Compactness of $\omega_1$

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Assume ZFC. Let $j : V \rightarrow M$ be a (nontrivial) elementary embedding with critical point $\text{crt}(j) = \kappa$. 

One can show that $\kappa$ is a large cardinal (e.g. inaccessible). Moreover, the "closer" $M$ is to $V$, the "stronger" the large cardinal property of $\kappa$ is. For instance, $\kappa$ is a measurable cardinal. Note that $V_{\kappa+1} = V_M$. 

If for some $\lambda \geq \kappa$ such that $\lambda < j(\kappa)$, there is some $x \in M$ such that $j(\lambda) \subseteq x$ and $|x| < j(\kappa)$ in $M$, then $\kappa$ is said to be $\lambda$-strongly compact. 

If for some $\lambda \geq \kappa$ such that $\lambda < j(\kappa)$, $j(\lambda) \in M$ (or equivalently $M^\lambda \subseteq M$), then $\kappa$ is $\lambda$-supercompact. 

We say that $\kappa$ is supercompact/strongly compact if $\kappa$ is $\lambda$-supercompact/$\lambda$-strongly compact for all $\lambda$. Clearly, supercompact $\rightarrow$ strongly compact $\rightarrow$ measurable.
Elementary embeddings and large cardinals

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- if for some $\lambda \geq \kappa$ such that $\lambda < j(\kappa)$, there is some $x \in M$ such that $j'' \lambda \subseteq x$ and $(|x| < j(\kappa))^M$, then $\kappa$ is said to be $\lambda$-*strongly compact*. 
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Ultrafilters/measures

Recall the definition of an ultrafilter/measure.

**Definition**

μ is a measure on a set X if

\[ \mu : \mathcal{P}(X) \rightarrow \{0, 1\} \]

such that

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$\mu$ is **nonprincipal** if there is no nonempty set $Y \subseteq X$ such that if $\mu(A) = 1$ then $Y \subseteq A$. 
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$\mu$ is **$\kappa$-complete** if for every $\eta < \kappa$ and for every $\langle A_\alpha : \alpha < \eta \rangle \subseteq \mathcal{P}(X)$ such that $\mu(A_\alpha) = 1$ for all $\alpha < \eta$,

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Compactness ultrafilters

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Strong compactness was introduced by Keisler and Tarski (1963/64) and it turns out that under ZFC, the two notions of strong compactness are equivalent. Without the Axiom of Choice, this is not true.
Compactness ultrafilters (cont.)

Let $\kappa, X$ be as above. Let $\mu$ be a fine, $\kappa$-complete measure on $\mathcal{P}_\kappa(X)$. Let $(A_x : x \in X)$ be a sequence of sets in $\mu$. Then

$$\bigtriangleup_x A_x = \{ \sigma : \sigma \in \bigcap_{x \in \sigma} A_x \}.$$ 

We say that $\mu$ is normal if and only if for every sequence $(A_x : x \in X)$ as above,

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Let $\kappa, X$ be as above. We say that $\kappa$ is **$X$-supercompact** if there is a $\kappa$-complete, fine, normal measure on $\mathcal{P}_\kappa(X)$.

Supercompactness was introduced by Reinhardt and Solovay (1978). Again, under ZFC, the two notions of supercompactness are equivalent.
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**Open problem:** (ZFC) Is strong compactness equiconsistent with supercompactness?
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In particular, we are interested in the following two classes of problems:

1. Is $\omega_1$ is strongly compact equiconsistent with $\omega_1$ is supercompact? More locally, for a given $X$, is $\omega_1$ is $X$-strongly compact equiconsistent with $\omega_1$ is $X$-supercompact?

2. What are the "canonical" (e.g. "minimal") models of $\omega_1$ is $X$-compact for a given $X$?
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X-strong compactness of $\omega_1$ versus X-supercompactness of $\omega_1$

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Question 1 is more tractable than the corresponding ZFC question. Both questions arise in relation with recent development in descriptive inner model theory; as compactness measures are important in studying canonical structures of large cardinals in determinacy settings.
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In this talk, focus on "\( \omega_1 \) is \( \mathbb{R} \)-compact", "\( \omega_1 \) is \( \mathcal{P}(\mathbb{R}) \)-compact", and "\( \omega_1 \) is (fully) compact".
We work in ZF + DC from now on. We are interested in compactness properties of \(\omega_1\). (Why DC?)

In particular, we are interested in the following two classes of problems:

1. Is "\(\omega_1\) is strongly compact" equiconsistent with "\(\omega_1\) is supercompact"? More locally, for a given \(X\), is "\(\omega_1\) is \(X\)-strongly compact" equiconsistent with "\(\omega_1\) is \(X\)-supercompact"?

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In this talk, focus on "\(\omega_1\) is \(\mathbb{R}\)-compact", "\(\omega_1\) is \(P(\mathbb{R})\)-compact", and "\(\omega_1\) is (fully) compact".
Recall $\text{AD}_X$ is the statements that infinite games of perfect information on $X$ is determined. So for $A \subseteq X^\omega$, the game $G_A$ is determined under $\text{AD}_X$. AD is $\text{AD}_\omega$. 
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Martin shows that the cone filter $\mathcal{F}$ on the Turing degrees is an ultrafilter. Now define $\mu$ on $\mathcal{P}_{\omega_1}(\mathbb{R})$ as follows: for $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R})$,

$$A \in \mu \iff \text{for a cone of } d, \{x \in \mathbb{R} : x \leq_T d\} \in A.$$
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It is easy to check that $\mu$ is countably complete and fine. So $\omega_1$ is $\mathbb{R}$-strongly compact.
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(Solovay) For $A \subseteq P_{\omega_1}(\mathbb{R})$. Play the following game $G_A$: I and II take turns to play finite sets of reals $(s_i : i < \omega)$. II wins the play if the set $\sigma := \bigcup \{s_i : i < \omega\} \in A$. Then define

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What about measures on $\mathcal{P}_{\omega_1}(X)$ for $X$ "bigger" than $\mathbb{R}$?
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By Solovay, DC implies $\text{cof}(\Theta) > \omega$. So, let us assume $\text{cof}(\Theta) = \omega_1$. Let $\nu$ be the (club) measure on $\omega_1$ (Solovay). Let $f : \omega_1 \to \Theta$ be cofinal, increasing, continuous.
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For $\alpha < \Theta$, let $\Gamma_\alpha = \{A : w(A) < \alpha\}$, where $w(A)$ is the Wadge rank of $A$. Let $\mu_\alpha$ be the measure on $\mathcal{P}_{\omega_1}(\Gamma_\alpha)$ induced by the Solovay measure (unique by Woodin).
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So we get $\omega_1$ is $P(R)$-strongly compact. To get a normal measure on $P_{\omega_1}(P(R))$, we seem to need $\Theta$ is measurable. It is known that $\text{AD}_R + \text{DC}$ is not enough.
Suppose $V \models \text{ZFC}+$ there is a measurable cardinal. Let $\kappa$ is a measurable witnessed by $\mu$, $j : V \to M$ be the $\mu$-ultrapower map, and $G \subseteq \text{Col}(\omega, < \kappa)$. 

Classical constructions of models with $\omega_1$ being $\mathbb{R}$-compact
Suppose $V \models \text{ZFC}+$ there is a measurable cardinal. Let $\kappa$ is a measurable witnessed by $\mu$, $j : V \to M$ be the $\mu$-ultrapower map, and $G \subseteq \text{Col}(\omega, < \kappa)$.

Let $R_G = R^{V[G]}$. Define a filter $F$ in $V[G]$ as follows: for $A \subseteq \mathcal{P}_{\omega_1}(R_G)$,

$$A \in F \iff V[G] \models \text{Col}(\omega, < j(\kappa)) R_G \in j(A).$$

One can show that $L(\mathbb{R}, F) \models "\omega_1 \text{ is } \mathbb{R}-\text{supercompact}".$
Suppose $V \models \text{ZFC}^+$ there is a measurable cardinal. Let $\kappa$ is a measurable witnessed by $\mu$, $j : V \to M$ be the $\mu$-ultrapower map, and $G \subseteq \text{Col}(\omega, < \kappa)$.

Let $\mathbb{R}_G = \mathbb{R}^{V[G]}$. Define a filter $F$ in $V[G]$ as follows: for $A \subseteq \mathcal{P}_{\omega_1}(\mathbb{R}_G)$,

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One can show that $L(\mathbb{R}, F) \models \text{“} \omega_1 \text{ is } \mathbb{R}\text{-supercompact.} \text{”}$

Though, for example, if $V = L[\mu]$, the minimal model of a measurable cardinal, then $L(\mathbb{R}, F)$ fails to satisfy AD.
With or without AD

Without AD,

**Theorem**

*The following are equiconsistent.*

- $\omega_1$ is $\mathbb{R}$-strongly compact;
- $\omega_1$ is $\mathbb{R}$-supercompact;
- $\text{ZFC}+$ there is a measurable cardinal.
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With AD, we have some separation of the two.

**Theorem**

*The following are equiconsistent.*

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**Theorem**

The following are equiconsistent.

1. AD.
2. AD + $\omega_1$ is $\mathbb{R}$-strongly compact.
By Woodin, the above are equiconsistent with “ZFC + $\exists \omega$ many Woodin cardinals". $\mathbb{R}$-supercompactness requires $\omega^2$ many Woodin cardinals.
By Woodin, the above are equiconsistent with “ZFC + \( \exists \omega \) many Woodin cardinals". R-supercompactness requires \( \omega^2 \) many Woodin cardinals.

**Theorem (Woodin)**

The following are equiconsistent.

1. AD + \( \omega_1 \) is R-supercompact.
2. There are \( \omega^2 \) many Woodin cardinals.
By Woodin, the above are equiconsistent with “ZFC + ∃ω many Woodin cardinals". 
R-supercompactness requires ω² many Woodin cardinals.

Theorem (Woodin)

The following are equiconsistent.

1. AD + ω₁ is R-supercompact.
2. There are ω² many Woodin cardinals.

Corollary

“AD + ω₁ is R-supercompact" is strictly stronger (consistencywise) than “AD + ω₁ is R-strongly compact".
Under $\text{AD}_\mathbb{R}$, Woodin (early 1980’s) has shown that the Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is unique and asked about uniqueness of models of the form $L(\mathbb{R}, \mu) \models \ \text{"\mu is a supercompact measure on } \mathcal{P}_{\omega_1}(\mathbb{R}) \ \text{(under AD).}$. 

Note: Los theorem fails for the ultrapower embedding induced by $\mu$ on $V$. 

The combinatorial heart of the above results come from the following fact: in $L(\mathbb{R}, \mu)$ where $\mu$ witnesses $\omega_1$ is $\mathbb{R}$-supercompact, let $M_\sigma = HOD_\sigma \cup \{\sigma\}$ and $M = \prod_\sigma M_\sigma / \mu$. Then Los theorem holds for this ultraproduct. The key to the proof is the use of normality of $\mu$. 

Nam Trang
Under AD$_\mathbb{R}$, Woodin (early 1980’s) has shown that the Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is unique and asked about uniqueness of models of the form $L(\mathbb{R}, \mu) \models \text{“}\mu \text{ is a supercompact measure on } \mathcal{P}_{\omega_1}(\mathbb{R})\text{ (under AD).}$$

Without AD, there may be more than one model of the form $L(\mathbb{R}, \mu)$ (D. Rodriguez). With AD, Woodin (early 1980’s) conjectured that there is at most one model of the form $L(\mathbb{R}, \mu)$. 

Theorem (Rodriguez-Trang, 2015) 

Assume AD, then there is at most one model of the form $V = L(\mathbb{R}, \mu)$ that satisfies AD + “$\mu$ witnesses $\omega_1$ is $\mathbb{R}$-supercompact.”

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Canonically models of $\omega_1$ is $\mathbb{R}$-supercompact

Nam Trang
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Under AD$_R$, Woodin (early 1980’s) has shown that the Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is unique and asked about uniqueness of models of the form $L(\mathbb{R}, \mu) \models \text{“} \mu \text{ is a supercompact measure on } \mathcal{P}_{\omega_1}(\mathbb{R}) \text{ (under AD).} \text{”}$

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The combinatorial heart of the above results come from the following fact: in $L(\mathbb{R}, \mu)$ where $\mu$ witnesses $\omega_1$ is $\mathbb{R}$-supercompact, let $M_\sigma = HOD_{\sigma \cup \{ \sigma \}}$ and $M = \prod_\sigma M_\sigma / \mu$. Then Los theorem holds for this ultrapower. The key to the proof is the use of normality of $\mu$. 

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**Remarks:**

- **Los theorem** fails for the ultrapower embedding induced by $\mu$ on $V$.
- **Normality** of $\mu$ is crucial in the proof.
- The proof involves advanced techniques in set theory.
Under $\text{AD}_R$, Woodin (early 1980’s) has shown that the Solovay measure on $\mathcal{P}_{\omega_1}(R)$ is unique and asked about uniqueness of models of the form $L(R, \mu) \models " \mu \text{ is a supercompact measure on } \mathcal{P}_{\omega_1}(R) \text{ (under } \text{AD}).$

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**Theorem (Rodriguez-Trang, 2015)**

*Assume AD, then there is at most one model of the form $V = L(R, \mu)$ that satisfies "AD + $\mu$ witnesses $\omega_1$ is $R$-supercompact".*

Rodriguez subsequently proved the conclusion of the above theorem also holds assuming ZFC.

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Under AD_{\mathbb{R}}, Woodin (early 1980’s) has shown that the Solovay measure on \mathcal{P}_{\omega_1}(\mathbb{R}) is unique and asked about uniqueness of models of the form \textit{L}(\mathbb{R}, \mu) \models "\mu \text{ is a supercompact measure on } \mathcal{P}_{\omega_1}(\mathbb{R})" \text{ (under AD).}

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**Theorem (Rodriguez-Trang, 2015)**

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The combinatorial heart of the above results come from the following fact: in \textit{L}(\mathbb{R}, \mu) where \mu witnesses \omega_1 is \mathbb{R}-supercompact, let \textit{M}_\sigma = HOD_{\sigma \cup \{\sigma\}} \text{ and } \textit{M} = \prod_\sigma \textit{M}_\sigma / \mu. Then Los theorem holds for this ultrapower. The key to the proof is the use of normality of \mu.

**Note:** Los theorem fails for the ultrapower embedding induced by \mu on \textit{V}. 
Assume $\text{AD}_\mathbb{R} + \text{DC}$. Recall that working in a minimal model of $\text{AD}_\mathbb{R} + \text{DC}$ (so $\text{cof}(\Theta) = \omega_1$), we can construct a countably complete, fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$ by “integrating the Solovay measure along a cofinal, continuous function $f : \omega_1 \to \Theta$".
Assume $\text{AD}_R + \text{DC}$. Recall that working in a minimal model of $\text{AD}_R + \text{DC}$ (so $\text{cof}(\Theta) = \omega_1$), we can construct a countably complete, fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(R))$ by “integrating the Solovay measure along a cofinal, continuous function $f : \omega_1 \to \Theta$".

**Theorem (Trang-Wilson, 2014-2015)**

The following are equiconsistent.

- $\text{AD}_R + \text{DC}$.
- $\text{ZF} + \text{DC} + \omega_1$ is $\mathcal{P}(R)$-strongly compact.

These theories are strictly weaker than

- $\text{ZF} + \text{DC} + \omega_1$ is $\mathcal{P}(R)$-supercompact compact.\(^a\)

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From ZF + DC + "$\omega_1$ is $\mathcal{P}(\mathbb{R})$-supercompact", one obtains the sharp for a model of $\text{AD}_\mathbb{R} + \text{DC}$. 
From ZF + DC + “\(\omega_1\) is \(\mathcal{P}(\mathbb{R})\)-supercompact"", one obtains the sharp for a model of \(\text{AD}_\mathbb{R} + \text{DC}\).

To see this, note that from the proof of the above theorem, we get a model \(L(\Omega^*, \mathbb{R}) \models \text{AD}_\mathbb{R} + \text{DC}\), where \(\Omega^* \subseteq \mathcal{P}(\mathbb{R})\). Fix a countably complete, fine, normal measure \(\mu\) on \(\mathcal{P}_{\omega_1}(\Omega^*)\). Then note that by normality,

\[
\forall^* \sigma \ M_\sigma = L(\Omega^*_\sigma, \mathbb{R}_\sigma) \models \text{AD}_\mathbb{R} + \text{DC},
\]

where we have that \(\Omega^* = [\sigma \mapsto \Omega^*_\sigma]_\mu\) and \(\mathbb{R} = [\sigma \mapsto \mathbb{R}_\sigma]_\mu\).
From ZF + DC + “ω₁ is \( \mathcal{P}(\mathbb{R}) \)-supercompact", one obtains the sharp for a model of AD\(\mathbb{R}\) + DC.

To see this, note that from the proof of the above theorem, we get a model 
\( L(\Omega^*, \mathbb{R}) \models AD_{\mathbb{R}} + DC \), where \( \Omega^* \subseteq \mathcal{P}(\mathbb{R}) \). Fix a countably complete, fine, normal measure \( \mu \) on \( \mathcal{P}_{\omega_1}(\Omega^*) \). Then note that by normality,

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where we have that \( \Omega^* = [\sigma \mapsto \Omega^*_\sigma]_\mu \) and \( \mathbb{R} = [\sigma \mapsto \mathbb{R}_\sigma]_\mu \).

Now, \( \forall^* \sigma (\Omega^*_\sigma, \mathbb{R}_\sigma)^\# \) exists (because \( \omega_1 \) is measurable); by normality again, the sharp for 
\( L(\Omega^*, \mathbb{R}) \) exists. This demonstrates that the theory ZF + DC + “ω₁ is \( \mathcal{P}(\mathbb{R}) \)-supercompact" is strictly stronger than ZF + DC + “ω₁ is \( \mathcal{P}(\mathbb{R}) \)-strongly compact".
From ZF + DC + "ω₁ is ℙ(ℝ)-supercompact"}, one obtains the sharp for a model of ADₐ + DC.

To see this, note that from the proof of the above theorem, we get a model
L(Ω*, ℝ) ⊨ ADᵣ + DC, where Ω* ⊆ ℙ(ℝ). Fix a countably complete, fine, normal measure μ on ℙₙ₁(Ω*). Then note that by normality,

∀*µ σ Mσ = L(Ω*, Rσ) ⊨ ADᵣ + DC,

where we have that Ω* = [σ ↦ Ω*']μ and R = [σ ↦ Rσ]μ.

Now, ∀*µ σ (Ω*, Rσ)# exists (because ω₁ is measurable); by normality again, the sharp for
L(Ω*, ℝ) exists. This demonstrates that the theory ZF + DC + "ω₁ is ℙ(ℝ)-supercompact" is
strictly stronger than ZF + DC + "ω₁ is ℙ(ℝ)-strongly compact".
Some determinacy theories

Recall $\Theta$ is the supremum of $\alpha$ such that there is a surjection of $\mathbb{R}$ onto $\alpha$. 
Recall Θ is the supremum of α such that there is a surjection of \( \mathbb{R} \) onto α.

Definition (AD + DC\( \mathbb{R} \))

The Solovay sequence is a sequence \((\theta_\alpha : \alpha \leq \Omega)\) such that

1. \(\theta_0\) is the sup of α such that there is an OD surjection from \( \mathbb{R} \) onto α.
2. \(\theta_\Omega = \Theta\).
3. \(\theta_\alpha\) is the sup of \(\theta_\beta\) for \(\beta < \alpha\) and \(\alpha\) is limit.
4. For \(\alpha < \Omega\), let A be of Wadge rank \(\theta_\alpha < \Theta\), \(\theta_{\alpha+1}\) is the sup of α such that there is an OD(A) surjection from \( \mathbb{R} \) onto α.
Recall $\Theta$ is the supremum of $\alpha$ such that there is a surjection of $\mathbb{R}$ onto $\alpha$.

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4. For $\alpha < \Omega$, let $A$ be of Wadge rank $\theta_\alpha < \Theta$, $\theta_{\alpha+1}$ is the sup of $\alpha$ such that there is an OD($A$) surjection from $\mathbb{R}$ onto $\alpha$.

Here are some determinacy theories in increasing strength: (1) AD, (2) AD$^+ + \Theta > \theta_0$, (3) AD$\mathbb{R}$, (4) AD$\mathbb{R}$ + DC, (5) AD$\mathbb{R}$ + $\Theta$ is regular, (6) AD$\mathbb{R}$ + $\Theta$ is measurable, (7) AD$\mathbb{R}$ + $\Theta$ is Mahlo, (8) AD$^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal (LSA).
Recall $\Theta$ is the supremum of $\alpha$ such that there is a surjection of $\mathbb{R}$ onto $\alpha$.

**Definition (AD + DC$_{\mathbb{R}}$)**

The Solovay sequence is a sequence $(\theta_\alpha : \alpha \leq \Omega)$ such that

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Here are some determinacy theories in increasing strength: (1) AD, (2) AD$^+ + \Theta > \theta_0$, (3) AD$_{\mathbb{R}}$, (4) AD$_{\mathbb{R}} + $ DC, (5) AD$_{\mathbb{R}} + \Theta$ is regular, (6) AD$_{\mathbb{R}} + \Theta$ is measurable, (7) AD$_{\mathbb{R}} + \Theta$ is Mahlo, (8) AD$^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal (LSA).
ZFC and large cardinals
Two classes of problems
When $X = \mathbb{R}$
Beyond $\mathbb{R}$-compactness
Some questions

Hierarchies

- Large Cardinals
  - Supercompact
  - WLW
  - ?
  - $\Theta$ reg-hypo
  - non-domestic
  - $\text{AD}_\mathbb{R}$-hypo
  - $\omega$ Woodins

- Determinacy
  - $\text{AD}_\mathbb{R} + \Theta$ regular
  - $\text{AD}_\mathbb{R} + \text{DC}$
  - $\text{AD}_\mathbb{R}$
  - $\text{AD}$

- HOD
  - Regular limit of Wdns
  - $\omega_1$ Woodins
  - $\omega$ Woodins
  - 1 Woodin

- Combinatorial Theories
  - $\text{PFA}$
  - $\omega_1$ is (str/super)compact
  - $\omega_1$ is $\mathcal{P}(\mathbb{R})$-spct
  - $\omega_1$ is $\mathcal{P}(\mathbb{R})$-str.cpct.
  - $\text{AD} + \omega_1$ is $\mathbb{R}$-str.cpct.

Nam Trang
Compactness of $\omega_1$
The Chang$^+$ model

For each $\lambda \geq \omega$, let $\mathcal{F}_\lambda$ be the club filter on $\mathcal{P}_{\omega_1}(\lambda^\omega)$, and define the Chang$^+$ model

$$
\mathcal{C}^+ = L[\bigcup_\lambda \lambda^\omega][ \mathcal{F}_\lambda : \lambda \in ON ].
$$

$\mathcal{C}^+$ satisfies ZF + DC.
The Chang$^+$ model

For each $\lambda \geq \omega$, let $\mathcal{F}_\lambda$ be the club filter on $\mathcal{P}_{\omega_1}(\lambda^\omega)$, and define the Chang$^+$ model

$$C^+ = L[\bigcup \mathcal{F}_\lambda][F_\lambda : \lambda \in \text{ON}].$$

$C^+$ satisfies ZF + DC.

**Theorem (Woodin)**

*Suppose there is a proper class of Woodin limits of Woodin cardinals. Then $C^+ \models \omega_1$ is supercompact. Furthermore, $C^+ \models AD_\mathbb{R}$.***
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*Suppose there is a proper class of Woodin limits of Woodin cardinals. Then $C^+ \models \omega_1$ is supercompact. Furthermore, $C^+ \models \text{AD}_\mathbb{R}$.***

**Theorem**

- (Trang) $\text{Con}(\omega_1$ is supercompact) implies $\text{Con}(\text{AD}_\mathbb{R} + \Theta$ is regular).
- (Sargsyan-Trang) $\text{Con}(\text{AD} + \omega_1$ is supercompact) implies $\text{Con}(\text{LSA})$. 
The Chang$^+$ model

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**Theorem (Woodin)**

Suppose there is a proper class of Woodin limits of Woodin cardinals. Then $C^+ \models \omega_1$ is supercompact. Furthermore, $C^+ \models AD_\mathbb{R}$.

**Theorem**

- (Trang) $\text{Con}(\omega_1$ is supercompact) implies $\text{Con}(AD_\mathbb{R} + \Theta$ is regular).
- (Sargsyan-Trang) $\text{Con}(AD + \omega_1$ is supercompact) implies $\text{Con}(LSA)$. 
Some questions

Rodriguez’s construction of distinct models of the form $L(\mathbb{R}, \mu)$ needs a measurable of Mitchell order 2.
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Rodriguez’s construction of distinct models of the form $L(\mathbb{R}, \mu)$ needs a measurable of Mitchell order 2.

**Question**

*Can one construct distinct models of “$\omega_1$ is $\mathbb{R}$-supercompact” from a measurable cardinal?*
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**Conjecture**

*The following are equiconsistent.*

- $\text{ZF} + \text{DC} + (\text{AD}/\text{AD}_\mathbb{R}) + \omega_1$ is strongly compact.
- $\text{ZF} + \text{DC} + (\text{AD}/\text{AD}_\mathbb{R}) + \omega_1$ is supercompact.
- $\text{ZFC}+$ there is a proper class of Woodin limits of Woodins.
Some questions

Rodriguez’s construction of distinct models of the form $L(\mathbb{R}, \mu)$ needs a measurable of Mitchell order 2.

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- $\text{ZFC}^+ \text{ there is a proper class of Woodin limits of Woodins.}$
Thank you!