Math 161 Modern Geometry Homework 3

Submit your answers to the following questions at the discussion on *Thursday* May 4, 2017

Guidelines:

- You should write legibly.
- You should write down your proof carefully and clearly state how you start the proof (direct proof, proof by contrapositive, by contradiction etc.) and write complete sentences.
- You can discuss homework with your classmates. But please clearly state who you work with on your homework. The cooperation stops at exchanging ideas. You must write the solutions on your own and not copy from your friend's homework.

1. Two circles meet at points P and Q. Let AP and BP be diameters of the circles. Prove that AB passes through the other intersection point Q.

2. Let AD and BC be two chords of a circle that intersect at P. Show that $(AP)(PD) = (BP)(PC)$. *Hint: use similar triangles.*

3. (a) Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be points. Show that any point on the line segment joining A and B has co-ordinates $((1-t)a_1 + tb_1, (1-t)a_2 + tb_2)$ where $0 \leq t \leq 1$.

   (b) Show that the midpoint of A and B has co-ordinates $\left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)$.
   You'll probably find it easiest to think in terms of vectors for both parts...

4. Given a quadrilateral $ABCD$, let $WXYZ$ be the midpoints of the sides AB, BC, CD, DA respectively. Use vectors to prove that $WXYZ$ is a parallelogram.

5. In this question you *only* have definitions of sine and cosine for angles between 0 and π radians. The challenge is to be rigorous *without* saying things like $\sin(-\beta) = -\sin\beta$.

   (a) Let $A = (\cos\alpha, \sin\alpha)$ and $B = (\cos\beta, \sin\beta)$, where $0 \leq \beta \leq \alpha \leq \pi$ radians. Use the dot product result $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ to prove that

   $$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

   (b) In the same manner as part (a), draw a picture which justifies

   $$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

   (c) If $0 \leq \theta \leq \frac{\pi}{2}$ radians, prove that $\sin\theta = \cos(\frac{\pi}{2} - \theta)$. What is the corresponding formula if $\frac{\pi}{2} \leq \theta \leq \pi$? Can you prove it?

   (d) Use this to prove that

   $$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
   $$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

   for any angles satisfying $0 \leq \beta \leq \alpha \leq \alpha + \beta \leq \pi$.