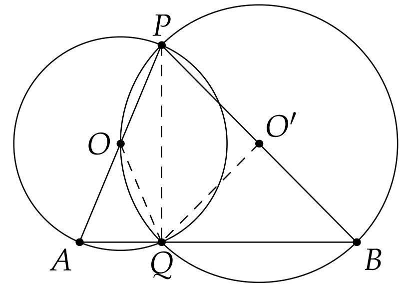
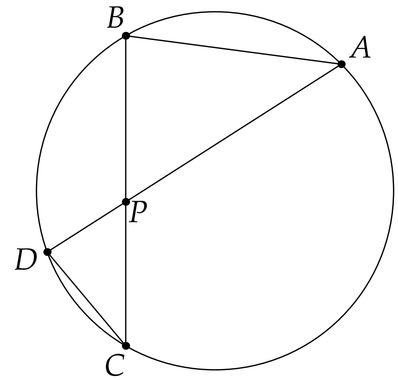
Math 161 Modern Geometry Homework Answers 3

1. Consider the picture. Since PA is a diameter, $\triangle PQA$ is a triangle in a semi-circle and so $\angle PQA$ is a right angle. Similarly $\angle PQB$ is a right angle. It follows that $\angle AQB$ is a straight edge.
   If you forgot this result, you can prove it along the way by cutting up everything into isosceles triangles and comparing angles.





2. Note that $\angle ADC = \angle ABC$ since the angles share the same arc BD. Similarly $\angle ABC = \angle ADC$ by sharing the arc AC. Finally $\angle APB = \angle CPD$ since these angles are vertical. It follows that we have similar triangles $\triangle ABP \sim \triangle CDP$. Whence the side lengths are in ratio:

$$\frac{AP}{BP} = \frac{PC}{PD} \implies (AP)(PD) = (BP)(PC)$$

3. (a) The vector joining A to B is $\mathbf{v} = \binom{b_1 - a_1}{b_2 - a_2}$. We must first get to the point A. Thus any point on the segment between A and B has position vector

$$\vec{A} + t\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = (1-t)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

   where t is some real number. Clearly $t = 0$ corresponds to A, and $t = 1$ to B. Any number $0 < t < 1$ corresponds to traversing a fraction of the line segment from A to B and is thus between the points.

   (b) By part (a), choosing $t = \frac{1}{2}$ gives the point $M := \left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)$ as being between A and B. No we need only check that it is the midpoint. For this, compute the distance

$$|AM| = \sqrt{\left(a_1 - \frac{a_1 + b_1}{2}\right)^2 + \left(a_2 - \frac{a_2 + b_2}{2}\right)^2} = \sqrt{\left(\frac{a_1 - b_1}{2}\right)^2 + \left(\frac{a_2 - b_2}{2}\right)^2}$$
$$= \frac{1}{2}\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} = \frac{1}{2}|AB|.$$

   Thus M is half way between A and B.

4. Let A, B, C, D, have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Then W, X, Y, Z have position vectors

$$\mathbf{w} = \frac{1}{2}(\mathbf{a} + \mathbf{b}), \quad \mathbf{x} = \frac{1}{2}(\mathbf{b} + \mathbf{c}), \quad \mathbf{y} = \frac{1}{2}(\mathbf{c} + \mathbf{d}), \quad \mathbf{z} = \frac{1}{2}(\mathbf{d} + \mathbf{a}),$$

   respectively. Now compute:

$$\overrightarrow{WX} = \mathbf{x} - \mathbf{w} = \frac{1}{2}(\mathbf{c} - \mathbf{a}) = \mathbf{y} - \mathbf{z} = \overrightarrow{ZY}.$$

Similarly

$$\overrightarrow{XY} = \mathbf{y} - \mathbf{x} = \frac{1}{2}(\mathbf{d} - \mathbf{b}) = \mathbf{z} - \mathbf{w} = \overrightarrow{WZ}.$$

Thus opposite sides are the same vector. We have a parallelogram.

5. (a) The first picture below illustrates the problem. Take $\mathbf{u} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$. Then

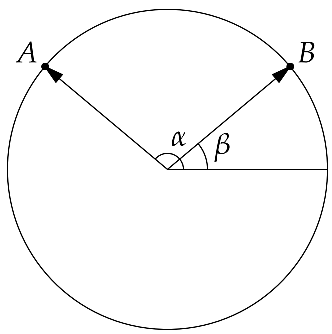
$$\mathbf{u} \cdot \mathbf{v} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

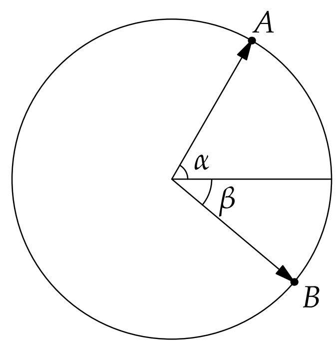
and

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos(\alpha - \beta) = \cos(\alpha - \beta)$$

since both vectors have length 1.

(b) The second picture shows what happens. This time, we take $B = (\cos\beta, -\sin\beta)$ before computing the dot product. The result is immediate.



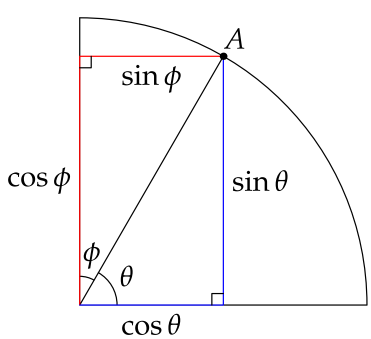


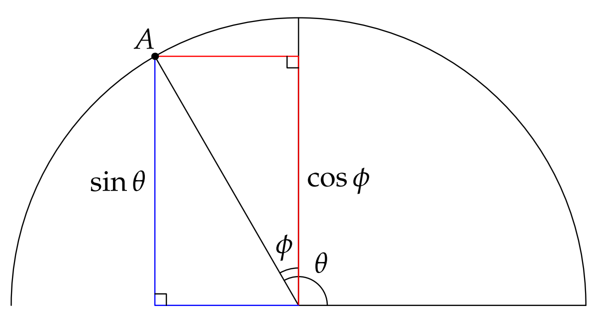
(c) Consider the first picture below. Let $\phi = \frac{\pi}{2} - \theta$. Using the red and blue triangles we may compute the side lengths of the combined rectangle in two ways. It should be obvious that we have a rectangle, since the red and blue triangles are right-angled, and $\theta + \phi = \frac{\pi}{2}$: the fourth angle must also be a right angle. Two formulæ follow immediately:

$$\sin\theta = \cos\phi = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\cos\theta = \sin\phi = \sin\left(\frac{\pi}{2} - \theta\right)$$

According to the definition, if $\frac{\pi}{2} \le \theta \le \pi$, then $\sin\theta = \sin(\pi - \theta)$. From the fourth picture, we see that

$$\sin\theta = \sin(\pi - \theta) = \cos\phi = \cos\left(\frac{\pi}{2} - (\pi - \theta)\right) = \cos\left(\theta - \frac{\pi}{2}\right).$$





(d) Suppose that $\alpha + \beta < \frac{\pi}{2}$. Then we certainly have $\frac{\pi}{2} - \alpha < \frac{\pi}{2}$ and $\beta \leq \frac{\pi}{2}$. Thus

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos(\tfrac{\pi}{2} - (\alpha+\beta)) = \cos\left((\tfrac{\pi}{2} - \beta) - \alpha\right) \\
&= \cos(\tfrac{\pi}{2} - \beta)\cos\alpha + \sin(\tfrac{\pi}{2} - \beta)\sin\alpha \\
&= \sin\beta\cos\alpha + \cos\beta\sin\alpha \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta.
\end{aligned}$$

If $\frac{\pi}{2} < \alpha + \beta \leq \pi$, then, by our assumption in the question that $\beta \leq \alpha$, we see that $\beta \leq \frac{\pi}{2}$. The calculation above is almost identical:

$$\begin{aligned}
\sin(\alpha+\beta) &= \cos((\alpha+\beta) - \tfrac{\pi}{2}) = \cos\left(\alpha - (\tfrac{\pi}{2} - \beta)\right) \\
&= \cos\alpha\cos(\tfrac{\pi}{2} - \beta) + \sin\alpha\sin(\tfrac{\pi}{2} - \beta) \\
&= \cos\alpha\sin\beta + \sin\alpha\cos\beta \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta.
\end{aligned}$$

Similar arguments work for $\sin(\alpha - \beta)$. If $\alpha - \beta \leq \frac{\pi}{2}$, then

$$\sin(\alpha-\beta) = \cos(\tfrac{\pi}{2} - (\alpha-\beta)) = \begin{cases} \cos((\tfrac{\pi}{2} - \alpha) + \beta)) & \text{if } \alpha \leq \tfrac{\pi}{2} \\ \cos(\beta - (\alpha - \tfrac{\pi}{2})) & \text{if } \alpha > \tfrac{\pi}{2} \end{cases}$$

while if $\alpha - \beta > \frac{\pi}{2}$, then $\alpha > \frac{\pi}{2} > \beta$, and so

$$\sin(\alpha-\beta) = \cos((\alpha-\beta) - \tfrac{\pi}{2}) = \cos((\alpha - \tfrac{\pi}{2}) - \beta)).$$

In either case, following the multiple angle formulae for cosine and translating back yields the results.