Math 161 Homework 5 Solutions

(1) Identify the product f of a reflection in the line $y = -x$, a rotation through $\pi/3$, and a reflection in the y-axis. Make sure you specify the exact nature of f and provide the matrix representation for f.

Proof. Reflection in $y = x$ has matrix $M_{\pi/2}$ because $y = x$ makes angle $\pi/4$ with the positive x-axis.

Rotation by $\pi/3$ has matrix $R_{\pi/3}$.

Reflection in y-axis has matrix M_π because the y-axis has angle $\pi/2$ relative to the positive x-axis.

Multiplying $M_\pi R_{\pi/3} M_{\pi/2} = M_\pi M_{\pi/3+\pi/2} = R_{\pi-\pi/3-\pi/2} = R_{\pi/6}$.

Conclusion: f is a rotation by angle $\pi/6$ about the origin. □

(2) Identify the product f of a reflection in the line $y = -x$, a rotation through $\pi/3$, and a reflection in the y-axis. Make sure you specify the exact nature of f and provide the matrix representation for f.

Proof. Reflection in $y = -x$ has matrix $M_{3\pi/2}$ because $y = -x$ makes angle $3\pi/4$ with the positive x-axis.

Rotation by $\pi/3$ has matrix $R_{\pi/3}$.

Reflection in y-axis has matrix M_π because the y-axis has angle $\pi/2$ relative to the positive x-axis.

Multiplying $M_\pi R_{\pi/3} M_{3\pi/2} = M_\pi M_{\pi/3+3\pi/2} = R_{\pi-\pi/3-3\pi/2} = R_{-5\pi/6}$.

Conclusion: f is a rotation by angle $-5\pi/6$ about the origin. □

(3) Identify the product of a rotation through $\pi/6$ about the origin followed by a rotation through $\pi/3$ about the point $A = (1, 0)$. **Hint:** It's fairly clear that this is a rotation; the main thing is to compute the center of the rotation.

Proof. This is clearly a 90° rotation. Why? The first map has matrix $R_{\pi/6}$ and the second map is $\tau R_{\pi/3}$ for some translation τ (this has been discussed carefully in class). Now $\tau R_{\pi/3} R_{\pi/6} = \tau R_{\pi/2}$.

The only question is: what is the centre? In other words, what is τ? In the following, draw a picture to convince yourself.

The origin O rotates to the point B where $AO = AB$ and the angle OAB is 60°. The centre must be the point C such that $CO = CB$ and $\angle OCB = 90°$.

Since $\triangle OAB$ is an isosceles triangle with a 60° angle it?s equilateral, and so $OB = 1$. $\triangle OCB$ is a right-angled triangle and so $OC = 1/\sqrt{2}$. Angle $AOC = 60° - 45° = 15°$. □

(4) Identify the product of the reflection in the line $x + y = 1$ followed by the rotation through $\pi/4$ about the point $(1, 0)$.

Proof. $R_{\pi/4} M_{\pi/2} = M{-}\pi/4$ so the product must be a reflection in a line inclined to the positive x-axis by the angle $-22.5°$. Since the point $(1, 0)$ lies on the axis of reflection it is fixed by the product. Therefore the product must be a reflection in the line through $(1, 0)$ at an angle of $-22.5°$. □

(5) The point $P = (1, 1)$ is rotated through $\pi/6$ about the point $(2, 3)$ and then translated in the direction of $(1, 2)$ (i.e. translated in the direction of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$) through a distance of 3 units. Find the coordinates of the resulting point.

Proof. Translate $(2, 3)$ to the origin. So $(1, 1) \to (1, -1)$. The rotation matrix for a 30° rotation about the origin is $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$, so $(1, -1) \to$ $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1+\sqrt{3}}{2} \\ \frac{\sqrt{3}-1}{2} \end{bmatrix}$.

Now translate the origin back to $(2, 3)$. The point now moves to $(\frac{3-\sqrt{3}}{2}, \frac{\sqrt{3}+5}{2})$. Now translate by $\frac{3}{\sqrt{5}}(1, 2)$ to get $(\frac{3-\sqrt{3}}{2} + \frac{3\sqrt{5}}{5}, \frac{\sqrt{3}+5}{2} + \frac{6\sqrt{5}}{5})$. □

(6) $ABCD$ is a unit square and a point P is successively rotated through $\pi/2$ about each of the four points, in the given order. Show that, after the four rotations, the net effect will be to translate P in the direction AD through a distance of 4 units.

Proof. Let the points A, B, C, D be represented by vectors $\mathbf{0}, \mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{b}$ respectively, where $\mathbf{a}, \mathbf{b}$ is an orthonormal basis of $\mathbb{R}^2$. Let $R = R_{\pi/2}$ be the matrix of a 90° rotation about the origin. So here, we put A at the origin.

Then $R\mathbf{a} = \mathbf{b}$ and $R\mathbf{b} = -\mathbf{a}$. Let P be represented by vector $\mathbf{v}$, then the successive positions are:
$\mathbf{v} \to \mathbf{Rv}$
$\to \mathbf{R}(\mathbf{Rv} - \mathbf{a}) + \mathbf{a} = \mathbf{Rv} - \mathbf{Ra} + \mathbf{a} = -\mathbf{v} - \mathbf{b} + \mathbf{a}$
$\to \mathbf{R}[-\mathbf{v} - \mathbf{b} + \mathbf{a} - (\mathbf{a} + \mathbf{b})] + \mathbf{a} + \mathbf{b} = -\mathbf{Rv} - \mathbf{2Rb} + \mathbf{a} + \mathbf{b} = -\mathbf{Rv} + \mathbf{3a} + \mathbf{b}$
$\to \mathbf{R}[-\mathbf{Rv} + \mathbf{3a} + \mathbf{b} - \mathbf{b}] + \mathbf{b} = -\mathbf{R2v} + \mathbf{3Ra} + \mathbf{b} = \mathbf{v} + \mathbf{4b}$.

So the product of the four rotations is a translation in the direction AD through a distance of 4. It?s clear that the product of four 90° rotations about any center will result in directed lines being fixed and so will be a translation. If we take A and carry out the rotations we can see geometrically that the direction and distance of the translation are as claimed.] □