Math 161 Modern Geometry Homework Answers 3 - Extras

(1) (a) $\frac{1}{2i} = \frac{1}{2i} \cdot \frac{-2i}{-2i} = \frac{-2i}{4} = -\frac{1}{2}i$

$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+2i-1}{1+1} = i$

$\frac{1}{2+4i} = \frac{2-4i}{(2+4i)(2-4i)} = \frac{2-4i}{4+16} = \frac{1}{10} - \frac{1}{5}i$

(b) Let $z = x + iy$, where $x, y \in \mathbb{R}$ aare not both zero. Then

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

Thus $\frac{1}{z} \in \mathbb{C}$ and so $\mathbb{C}$ is closed under multiplicative inverses.

(2) (a) Just compute:

$$\begin{aligned} e^{a+i\theta}e^{b+i\phi} &= e^a(\cos\theta + i\sin\theta)e^b(\cos\phi + i\sin\phi) \\ &= e^a e^b\left(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\right) \\ &= e^{a+b}\left(\cos(\theta+\phi) + i\sin(\theta+\phi)\right) \\ &= e^{a+b+i(\theta+\phi)} \\ &= e^{(a+i\theta)+(b+i\phi)} \end{aligned}$$

(b) Base case $n = 1$: $e^z = e^z$ is trivial.
Induction step: Fix $n \in \mathbb{N}$ and assume that $e^{nz} = (e^z)^n$. Then

$$e^{(n+1)z} = e^{nz+z} = e^{nz}e^z = (e^z)^n e^z = (e^z)^{n+1}.$$

Thus, by induction, $e^{nz} = (e^z)^n$ for all $n \in \mathbb{N}$.

(c) Using $n = 3$ in the expression in part (b), prove that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

Find a corresponding trigonometric identity for $\sin 3\theta$. Just take $n = 3$ in Euler's formula:

$$\begin{aligned} \cos 3\theta + i\sin 3\theta = e^{3i\theta} &= (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) \end{aligned}$$

Equating real and imaginary parts we obtain:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

(3) Let $z = 1 + i$. To find $P = (X, Y, Z)$, use the formula derived in class (lecture notes posted on class website). We have:

$$(X, Y, Z) = \left(\frac{2.1}{1^2 + 1^2 + 1}, \frac{2.1}{1^2 + 1^2 + 1}, \frac{1^2 + 1^2 - 1}{1^2 + 1^2 + 1}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

Now let $Q = (1/2, 1/2, 1/\sqrt{2})$. To compute $w = c + id$, we need to do a similar calculations as the one done in class. First, identify w with the point $R = (c, d, 0)$ in $\mathbb{R}^3$. Then we write the vector equation of the line (l) going through $(0, 0, 1)$ and Q. The equation of (l) is:

$$(0,0,1) + t(1/2, 1/2, 1/\sqrt{2} - 1) = (1/2t, 1/2t, (1/\sqrt{2} - 1)t + 1)$$

Now R is the intersection of (l) with the xy-plane. Equate $c = 1/2t, d = 1/2t, 0 = 1 + (1 - 1/\sqrt{2})t$. Solve the third equation for t, we get $t = \frac{-2}{2-\sqrt{2}}$. So $c = d = 1/2t = \frac{-1}{2-\sqrt{2}}$.

So $w = \frac{-1}{2-\sqrt{2}} + i\frac{-1}{2-\sqrt{2}}$.

(4) (a) Note that this problem was given on Quiz 3. It is enough to prove $(||\vec{a}|| + ||\vec{b}||)^2 \geq ||\vec{a} + \vec{b}||^2$.
Now, $(||\vec{a}|| + ||\vec{b}||)^2 = ||\vec{a}||^2 + ||\vec{b}||^2 + 2||\vec{a}||||\vec{b}||$.
$||\vec{a}+\vec{b}||^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+2\vec{a}\cdot\vec{b} = ||\vec{a}||^2+||\vec{b}||^2+2||\vec{a}||||\vec{b}||\cos\theta$, where θ is the angle between $\vec{a}, \vec{b}$.
Since $\cos\theta \leq 1$, $2||\vec{a}||||\vec{b}||\cos\theta \leq 2||\vec{a}||||\vec{b}||$; hence, $(||\vec{a}|| + ||\vec{b}||)^2 \geq ||\vec{a} + \vec{b}||^2$.

(b) This is an application of Cauchy-Schwarz.

$(1/x+1/y+1/z)(x+y+z) = (1/(\sqrt{x})^2+1/(\sqrt{y})^2+1/(\sqrt{y})^2)((\sqrt{x})^2 + (\sqrt{y})^2+(\sqrt{z})^2) \geq ((1/(\sqrt{x})^2)\sqrt{x}^2+(1/(\sqrt{y})^2)\sqrt{y}^2+(1/(\sqrt{y})^2)\sqrt{y}^2)^2 = 3^2 = 9$.

So $9 \leq (1/x + 1/y + 1/z)(x + y + z) \leq 3(1/x + 1/y + 1/z)$. So $(1/x + 1/y + 1/z) \geq 3$.

(c) This is an application of dot product's formula: $\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}||\cos\theta$, where θ is the angle between $\vec{v}, \vec{w}$; by convention, $0 \leq \theta \leq \pi$.
By definition $\vec{v} \cdot \vec{w} = 1.2 + 1.(-1) + 2.1 = 3$.
$||\vec{v}||||\vec{w}||\cos\theta = \sqrt{6}\sqrt{6}\cos\theta = 6\cos\theta$.
So $6\cos\theta = 3$. So $\cos\theta = 1/2$. Therefore, $\theta = \pi/6$.