ and $0 = (\delta)$. So by Theorem 28F, $st(\gamma) = st(\epsilon \ {}^*\!+\delta) = st(\epsilon) + st(\delta) = 0 + 0 = 0$. In other words, γ satisfies (b). So we are done.