Math 161 Modern Geometry Homework Questions 3 - Extras

Submit nothing - just extra practice for the midterm

(1) (a) Express each of the following fractions as complex numbers by rationalizing the denominator (multiplying through by the complex conjugate...)
$$\frac{1}{2i}, \quad \frac{1+i}{1-i}, \quad \frac{1}{2+4i}$$
(b) Prove that $\mathbb{C}$ is closed under multiplicative inverses: i.e., $\forall z \in \mathbb{C} \setminus \{0\}$, prove that $\frac{1}{z} \in \mathbb{C}$.

(2) (a) Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, prove that
$$e^{a+i\theta}e^{b+i\phi} = e^{(a+i\theta)+(b+i\phi)}$$
for any complex numbers $a + i\theta$, $b + i\phi$.

(b) Prove that i^i is a complex number.

(c) Prove by induction that, for any complex number z, we have
$$\forall n \in \mathbb{N},\ e^{nz} = (e^z)^n$$
(d) Using $n = 3$ in the expression in part (b), prove that
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

(3) Consider the stereographic projection that associates points on the complex plane with points on the unit sphere $x^2 + y^2 + z^2 = 1$ as in the lectures. Let $z = 1 + i$, compute the corresponding point $P = (X, Y, Z)$ on the unit sphere. Let $Q = (1/2, 1/2, 1/\sqrt{2})$ be a point on the unit sphere. Compute the corresponding complex number $w = c + id$.

(4) (Vectors)

(a) Prove the triangle inequality: for any vectors $\vec{a}, \vec{b}$, $||\vec{a}|| + ||\vec{b}|| \geq ||\vec{a} + \vec{b}||$. When does equality occur?

(b) Let x, y, z be three positive real numbers such that $x + y + z \leq 3$. Prove that $1/x + 1/y + 1/z \geq 3$.

(c) Find the angle between vectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.