Math 161 Modern Geometry Homework Questions 4

Due : Thursday, May 18, 2017

(1) Consider the stereographic projection described in class.
  (a) Consider the circle $(x-5)^2+(y-3)^2=1$ on the xy-plane (recall this is identified with the complex numbers). To what point (X,Y,Z) on the unit sphere is the center of this circle mapped by the stereographic projection?
  (b) Consider the plane $-7X+2Y+\frac{3}{2}Z=\frac{5}{2}$ in $\mathbb{R}^3$. Show that this plane intersects the unit sphere (**Hint:** compute the distance between $(0,0,0)$ and this plane). Let the intersection be the circle (c). Compute the coordinates of the center and the radius of the corresponding circle on the xy-plane by the stereographic projection (i.e. compute the equation of the image of the circle (c) under the stereographic projection).

(2) Consider points in the plane as ordered pairs (x,y) and consider the function f on the plane defined by $f(x,y)=(kx+a,ky+b)$ where k,a,b are fixed real constants and $k\neq 0$. Is f a transformation? Is f an isometry?

(3) Show that the matrix for the reflection map about the line through the origin that is inclined at the angle θ to the positive x-axis is

$$M_{2\theta}=\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix}.$$

(4) Let f be the composition of the reflection through the line $y=x$, followed by a rotation by $\pi/3$, and followed by a reflection through the y-axis. Identify f (i.e. determine whether f is a rotation or a reflection).

(5) We saw in class that every isometry can be thought of as a function $f_{A,\mathbf{c}}$: $\mathbf{x}\mapsto \mathbf{Ax}+\mathbf{c}$ where A is an orthogonal matrix and $\mathbf{c}$ is a constant vector. That is, every isometry is a combination of a rotation/reflection (multiplying by A) and a translation (adding $\mathbf{c}$). A rotation/reflection would have $\mathbf{c}=\mathbf{0}$, while a pure translation would have $A=I$ (the identity matrix).
  (a) Prove that composition works as follows $f_{A,\mathbf{c}}\circ f_{B,\mathbf{d}}=f_{AB,A\mathbf{d}+\mathbf{c}}$. Thus the composition of any two isometries is an isometry.
  (b) What is the inverse of the isometry $f_{A,\mathbf{c}}$? That is, if $f_{A,\mathbf{c}}\circ f_{B,\mathbf{d}}=f_{I,\mathbf{0}}$, where I is the identity matrix, then what are $B,\mathbf{d}$?
  (c) Compute the composition $f_{A,\mathbf{c}}\circ f_{I,\mathbf{d}}\circ f^{-1}_{A,\mathbf{c}}$,. *You should obtain a pure translation. This shows that translations form a normal subgroup of the group of isometries.*