Math 161 Modern Geometry Homework Answers 2

1. Argue by contrapositive. Suppose that the two angles at the base of $\triangle ABC$ are congruent: let these corners be B and C. Then $\triangle ABC$ and $\triangle ACB$ have congruent angle-side-angle and are thuse congruent triangles. It follows that AB and AC have the same length.

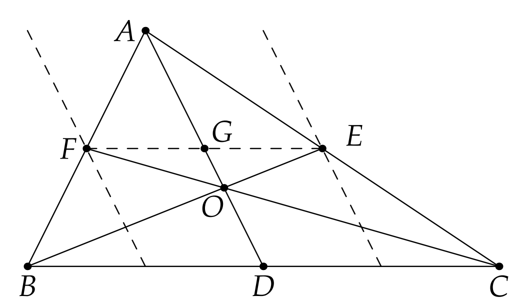
2. Counterexample to SSA: Consider $\triangle ABC$ such that there is a point D on BC such that $|AB| = |AD|$. Triangles $\triangle ABC$ and $\triangle ADC$ satisfy SSA: $|AB| = |AD|, |AC| = |AC|, \angle ACB = \angle ACD$. But these triangles are clearly not congruent.

   AAS is a criterion for congruent triangles. Suppose $\triangle ABC$ and $\triangle DEF$ are such that: $\angle A = \angle D$, $\angle B = \angle E$, and $|BC| = |EF|$. Notice: $m(\angle C) = \pi - m(\angle A) - m(\angle B) = \pi - m(\angle D) - m(\angle E) = m(\angle F)$. Now observe BC is between $\angle B$ and $\angle C$ and EF is between $\angle E$ and $\angle F$. Now we can use ASA to conclude $\triangle ABC$ is congruent to $\triangle DEF$.

3. (a) Consider the picture below. Since $\frac{AF}{FB} = \frac{AE}{EC}$ (indeed both are 1) we conclude[1] that $FE \parallel BC$. It follows that $\angle AFE = \angle ABC$ and that $\angle AEF = \angle ACB$. Thus the triangles $\triangle AFE$. $\triangle ABC$ are similar. Two of their sides are in the ratio 1:2, whence so are their other sides. Thus $FE : BC = 1 : 2$.

   (b) Use Ceva's Theorem. Each median is a cevian, and in each case divides each side in the ration 1:1. Thus each of the three ratios in Ceva's Theorem is 1, whence so is their product. The medians therefore meet at a point.

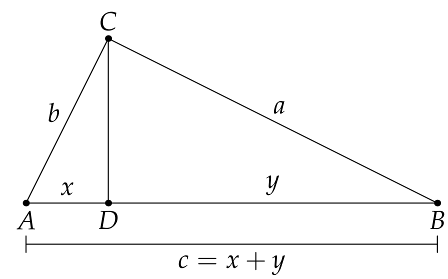
   (c) Let G be the intersection of FE and the median AD. Since $GE \parallel DC$, the triangles $\triangle ADC$ and $\triangle AGE$ have the same angles and are thus similar, indeed also in a ratio of 2:1. Whence AG is the median of $\triangle AFE$.
   Now draw parallels to AD through E and F. Since $FG = EG$ it follows that the altitudes (perpendiculars) from AG of the triangles $\triangle AFG$ and $\triangle AEG$ are the same length. Since they share a common base AG, these triangles have the same area.
   We have therefore shown that adjacent subtriangles have the same area. By permuting this around the original triangle we have the result.



   (d) Consider the picture again, now with the centroid labelled O. The triangle $\triangle AOB$ has twice the area as $\triangle AOE$, but has the same height (altitude from A to BE). Thus the bases must be in the ratio $BO : OE = 2 : 1$. The same is true for each median.

[1] Or use Side-Angle-Side Similarity to see that $\triangle ABC \sim \triangle AFE$ in a side length ratio of 2:1, though in the book this is a corollary of the result we're using.

4. (a) Simply label angles and use the fact that angles in a triangle add to a straight edge. It follows that $\angle CAD = \angle BCD$ and $\angle ACD = \angle CBD$. Since the other angle in each triangle is a right-angle, all three triangles are similar.



(b) Since the three triangles are similar, we know that the ratios of certain sides are equal: in particular

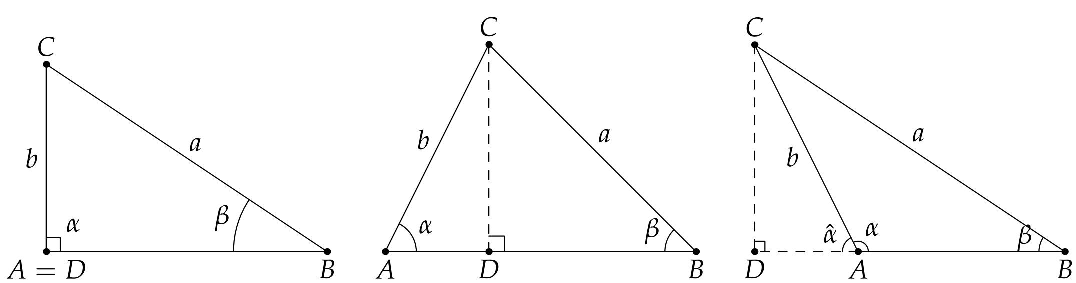
$$\frac{c}{a} = \frac{a}{y}, \qquad \frac{c}{b} = \frac{b}{x}.$$

It follows that

$$a^2 + b^2 = cx + cy = c^2.$$

5. (a) Suppose that a triangle has two or more obtuse angles. Then its interior angles sum to more than a two right-angles and thus more than a straight edge. A condtradiction. Therefore a triangle may have at most one obtuse angle.
If it has one obtuse angle then the remaining two must sum to less than a right-angle. Thus both are themselves acute. This is case 3.
If it has no obtuse angles, then consider the largest of them. At most, this is a right-angle. The remaining two sum to a right-angle and so each is acute. This is case 1. Otherwise the largest angle is acute, and so therefore are the other two. This is case 2.

(b) Drop a perpendicular from C to AB at D. Without loss of generality there are three cases to consider, depending on the location of the point D; those of part (a) yield the following three pictures.



Case 1 Case 2 Case 3

Now consider the formlae for sine in each case:

Case 1: $\sin\alpha = 1$ and $\sin\beta = \frac{b}{a}$.
Case 2: $\sin\alpha = \frac{CD}{b}$ and $\sin\beta = \frac{CD}{a}$.
Case 3: $\sin\alpha = \sin\hat{\alpha} = \frac{CD}{b}$ and $\sin\beta = \frac{CD}{a}$.

In each case some quick algebra justifies the Sine Rule.