Math 161 Modern Geometry Practice Homework

(1) Show that two hyperbolic lines cannot have more than one common perpendicular.

Proof. Let $(l), (m)$ be hyperbolic lines. Suppose $(n), (n')$ are two lines that are perpendicular to both (l) and (m). Let A, A' be the intersection of (n) with $(l), (m)$ respectively; B, B' intersection of (n') with $(l), (m)$ respectively. Then $AA'B'B$ is a rectangle. This contradicts the fact that there are no rectangles in hyperbolic geometry. $\square$

(2) Prove that the summit is always larger than the base in a Saccheri quadrilateral.

Proof. Given a Saccheri quadrilateral, take one of the Lambert quadrilaterals $ABCD$ comprising half of it, where $\angle ADC < 90°$.
Choose a point M along CD (extended if necessary) so that $|CM| = |AB|$. Now join AM. We now have a Saccheri quadrilateral $ABCM$ with base BC and summit AM.
Since the summit angles $\angle BAM = \angle CMA$ are acute, it follows that the line segment AM lies inside $ABCD$ and M lies on the segment CD.
The original half-summit CD is longer than the original half-base AB.
$\square$

(3) Draw a cevian line for a triangle $\triangle ABC$. Prove that the angle defect (π radians minus the sum of the angles in the triangle) is equal to the sum of the defects of the two sub-triangles created by the cevian line.

Proof. This is easy and has been done in class. $\square$

(4) Prove that two Saccheri quadrilaterals with congruent bases and summit angles must be congruent.
Hint: suppose not and show that you can construct a rectangle.

Proof. Place the equal summits on top of each other so that the bases lie on the same side of the common summit. Suppose the quadrilaterals are not congruent. That means the sides of one quadrilateral are longer than the sides of the other (note that because we assume the summit angles are congruent; the sides of the quadrilaterals line up). This easily gives a rectangle. Contradiction. $\square$

(5) Let l and m intersect at O at an acute angle. Let $A, B \neq O$ be points on l and drop perpendiculars to m from A and B, intersecting m at A', B'. If $OA < OB$, show that $AA' < BB'$.

Proof. If $|AA'| = |BB'|$, then $AA'B'B$ is a Saccheri quadrilateral. Hence $\angle A'AB$ is acute. This means, $\angle OAA'$ is obtuse, being complementary to $\angle OAA'$. But then the sum of the interior angles of $\triangle OAA'$ easily adds up to be more than two right angles. Contradiction.

If $|AA'| > |BB'|$, let M be between A, A' so that $|A'M| = |BB'|$. Then as before, $A'MBB'$ is a Saccheri quadrilateral. So $\angle AMB$ is acute. By the exterior angle theorem, $\angle AMB > \angle A'AB$. So $\angle A'AB$ is acute. Exactly as above, we get a contradiction. $\square$

(6) Prove that two Saccheri quadrilaterals with equal bases and equal summit angles must be congruent.
Hint: suppose not and show that you can construct a quadrilateral with angles summing to $360°$.

Proof. Place the equal bases on top of each other so that the summits lie on the same side of the common base. Suppose that the common summit angle is α and that the quadrilaterals are not congruent. We then have a quadrilateral with angles $\alpha, \alpha, 180° - \alpha, 180° - \alpha$, summing to $360°$. A contradiction. $\square$

(7) The point $P = (1, 1)$ is rotated through angle $\pi/6$ about the point $(2, 3)$ and then translated in the direction of $(1, 2)$ through a distance of 3 units. Find the coordinates of the resulting point.

Proof. Let f be the rotation about the point $(2, 3)$ map and let g be the translation in the direction of $(1, 2)$ through a distance of 3 units. Now we want to write down the formula for f, g.

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{x}_{fix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By the formula in class,

$$f(\vec{v}) = R_{\pi/6}(\vec{v} - \vec{x}_{fix}) + \vec{x}_{fix},$$

here recall that $R_{\pi/6} = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ is the matrix of rotation by $\pi/6$ around the origin.

So $f(\vec{v}) = \begin{bmatrix} 1 - \sqrt{3}/2 \\ -1/2 - \sqrt{3} \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 - \sqrt{3}/2 \\ 3/2 - \sqrt{3} \end{bmatrix}$.

Now for any vector $\vec{w}$, $g(\vec{w}) = \vec{w} + \frac{3}{||\vec{u}||}\vec{u} = \vec{w} + \frac{3}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So $g(f(\vec{v})) = \begin{bmatrix} 3 - \sqrt{3}/2 \\ 3/2 - \sqrt{3} \end{bmatrix} + \begin{bmatrix} 3/\sqrt{5} \\ 6/\sqrt{5} \end{bmatrix}$. One could simplify further, but this is good enough. $\square$

(8) Identify the product, f, of a reflection in the line $y = x - 1$, the rotation by angle π about (1, 1) and a glide in the y-axis through vector $(1, 2)$.

Proof. Note first that the first action f_1, reflection in $y = x - 1$, is $\tau \circ \mu_{\pi/4}$, here τ is some translation isometry (we could calculate this exactly, but let's hold that for now) and $\mu_{\pi/4}$ is the central reflection about the line $y = x$.

The second action f_2, rotation by π around $(1, 1)$, is of the form $\sigma \circ \rho_\pi$, where σ is some translation and ρ_π is rotation by π around the origin. Here, see the above problem, technically, $f_2 = \sigma_1 \circ \rho_\pi \circ \sigma_2$ for some translations σ_1, σ_2 (to see this, recall from the previous problem, for any vector $\vec{v}$, $f_2(\vec{v}) = \rho_\pi(\vec{v} - \vec{x}_{fix}) + \vec{x}_{fix}$, here $\vec{x}_{fix}$ is the fixed point of f_2).Then we use the theorem proved in class to write $\rho_\pi \circ \sigma_2$ as $\sigma_2' \circ \rho_\pi$ for some translation σ_2'; call this fact $(*)$. Finally, $\sigma = \sigma_1 \circ \sigma_2'$.

For f_3, the glide in the y-axis through $(1, 2)$, $f_3 = \delta \circ \mu_{\pi/2}$, here δ is translation by $(1, 2)$.

Using $(*)$ several times, $f_3 \circ f_2 \circ f_1$ can be written as $\epsilon \circ \mu_{\pi/2} \circ \rho_\pi \circ \mu_{\pi/4}$, where ϵ is a translation to be determined.

Now $\mu_{\pi/2} \circ \rho_\pi \circ \mu_{\pi/4}$ is a composition of central isometries, using the formulas in class, we get: the matrix for this composition is $M_\pi R_\pi M_{\pi/2} = M_\pi M_{3\pi/2} = M_{-\pi/2}$. So this is a central reflection about the line $x+y=0$.

Now to determine ϵ, we will try to compute the image of, say the point $(0,0)$, under the composition map. First, $f_3 \circ f_2 \circ f_1(0,0) = (0,5)$. Now $(0,5) = \sigma \circ M_{-\pi/2}(0,0) = \sigma(0,0)$. So σ is translation by vector $(0,5)$.

Conclusion: The map is the glide in the line $x+y=0$ through vector $(0,4)$.

□

(9) Identify the product of the reflection in the line $y = x+3$ followed by the glide in the line $-x+y=2$ through vector $(1,1)$.

Proof. The proof is similar to the above. Let f_1 be the first action, reflection in $y = x+3$. $f_1 = \sigma \circ \mu_{\pi/4}$ where σ is some translation. f_2, the glide in $y = x+2$ through $(1,1)$, is $\tau \circ \mu_{\pi/4}$, for some translation τ.

So $f_2 \circ f_1 = \epsilon \circ \mu_{\pi/4} \circ \mu_{\pi/4}$.

The matrix for $\mu_{\pi/4} \circ \mu_{\pi/4}$ is $M_{\pi/2} M_{\pi/2} = M_0$. In other words, this is the reflection about the x-axis. Now to identify ϵ, as above, compute $f_2 \circ f_1(-3,0) = f_2(-2,-1) = (-3,0) + (1,1) = (-2,1)$ (you can take other points, if that makes your calculations easier).

Now, say ϵ is translation by $\vec{v}$, we have: $\epsilon \circ \mu_0(-3,0) = \epsilon(-3,0) = (-3,0) + \vec{v} = (-2,1)$. So $\vec{v} = (-2,1) - (-3,0) = (1,1)$.

Conclusion: The map is glide in the x-axis through vector $(1,1)$.

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