

Stability of coherent pattern formation through invasion in the FitzHugh-Nagumo system

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Abstract

We establish sharp nonlinear stability results for fronts that describe the creation of a periodic pattern through the invasion of an unstable state. The fronts we consider are critical, in the sense that they are expected to mediate pattern selection from compactly supported or steep initial data. We focus on pulled fronts, that is, on fronts whose propagation speed is determined by the linearization about the unstable state in the leading edge, only. We present our analysis in the specific setting of the FitzHugh-Nagumo system, where pattern-forming uniformly translating fronts have recently been constructed rigorously [18], but our methods can be used to establish nonlinear stability of pulled pattern-forming fronts in general reaction-diffusion systems. This is the first stability result of critical pattern-selecting fronts and provides a rigorous foundation for heuristic, universal wave number selection laws in growth processes based on a marginal stability conjecture. The main technical challenge is to describe the interaction between two separate modes of marginal stability, one associated with the spreading process in the leading edge, and one associated with the pattern in the wake. We develop tools based on far-field/core decompositions to characterize, and eventually control, the interaction between these two different types of diffusive modes. Linear decay rates are insufficient to close a nonlinear stability argument and we therefore need a sharper description of the relaxation in the wake of the front using a phase modulation ansatz. We control regularity in the resulting quasilinear equation for the modulated perturbation using nonlinear damping estimates.

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1 Introduction

Invasion into unstable states plays an important role in the development of complex coherent structures in many physical systems. Unstable states can be observed as a transient for instance after a change in system parameters, or after the introduction of a novel external agent to which the system is unstable. In spatially extended systems, one expects localized fluctuations of the now unstable background state to grow and spread, leaving a new stable state in the wake of a propagating *invasion front*. A fundamental problem then is to predict the speed of propagation of the invasion process as well as the new state selected in its wake.

Predictions for invasion speeds in the mathematical literature have historically been restricted to systems with comparison principles. Since comparison principles are incompatible with complex pattern formation, invasion fronts in these systems typically select a new spatially homogeneous equilibrium in their wake. On the other hand, many interesting invasion phenomena have been observed in experiments, simulations, and formal analyses in pattern-forming models describing a large variety of physical systems; see [75] for a comprehensive review. Pattern-forming systems usually admit families of periodic patterns, parameterized by the wave number. An invasion process spreading into an unstable state typically selects one distinguished pattern and wave number out of this family; see [24]. This wave number selection mechanism has been observed in many experiments across the sciences [75], and has promising applications to nanoscale manufacturing technologies [10, 56]. However, beyond the heuristics [24] and matched asymptotics [73], there do not appear to be mathematical results or techniques that describe these phenomena.

In a first approach, one would hope to make such predictions rigorous by finding a unique speed and wave number for which there is a stable traveling front solution connecting the unstable state to a periodic pattern in the wake. Existence of pattern-forming fronts has been rigorously established near the onset of a Turing instability [20, 25, 29, 37] and in phase separation problems [68, 69]. However, such fronts typically exist and are stable for a continuum of speeds and associated wave numbers, so that this simple approach does not predict which of these speed-wave number pairs are selected by localized initial data.

The *marginal stability conjecture* [9, 15, 16, 21, 24, 74, 75] asserts that speeds and associated wave numbers are determined by the distinguished front solutions which are *marginally spectrally stable* in an appropriate sense. There are two distinct scenarios for marginal spectral stability: the marginal stability may arise from marginal pointwise stability of the unstable state in the leading edge in a distinguished moving frame, or from marginally stable point spectrum of the invasion front itself. The former case is referred to as *pulled*, or linearly determined, propagation, while the latter case is referred to as *pushed*, or nonlinearly determined, propagation. Because the associated spectrum is marginally stable, or *critical*, selected fronts are sometimes referred to as critical fronts. By contrast, faster-traveling *supercritical* fronts arise for a continuum of speeds and associated wave numbers, and are often stable against restricted classes of perturbations. Stability of these supercritical fronts has been shown rigorously in some pattern-forming systems [27, 28], but they do not appear to be relevant to selection and propagation from compactly supported or steep initial data.

The marginal stability conjecture was recently established for systems of parabolic equations [2, 9] based on a novel, conceptual approach that does not rely on comparison principles, thus providing a promising avenue towards the analysis of pattern-forming systems that inherently do not possess ordering properties. The analysis there does however assume that the marginally stable front selects a spatially constant, exponentially attracting state in its wake, rather than a periodic pattern.

The results in [2, 9] demonstrate that the key ingredient to establishing the marginal stability conjecture is a sharp theory for nonlinear stability of invasion fronts against perturbations that do not alter the decay in the leading edge of the front. These sharp stability results predict the characteristic log-shift of the front position when starting from compactly supported or steep initial data and allow to close the matched asymptotics arguments in [73]. Our main result, informally stated below as Theorem 1.2, provides precisely these sharp stability estimates in the case of a uniformly translating, pulled front that creates a pattern in its wake.

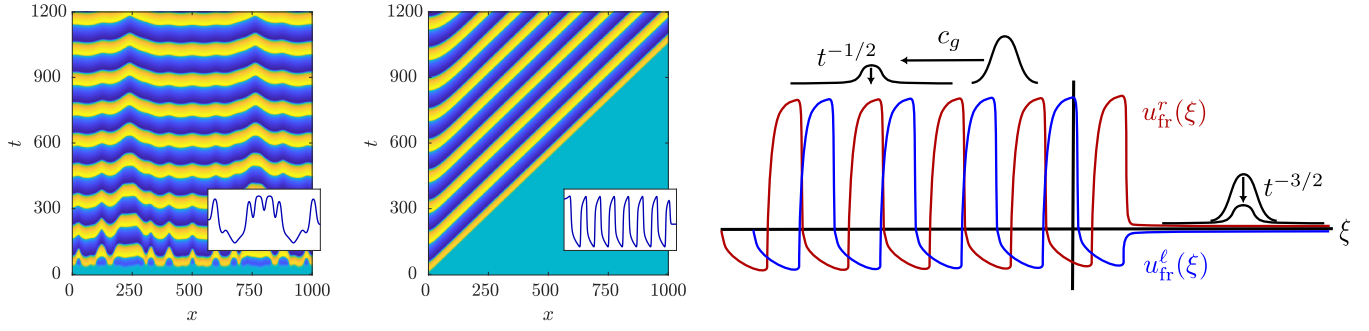


Figure 1: Left and center: Space-time plot of u from direct simulations of (1.2) with initial condition $u = w = 0$ perturbed by small white noise (left) and a small perturbation on the left end (center); $a = 0.4, \gamma = 0.1, \varepsilon = 0.005$. Insets show the u -component at time $t = 1000$. Right: schematic of first component u_{fr}^r (red) and u_{fr}^l (blue) of invasion fronts $\mathbf{u}_{\text{fr}}^{r/\ell}(\xi)$ in the co-moving frame, together with the dynamics of perturbations (black) in the leading edge and in the wake. Perturbations in the leading edge decay with improved rate $t^{-3/2}$, while perturbations in the wake are transported outward by the group velocity and decay with diffusive rate $t^{-1/2}$. Note also that the patterns in the wake are large amplitude, highly nonlinear structures which do not resemble a pure cosine wave.

To be concrete, we present our analysis in the setting of the FitzHugh-Nagumo system,

$$\begin{aligned} u_t &= u_{xx} + u(u + a)(1 - u - a) - w, \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \quad (1.1)$$

which we write abstractly as a system for $\mathbf{u} = (u, w)^\top$,

$$\mathbf{u}_t = D\mathbf{u}_{xx} + F(\mathbf{u}; a, \gamma, \varepsilon), \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F(\mathbf{u}; a, \gamma, \varepsilon) = \begin{pmatrix} u(u + a)(1 - u - a) - w \\ \varepsilon(u - \gamma w) \end{pmatrix}. \quad (1.2)$$

The FitzHugh-Nagumo system (1.2) models excitable and oscillatory media far from equilibrium and is ubiquitous across the sciences as both a phenomenological model and as a more rigorous simplification of more complex descriptions. It arose in this latter fashion first as a simplification of the Hodgkin-Huxley equations for signal propagation in nerve fibers and, together with variations and adaptations, has since been used to model, for instance, the onset of turbulence in fluids [12], carbon monoxide oxidation on platinum surfaces [11, 53], and cardiac arrhythmias [51]. Mathematically, (1.2) is a scalar reaction-diffusion equation coupled to a linear ODE and, thus, one of the simplest models which could, and does exhibit spatio-temporal pattern formation.

Existence of pulled fronts. We consider (1.1) in the oscillatory regime, $0 < a < \frac{1}{2}$, and $0 < \gamma < 4$, with $0 < \varepsilon \ll 1$. In this regime, $\mathbf{u} = (0, 0)$ is the unique spatially constant equilibrium, but is unstable with perturbations growing in any translationally invariant norm. Linearizing at this unstable equilibrium, one finds a linear spreading speed $c_{\text{lin}}(a, \gamma, \varepsilon) = 2\sqrt{a(1-a)} + O(\varepsilon)$ together with a characteristic exponential decay rate $-\eta_{\text{lin}}(a, \gamma, \varepsilon)$, at which the linear equation is marginally stable; see [18, Lemma 2.1] or Lemma 3.2 below for details, as well as [39] for background on linear spreading speeds. In addition to this spatially constant solution, (1.1) admits stable time-periodic, x -independent solutions, commonly referred to as relaxation oscillations, and spatially periodic modulations of these oscillations, which are traveling waves $\mathbf{u}(x, t) = \mathbf{u}_{\text{wt}}(x - ct)$ satisfying $\mathbf{u}_{\text{wt}}(\xi) = \mathbf{u}_{\text{wt}}(\xi + L)$ for some $L > 0$. These periodic traveling waves, or wave trains, exist for a range of periods $L_* < L < \infty$ with speeds $c = c(L)$. Traveling front solutions $\mathbf{u}(x, t) = \mathbf{u}(x - ct)$ that connect $\mathbf{u} = (0, 0)$ to a wave train $\mathbf{u}_{\text{wt}}$ have been constructed in [18, Theorem 1.2] using dynamical systems techniques, in particular geometric singular perturbation theory.

Theorem 1.1 (Existence of pulled pattern-forming fronts [18, Theorem 1.2]). *Fix $(3 - \sqrt{6})/6 < a < \frac{1}{2}$ and $0 < \gamma < 4$. There exists $\varepsilon_0 > 0$ such for all $0 < \varepsilon < \varepsilon_0$ (1.2) admits two pulled pattern-forming front*

solutions $\mathbf{u}(x, t) = \mathbf{u}_{\text{fr}}^\ell(x - c_{\text{lin}}t)$ and $\mathbf{u}(x, t) = \mathbf{u}_{\text{fr}}^r(x - c_{\text{lin}}t)$, which satisfy

$$\mathbf{u}_{\text{fr}}^{\ell/r}(\xi) = \left[\mathbf{u}_0^{\ell/r} \xi + \mathbf{u}_1^{\ell/r} + b^{\ell/r} \mathbf{u}_0^{\ell/r} \right] e^{-\eta_{\text{lin}} \xi} + O(e^{-(\eta_{\text{lin}} + \eta) \xi}), \quad \xi \rightarrow \infty, \quad (1.3)$$

and

$$\mathbf{u}_{\text{fr}}^{\ell/r}(\xi) = \mathbf{u}_{\text{wt}}^{\ell/r}(\xi) + O(e^{\eta \xi}), \quad \xi \rightarrow -\infty, \quad (1.4)$$

for some $b^{\ell/r} \in \mathbb{R}$, $\eta > 0$, $\mathbf{u}_0^{\ell/r} \in \mathbb{R}^2 \setminus \{0\}$, and $\mathbf{u}_1^{\ell/r} \in \mathbb{R}^2$, where $\mathbf{u}_{\text{wt}}^{\ell/r}$ is a wave train solution with selected period $L > 0$ and wavenumber $\frac{2\pi}{L}$, c_{lin} is the linear spreading speed, and η_{lin} is the characteristic exponential decay rate. Both fronts select the same wave trains, $\mathbf{u}_{\text{wt}}^\ell(\xi) = \mathbf{u}_{\text{wt}}^r(\xi + \xi_0)$ for some $\xi_0 \in \mathbb{R}$ but have opposite monotonicity in the leading edge, $\mathbf{u}_j^\ell = -\mathbf{u}_j^r$, $j = 0, 1$.

See Figure 1 for a schematic of the fronts. We shall discuss below results that establish that both fronts $\mathbf{u}_{\text{fr}}^r$ and $\mathbf{u}_{\text{fr}}^\ell$ are marginally stable due to marginal stability in the leading edge, that is, they are pulled fronts. We emphasize that the wave trains generated in the wake are “far-from-equilibrium patterns”, in the sense that they possess large amplitude and are highly nonlinear, in contrast to the low-amplitude, weakly nonlinear patterns selected by fronts near a Turing instability [20, 29, 37].

Nonlinear stability of pulled fronts — main result. To study stability of the front solutions established in Theorem 1.1, we consider the frame $\xi = x - c_{\text{lin}}t$ moving with the speed c_{lin} , in which (1.2) reads

$$\mathbf{u}_t = D\mathbf{u}_{\xi\xi} + c_{\text{lin}}\partial_\xi \mathbf{u} + F(\mathbf{u}; a, \gamma, \varepsilon), \quad (1.5)$$

so that the front is a stationary solution to (1.5). Our main result may be informally stated as follows.

Theorem 1.2. *Let $\mathbf{u}_{\text{fr}}$ be a uniformly translating pattern-forming front solution to (1.2), as established in Theorem 1.1. Assume that $\mathbf{u}_{\text{fr}}$ satisfies the following marginal spectral stability assumptions:*

- *The wave train generated by the front is diffusively spectrally stable, that is, its spectrum touches the origin in a single parabolic tangency and is otherwise stable;*
- *The group velocity of the wave train points to the left, away from the front interface;*
- *The linearization of (1.5) about $\mathbf{u}_{\text{fr}}$ possesses no unstable point spectrum, and no embedded eigenvalue at $\lambda = 0$.*

Then, $\mathbf{u}_{\text{fr}}$ is nonlinearly stable as a stationary solution to (1.5) against sufficiently localized perturbations, and perturbations decay pointwise in time with sharp diffusive rate $t^{-1/2}$, and with enhanced rate $t^{-3/2}$ in the leading edge of the front.

Before we prepare a more precise statement of this result in the next section (see Theorem 2.1), where we will also point out the conceptual nature of our methods with potential broader applicability, we mention some subtleties.

The assumptions in Theorem 1.2 are made precise in Hypotheses 1 through 3 below and are established for (1.2) in natural parameter ranges in the companion papers [3, 4]. The assumptions guarantee that the fronts studied here are pulled fronts, since they propagate at the linear spreading speed and do not possess marginally stable or unstable point spectrum. The diffusive spectral stability that we require for the wave trains in the wake is generic for wave trains in reaction-diffusion systems and a standard assumption for their nonlinear stability analysis [46, 66, 72]. The group velocity of the wave train describes the direction of transport of small perturbations; see [26] for further background. We assume that the group velocity points away from the front interface so that the front naturally acts as a source of patterns in the sense of [63].

Spreading at the linear spreading speed guarantees diffusive decay in exponentially weighted spaces in the leading edge; see Figure 1.

We reiterate that the fronts studied here are uniformly translating, i.e. equilibria in a comoving frame. In particular, they are not time-periodic in a comoving frame (or *modulated*) as is the case for many other pattern-forming fronts. Such modulated fronts have been constructed in the wake of a Turing instability [20] but also for large amplitudes in spinodal decomposition [69]. We do not treat such modulated fronts in this paper, but we expect that our methods and proofs can be adapted to this case; see also [14, 62] for stability results of modulated waves in the presence of essential spectrum. In this context, it is worth noting the stability result in the complex Ginzburg-Landau equation in [8], which does give sharp stability estimates towards a front that describes pattern formation near a Turing instability, albeit in an amplitude equation setting that averages oscillations so that the front is in fact uniformly translating, and which allows for coordinate choices that are not available in more general situations, in particular the present one.

The u -component of both fronts in Theorem 1.1 are monotone in the leading edge: for left fronts, u_{fr}^ℓ is increasing and for right fronts u_{fr}^r is decreasing. Both fronts select the same state in the wake (up to a phase shift of the wave train), a curious phenomenon not observed in order-preserving systems. Initial conditions will converge to left or to right fronts depending on their behavior in the leading edge. One indeed observes this selected wave number in simulations when the unstable state is perturbed by “shot” noise, locally in space. In contrast, spatially distributed white noise perturbations lead to long-wavelength modulations of spatially homogeneous oscillations with a slow coarsening process toward synchrony. In this sense, the fronts studied here mediate a rapid frequency synchronization in an otherwise disorganized system; see Figure 1. This work can be seen as a first instance where this rapid frequency synchronization through growth is mathematically corroborated.

Finally, we point out that the asymptotic estimate (1.3), while not explicitly stated in [18], follows from an extension of the analysis of [1, Section 4]; for a detailed proof, see for instance [5, Appendix A] in the context of the Fisher-KPP equation. The estimate (1.4) follows directly from the existence construction in [18] which finds the pattern-forming front in the unstable manifold of a hyperbolic periodic orbit representing the wave train in the corresponding traveling wave formulation.

2 Overview, challenges, setup, and main result

A standard approach to studying nonlinear stability of a given coherent structure $\mathbf{u}_*$, say in a reaction-diffusion system $\mathbf{u}_t = D\mathbf{u}_{\xi\xi} + c\mathbf{u}_\xi + f(\mathbf{u})$, is to derive an equation for the perturbation $\mathbf{w}(\xi, t) = \mathbf{u}(\xi, t) - \mathbf{u}_*(\xi)$, of the form

$$\mathbf{w}_t = \mathcal{A}\mathbf{w} + \mathcal{N}(\mathbf{w}),$$

where $\mathcal{A}$ is the linearization about $\mathbf{u}_*$ and $\mathcal{N}$ is the resulting quadratic nonlinear remainder. One then studies the behavior of $\mathbf{w}$ by analyzing the associated *variation of constants formula*

$$\mathbf{w}(t) = e^{\mathcal{A}t}\mathbf{w}_0 + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{N}(\mathbf{w}(s))ds,$$

where $\mathbf{w}_0(\xi) = \mathbf{u}(\xi, 0) - \mathbf{u}_*(\xi)$ is the initial perturbation, and $e^{\mathcal{A}t}$ is a strongly continuous semigroup generated by $\mathcal{A}$. In general, one needs to prove that $\mathcal{A}$ indeed generates a suitable semigroup, although this is automatic by standard results if, for instance, $\mathcal{A}$ is elliptic. One then hopes to establish decay of $\mathbf{w}(t)$ by combining decay estimates on $e^{\mathcal{A}t}$ with a contraction mapping or iterative argument on the variation of constants formula. Hence, the key first step is to obtain sharp linear decay estimates on the associated semigroup $e^{\mathcal{A}t}$. In Sections 2.1 and 2.2, we explain challenges to obtaining suitable linear and nonlinear estimates to close a nonlinear iteration argument in our present context, before setting up and precisely formulating our main result in Sections 2.3 through 2.5.

2.1 Linear diffusive stability

Classical semigroup estimates. If the spectrum of the linearization $\mathcal{A}$ is strictly contained in the left half-plane and a spectral mapping theorem holds, solutions to the linearized equation decay exponentially in time yielding nonlinear stability by classical arguments. If the spectrum is stable except for some discrete eigenvalues at the origin, typically related to translation invariance or other symmetries of the original equation, then one can define spectral projections which separate these neutral modes and recover nonlinear stability of the family of underlying traveling waves; see [38, 48] and references therein. In some cases when the essential spectrum is unstable or marginally stable, exponential weights can be used to push the essential spectrum into the left half-plane and return to a setting where classical arguments give stability [67].

Exponential weights. In our case, the essential spectrum of the linearization about $\mathbf{u}_{\text{fr}}$ is unstable in $L^2(\mathbb{R}, \mathbb{C}^2)$ due to the instability of the background state $\mathbf{u} \equiv 0$ to which $\mathbf{u}_{\text{fr}}$ converges at $\xi = +\infty$. Since the front is *critical*, traveling with the linear spreading speed, the essential spectrum cannot be fully stabilized with an exponential weight. The optimal choice of weight renders the spectrum marginally stable, touching the imaginary axis at the origin but otherwise contained in the left half-plane. To make this precise, we let

$$\mathcal{A}_{\text{fr}} = D\partial_\xi^2 + c_{\text{lin}}\partial_\xi + F'(\mathbf{u}_{\text{fr}})$$

denote the linearization of (1.5) about the invasion front $\mathbf{u}_{\text{fr}}$, and define a smooth, positive, monotonically increasing weight function ω satisfying

$$\omega(\xi; a, \gamma, \varepsilon) = \begin{cases} 1, & \xi \leq 0, \\ e^{\eta_{\text{lin}}(a, \gamma, \varepsilon)\xi}, & \xi \geq 1. \end{cases}$$

We will from now on suppress the dependence of ω and η_{lin} on a , γ and ε . We will restrict to perturbations $\mathbf{w}$ of $\mathbf{u}_{\text{fr}}$ for which $\omega\mathbf{w} \in L^2(\mathbb{R})$. The spectral problem for perturbations of this type is equivalent to the spectral problem for the conjugate operator

$$\mathcal{L}_{\text{fr}} = \omega\mathcal{A}_{\text{fr}}\omega^{-1}, \quad (2.1)$$

acting on $L^2(\mathbb{R})$. Here and everywhere after, ω^{-1} denotes the function $\xi \mapsto \frac{1}{\omega(\xi)}$, not the inverse of the function $\xi \mapsto \omega(\xi)$.

Critical diffusive modes. The essential spectrum of $\mathcal{L}_{\text{fr}}$ touches the imaginary axis at the origin in two different ways: one curve of essential spectrum is associated to the dynamics at $+\infty$, and has the expansion

$$\lambda_+(ik) = -D_{\text{eff}}^+ k^2 + O(k^3), \quad (2.2)$$

while the other is associated to dynamics near the wave train at $-\infty$, and has the expansion

$$\lambda_{\text{wt}}(ik) = -ic_g k - D_{\text{eff}}^{\text{wt}} k^2 + O(k^3), \quad (2.3)$$

where $D_{\text{eff}}^+, D_{\text{eff}}^{\text{wt}} > 0$, and $c_g < 0$. The expansion of the first curve naturally arises from marginal pointwise stability in the leading edge, which characterizes the linear spreading speed. The second expansion follows from standard diffusive spectral stability of the wave train in the wake with negative group velocity c_g in a frame moving with the linear spreading speed; see Figure 2, middle panel, for a depiction of these critical curves. These two curves, though both marginally stable, are quite different in that the first has a branch point at the origin while the second does not. We refer to the dynamics associated to the first curve as a *branched diffusive mode*, and those associated to the second as an *outgoing diffusive mode*. The fact that $c_g < 0$ means that perturbations to the wave train in the wake are, at least on the linear level, transported to the left, away from the front interface [26], which is why we refer to these modes as *outgoing*.

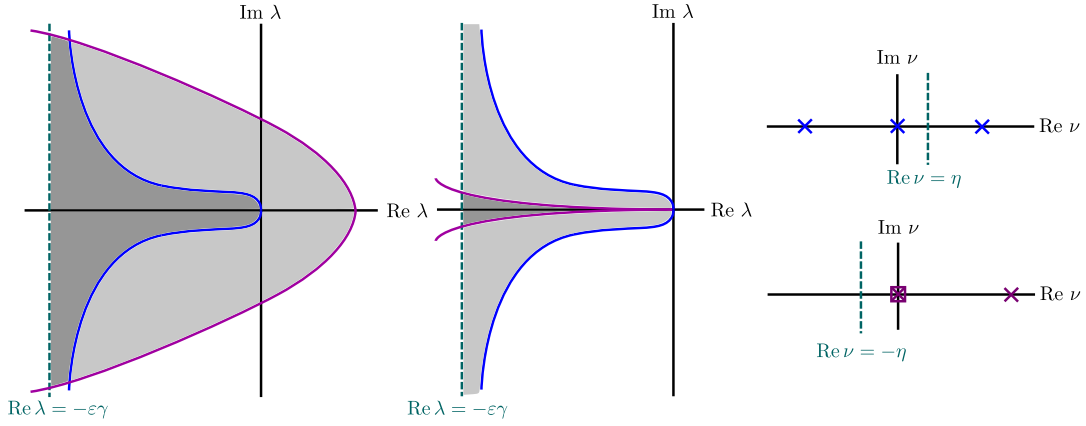


Figure 2: Spectrum of the unweighted linearization $\mathcal{A}_{\text{fr}}$ (left) and the weighted linearization $\mathcal{L}_{\text{fr}}$ (middle). The purple and blue curves denote the essential spectrum of the limiting operators at $\pm\infty$, respectively. These curves are the Fredholm borders of $\mathcal{A}_{\text{fr}}/\mathcal{L}_{\text{fr}}$: in the shaded regions, the operator is Fredholm but with index -1 (light grey) or -2 (darker grey), with the index changing across the Fredholm borders. Top right: spatial Floquet exponents of $\mathcal{L}_{\text{wt}}$. Bottom right: spatial eigenvalues of $\mathcal{L}_+$; the box indicates a double spatial eigenvalue at the origin. See Section 3 for details about spatial Floquet exponents and eigenvalues.

Branched diffusive modes occur naturally in the stability of pulled invasion fronts [2, 7, 9, 31, 32], and also of stationary periodic patterns [71, 72], degenerate viscous shock waves [40], layer solutions in phase separation models [42], and contact defects [63, 64]. Outgoing diffusive modes arise naturally in the study of the stability of undercompressive shock waves [77], traveling periodic waves [26, 45, 46], and source defects [13, 63].

The curve (2.2) of essential spectrum associated to the branched diffusive mode cannot be stabilized with an exponential weight, since a stable and an unstable spatial eigenvalue must collide at the branch point; in the language of [61], there is *absolute spectrum* at the origin. See Section 3 for further details on spatial eigenvalues and their relation to essential spectrum. At the linear level, the curve (2.3) of essential spectrum associated to the outgoing diffusive mode can be moved into the left half-plane with exponential weights, but these weights turn out to be incompatible with a nonlinear argument due to the outgoing group velocity. Indeed, conjugating the equation with an exponential weight which stabilizes this spectral curve introduces coefficients in the nonlinearity which grow exponentially as $\xi \rightarrow -\infty$, and so one is not able to control the nonlinearity with this approach. One therefore has no choice but to address a stability problem where continuous curves of essential spectrum touch the imaginary axis, and, as a result, work with algebraic decay of perturbations in the chosen function space.

When the underlying coherent structure is a purely periodic wave train, one can represent the corresponding semigroup via the Bloch wave transform, analogous to the Fourier representation of the heat semigroup, and use this representation to obtain algebraic time decay estimates for perturbations in the linearized equation [44, 45, 46, 71, 72]. This approach does not work for front solutions connecting two different end states, since no analogue of the Bloch or Fourier transform is available. An exception is in the study of supercritical pattern-forming fronts, that is, fronts traveling faster than the selected speed predicted by the marginal stability conjecture. In this case, the dynamics associated to the front tail and interface are exponentially stable, and the stability problem can be reduced to the stability of the periodic pattern in the wake [27, 28]. We emphasize that these supercritical fronts are not observed when invasion originates from highly localized initial conditions, and the methods of [27, 28] do not appear to generalize to critical invasion fronts.

Pointwise semigroup estimates. Originally developed to analyze stability of undercompressive viscous shock waves [77], pointwise semigroup methods are a powerful tool for establishing linear and nonlinear

stability of front-like solutions in the presence of critical diffusive modes. Since the obstruction to obtaining linear stability estimates via a classical approach is the essential spectrum near the origin, the idea is to closely analyze the behavior of the resolvent operator near the critical essential spectrum. The hope is that with detailed enough information on this resolvent, one can deform the inverse Laplace integration contours arbitrarily close to or even into the essential spectrum, and recover algebraic decay estimates. In the now-seminal pointwise semigroup approach, the resolvent is analyzed by constructing its integral kernel (or spatial Green's function) $G_\lambda(x, y)$, which solves

$$(\mathcal{L} - \lambda)G_\lambda(x, y) = \delta(x - y)I, \quad (2.4)$$

where $\mathcal{L}$ is the linearization about the coherent structure of interest, δ is the Dirac delta function, and I is the identity. The solution to the linearized equation $u_t = \mathcal{L}u$, with initial data u_0 , is then represented by

$$u(x, t) = \int_{\mathbb{R}} G(x, y, t)u_0(y)dy,$$

where $G(x, y, t)$ is the temporal Green's function, expressed through the inverse Laplace transform

$$G(x, y, t) = -\frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} G_\lambda(x, y) d\lambda, \quad (2.5)$$

and Γ is some contour initially lying to the right of the spectrum of $\mathcal{L}$. Of course, the resolvent operator $(\mathcal{L} - \lambda)^{-1}$ is singular near the spectrum of $\mathcal{L}$. However, if all critical modes are outward diffusive modes, then the resolvent kernel $G_\lambda(x, y)$ remains analytic or at least meromorphic in λ for fixed x and y even as λ passes into the essential spectrum. The loss of analyticity of the resolvent is due to a loss of spatial localization of $G_\lambda(x, y)$, so that the resolvent is no longer a bounded operator between fixed function spaces. Such a loss of spatial localization can be repaired by conjugating with an exponential weight and, thus, $G_\lambda(x, y)$ remains pointwise, that is, for fixed x, y , analytic. Consequently, the integration contour Γ in (2.5) may be deformed into the essential spectrum of $\mathcal{L}$. Sharp linear decay estimates can then be obtained by deforming to pointwise contours $\Gamma_{x,y,t}$, conveniently chosen for each combination of x, y , and t , to extract spatio-temporal behavior of the inverse Laplace integral (2.5), using sharp estimates on the resolvent kernel $G_\lambda(x, y)$. Estimates on the resolvent kernel are typically established by solving (2.4) via a construction of the spatial Green's function through *exponential dichotomies* [77].

In the presence of branched diffusive modes, the resolvent kernel $G_\lambda(x, y)$ itself has a branch point at $\lambda = 0$ for each fixed $x, y \in \mathbb{R}$, and so integration contours can no longer be deformed past this singularity. Nonetheless, pointwise semigroup methods have been successfully adapted to problems with branched diffusive modes, using pointwise defined contours which pass near the essential spectrum but remain to the right of the branch point [31, 32, 40, 42, 43].

From this perspective, our results rely on a linear problem with outgoing and branched modes as in [43], albeit in a context where both the linear and the nonlinear argument are considerably more delicate as we shall explain in the remainder of this introduction. Closer to the problems that we encounter here are results on the stability of invasion fronts in the real Ginzburg-Landau equation [8, 17, 30]. The pulled front there is special in the sense that, although it selects a spatially constant state rather than a periodic pattern, the selected constant state is only diffusively stable instead of exponentially stable. This relates to the fact that the real Ginzburg-Landau equation is a universal modulation equation for pattern-forming systems near a Turing instability, and the pulled front inherits spectral properties of pattern-forming invasion fronts. In [8], the stability of fronts and the interaction of branched and outgoing modes on the linear level are handled by a far-field/core decomposition of the resolvent, a technique that we rely on here as well, and that we describe next.

Linear estimates via resolvent decompositions. As an alternative to pointwise estimates, we rely on a somewhat more direct and efficient functional analytic approach to stability problems developed in [7, 8].

Following the strategy outlined above based on an inverse Laplace transform, we carefully analyze the behavior of the resolvent near the origin, and then deform contours close to the essential spectrum to extract temporal decay. We do not however construct the very detailed and at times cumbersome pointwise integral kernel $G_\lambda(x, y)$ and associated temporal Green's function, but rather solve the resolvent equation

$$(\mathcal{L}_{\text{fr}} - \lambda)\mathbf{u} = \mathbf{g} \quad (2.6)$$

by decomposing both the data $\mathbf{g}$ and the solution $\mathbf{u}$ using a partition of unity, into two far-field parts which are supported on the wake and on the leading-edge of the front, and a core part which is strongly localized. We then rely on somewhat straightforward Fredholm properties in exponentially localized spaces rather than the more subtle constructions based on for instance the Gap Lemma [35, 49]; see for instance [65] for comparisons.

We naturally arrive at an explicit spatial decomposition of the solution to the resolvent equation (2.6), in which some terms only “see” the outgoing diffusive mode, some terms “see” the branched diffusive mode, and another, crucial term encodes their interaction. We can then adapt our integration contours specifically to each term in this decomposition, using classical pointwise semigroup contours for the terms which only see the outgoing mode, contours adapted to branched modes as in [7, 9] for the branched mode, and contours adapted to the interaction terms. Somewhat more explicitly, the decomposition of the linear semigroup $e^{\mathcal{L}_{\text{fr}} t}$ that we obtain is of the form

$$[e^{\mathcal{L}_{\text{fr}} t} \mathbf{g}](\xi) = \mathbf{u}'_{\text{fr}}(\xi)[S_p(t)\mathbf{g}](\xi) + [S_c(t)\mathbf{g}](\xi) + [S_e(t)\mathbf{g}](\xi), \quad (2.7)$$

where, roughly speaking, we have

$$\|S_p(t)\mathbf{g}\|_{L^\infty} \sim (1+t)^{-\frac{1}{2}}, \quad \|S_c(t)\mathbf{g}\|_{L^\infty} \sim (1+t)^{-1}, \quad \|S_e(t)\mathbf{g}\|_{L^\infty} \sim e^{-\mu t} \quad (2.8)$$

for some $\mu > 0$. We refer to Theorem 7.1 for precise linear estimates. For now, we point out that the first term in (2.7) is the slowest decaying, with diffusive rate $(1+t)^{-1/2}$, and is in fact supported on the wake of the front, $\xi < 0$, only.

2.2 Nonlinear diffusive stability

To study the nonlinear stability of the pulled pattern-forming front $\mathbf{u}_{\text{fr}}$, we analyze the dynamics of a perturbed solution to (1.5) of the form $\mathbf{u}(\xi, t) = \mathbf{u}_{\text{fr}}(\xi) + \mathbf{w}(\xi, t)$, with sufficiently localized initial perturbation $\mathbf{w}(\xi, 0) = \mathbf{w}_0(\xi)$. To enforce the exponential localization needed to stabilize the unstable state in the leading edge, we define the weighted perturbation $\tilde{\mathbf{v}}(\xi, t) = \omega(\xi)\mathbf{w}(\xi, t)$, with induced equation

$$\tilde{\mathbf{v}}_t = \mathcal{L}_{\text{fr}} \tilde{\mathbf{v}} + \tilde{\mathcal{N}}(\tilde{\mathbf{v}}), \quad (2.9)$$

where the nonlinearity $\tilde{\mathcal{N}}$ is given through

$$\tilde{\mathcal{N}}(\tilde{\mathbf{v}}) = \omega \tilde{\mathcal{N}}(\omega^{-1} \tilde{\mathbf{v}}), \quad \tilde{\mathcal{N}}(\mathbf{w}) = F(\mathbf{u}_{\text{fr}} + \mathbf{w}) - F(\mathbf{u}_{\text{fr}}) - F'(\mathbf{u}_{\text{fr}})\mathbf{w}.$$

A standard approach to nonlinear stability is to attempt to close an iteration argument based on a variation-of-constants formula for the perturbation equation (2.9).

From (2.8), we see that the decay rates exhibited by the full semigroup $e^{\mathcal{L}_{\text{fr}} t}$ are diffusive, that is, they coincide with the decay rates of the heat semigroup $e^{\partial_\xi^2 t}$. On the other hand, one finds that the nonlinearity $\tilde{\mathcal{N}}(\tilde{\mathbf{v}})$ in (2.9) contains quadratic terms in $\tilde{\mathbf{v}}$ which in general does not allow one to close a nonlinear argument, as is well known for instance in the case of the nonlinear heat equation $u_t = u_{xx} + u^2$ where all nonnegative nontrivial initial data blow up in finite time [34]. This difficulty does not arise in the stability of viscous shocks [43] since quadratic terms involve derivatives that induce stronger decay. It also does not

arise when establishing nonlinear stability of pulled fronts with an exponentially stable state in the wake since the linear diffusive decay in the leading edge is stronger, with rate $t^{-\frac{3}{2}}$, sufficient to close a nonlinear stability argument; see [7]. The improved decay results from the fact that the front interface provides an absorption mechanism similar to an absorbing boundary condition for the heat equation on the half line.

The weak stability in the wake at rate $t^{-\frac{1}{2}}$ does arise in the analysis of pulled fronts in the Ginzburg-Landau equation [8, 17, 30]. In this situation, there is however an astute change of coordinates that exploits the symmetry from the gauge invariance and exhibits gradients in the relevant parts of the nonlinearity. Explicitly, after switching to polar coordinates, the linearization diagonalizes with the outgoing diffusive mode only manifesting itself in the phase variable, and the associated nonlinearities at the relevant quadratic and cubic order carry derivatives with stronger associated decay. Such an explicit transformation does not seem to be available or convenient in our setting; see [70] for an analysis from this perspective of normal forms.

Forward and backward modulation of perturbations. Motivated by the fact that the terms in (2.7) with the slowest decay stem from the diffusively stable pattern in the wake of the front, we take inspiration from the nonlinear stability analysis of wave trains, see e.g. [44, 45, 46, 66] and references therein. It was observed in [26] that the most critical diffusive dynamics of perturbed wave trains can be captured by a spatio-temporal phase modulation. Intuitively speaking, such a phase modulation accounts for the translational invariance of the pattern, which causes spectrum to touch the imaginary axis at the origin. Such a phase modulation has been incorporated in the nonlinear stability analysis of wave trains, see [45, 46], by considering the *inverse-modulated perturbation*

$$\mathbf{v}(\xi, t) = \omega(\xi) (\mathbf{u}(\xi - \psi(\xi, t), t) - \mathbf{u}_{\text{fr}}(\xi)). \quad (2.10)$$

Here, the phase modulation function $\psi(t)$ is chosen a posteriori such that it accounts for the most critical terms in the Duhamel formulation for $\mathbf{v}(t)$. Our analysis shows that the ansatz (2.10) is also successful in the case of pulled pattern-forming fronts to control the diffusive dynamics of the periodic pattern in the wake. Key to this argument is the spatial decomposition (2.7) of the linearized dynamics, which shows that the slowest decaying terms are supported only in the wake of the invasion front. The resulting spatio-temporal modulation argument is then quite similar to that used for the nonlinear stability of pure wave trains [44, 45, 46, 66].

A natural alternative to modulating the perturbed solution $\mathbf{u}(t)$ is to modulate the pattern-forming front $\mathbf{u}_{\text{fr}}$ itself which leads us to the *forward-modulated perturbation*

$$\hat{\mathbf{v}}(\xi, t) = \omega(\xi) (\mathbf{u}(\xi, t) - \mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t))).$$

It has recently been shown in [76] that the $W^{k,p}$ -norms of the forward- and inverse-modulated perturbations are equivalent up to controllable norms of ψ_ξ . Thus, one finds that the forward- and inverse-modulated perturbations exhibit the same decay rates and it suffices to close a nonlinear argument for one of them. Nevertheless, it turns out to be advantageous to use both the forward- and inverse-modulated perturbation variables in the nonlinear argument. The reason is twofold. On the one hand, the equation for the inverse-modulated perturbation is quasilinear, introducing an apparent loss of derivatives in the nonlinear iteration scheme. On the other hand, the equation for the forward-modulated perturbation is semilinear, but contains terms which are not sufficiently rapidly decaying to close a nonlinear argument through iterative estimates on the associated Duhamel formulation, see [76, Section 5.2] for more details. We emphasize that these observations do not rely on the fact that the underlying solution is a periodic wave train and hold for any traveling wave, thus, in particular, for the pattern-forming fronts under consideration. Hence, we follow the approach, as proposed in [76], of establishing sharp bounds through iterative estimates on the Duhamel formula for the inverse-modulated perturbation $\mathbf{v}(t)$, while controlling regularity through nonlinear damping estimates on $\hat{\mathbf{v}}(t)$, thereby using the equivalence of $W^{k,p}$ -norms of $\mathbf{v}(t)$ and $\hat{\mathbf{v}}(t)$ modulo controllable errors.

Nonlinear damping estimates. We use energy estimates to effectively control H^k -norms of the perturbation $\hat{\mathbf{v}}(t)$ in terms of its L^2 -norm and the H^k -norm of its initial condition for $k \geq 0$. Such “nonlinear damping estimates” generally rely on damped high-frequency spectrum of the linearization. In fact, the same linear terms in the FitzHugh-Nagumo system (1.5) that yield high-frequency resolvent estimates in Appendix B are crucial for obtaining a nonlinear damping estimate. More concisely, high-frequency resolvent bounds are equivalent to *linear* damping estimates, which readily yield associated nonlinear damping estimates as long as solutions stay small; see for instance [59] for further discussion of this principle in the setting of the St. Venant equations. Here, the nonlinear damping estimate for $\hat{\mathbf{v}}(t)$ is induced by the second derivative $\partial_{\xi\xi}u_1$ in the first component and the term $-\epsilon\gamma u_2$ in the second component of (1.5). We remark that in reaction-diffusion systems of the form $\mathbf{u}_t = D\mathbf{u}_{xx} + F(\mathbf{u})$, with positive definite, nondegenerate diffusion matrix D , damping comes from the highest-order derivative $D\mathbf{u}_{xx}$, allowing control even of quasilinear terms so that, in that setting, one can directly obtain a nonlinear damping estimate for the inverse-modulated perturbation and it is not necessary to consider the forward-modulated perturbation, see [46].

2.3 Function spaces and notation

We introduce function spaces used in the analysis and in the precise statement of our main result.

Exponentially weighted spaces. Given $\eta_{\pm} \in \mathbb{R}$, we define a smooth positive weight function ω_{η_-, η_+} satisfying

$$\omega_{\eta_-, \eta_+}(\xi) = \begin{cases} e^{\eta_- \xi}, & \xi \leq -1, \\ e^{\eta_+ \xi}, & \xi \geq 1. \end{cases}$$

Given additionally non-negative integers k and m , a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and a real number $1 \leq p \leq \infty$, we define a corresponding weighted Sobolev space $W_{\exp, \eta_-, \eta_+}^{k, p}(\mathbb{R}, \mathbb{F}^m)$ through the norm

$$\|f\|_{W_{\exp, \eta_-, \eta_+}^{k, p}} = \|\omega_{\eta_-, \eta_+} f\|_{W^{k, p}},$$

where $W^{k, p}(\mathbb{R}, \mathbb{F}^m)$ is the standard Sobolev space with differentiability index k and integrability index p . When $p = 2$, we write $W_{\exp, \eta_-, \eta_+}^{k, 2}(\mathbb{R}, \mathbb{F}^m) = H_{\exp, \eta_-, \eta_+}^k(\mathbb{R}, \mathbb{F}^m)$. When domain and codomain can be readily inferred, we write $W_{\exp, \eta_-, \eta_+}^{k, p}(\mathbb{R}, \mathbb{F}^m) = W_{\exp, \eta_-, \eta_+}^{k, p}$. Finally, we write $W_{\exp, \eta_-, \eta_+}^{0, p} = L_{\exp, \eta_-, \eta_+}^p$.

Given $\eta \in \mathbb{R}$ and $1 \leq p \leq \infty$, we define $X_{\eta}^p = L_{\exp, -\eta, \eta}^p(\mathbb{R}, \mathbb{C}^2)$ and let

$$Y_{\eta}^p = W_{\exp, -\eta, \eta}^{2, p}(\mathbb{R}, \mathbb{C}) \times W_{\exp, -\eta, \eta}^{1, p}(\mathbb{R}, \mathbb{C})$$

denote the domain of the linearization $\mathcal{L}_{\text{fr}}$, given by (2.1), on X_{η}^p , so that $\mathcal{L}_{\text{fr}} : Y_{\eta}^p \subset X_{\eta}^p \rightarrow X_{\eta}^p$ is a closed operator.

Algebraically weighted spaces. To exert finer control over the spatial localization of perturbations, we also introduce algebraically weighted spaces. Given $r_{\pm} \in \mathbb{R}$, we define a smooth positive weight function

$$\rho_{r_-, r_+}(\xi) = \begin{cases} |\xi|^{r_-}, & \xi \leq -1, \\ |\xi|^{r_+}, & \xi \geq 1. \end{cases}$$

As in the definition of exponentially weighted spaces, given additionally non-negative integers k and m , a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, and a real number $1 \leq p \leq \infty$, we define a corresponding algebraically weighted Sobolev space $W_{r_-, r_+}^{k, p}(\mathbb{R}, \mathbb{F}^m)$ through the norm

$$\|f\|_{W_{r_-, r_+}^{k, p}} = \|\rho_{r_-, r_+} f\|_{W^{k, p}}.$$

We again suppress the notation for the domain and codomain when clear in context, and we write $W_{r_-, r_+}^{0,p} = L_{r_-, r_+}^p$.

Additional function spaces. An additional technical challenge we must address is that the FitzHugh-Nagumo system is not fully parabolic, so the linearization $\mathcal{L}_{\text{fr}}$ is not a sectorial operator. Spectral mapping in such contexts is not automatic. We will first prove spectral mapping properties for initial data $\mathbf{u}_0 \in C_c^\infty(\mathbb{R})$, and then extend the semigroup to larger function spaces. To carry out this extension, we consider spaces in which such test functions are dense. Therefore, given non-negative integers k and m and a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we let $C_0^k(\mathbb{R}, \mathbb{F}^m)$ denote the space of bounded continuous functions $\mathbf{u} : \mathbb{R} \rightarrow \mathbb{F}^m$ satisfying

$$\lim_{\xi \rightarrow \pm\infty} |\partial_\xi^\ell \mathbf{u}(\xi)| = 0$$

for $\ell = 0, 1, \dots, k$. Note that such functions (and their ℓ -th derivatives) are automatically uniformly continuous. We equip this space with the $W^{k,\infty}$ norm and recall the well-known fact that test functions are dense in this space.

We will use a mix of C_0 and L^2 -based spaces in our nonlinear argument, and therefore define the spaces

$$Z_k(\mathbb{R}, \mathbb{F}^m) := H^k(\mathbb{R}, \mathbb{F}^m) \cap C_0^k(\mathbb{R}, \mathbb{F}^m),$$

endowed with norm $\|\cdot\|_{H^k} + \|\cdot\|_{W^{k,\infty}}$, for any non-negative integers k and m and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Spaces involving time. Given an interval $I \subseteq [0, \infty)$ and a Banach space X , we let $C^k(I, X)$ denote the space of k -continuously differentiable functions on I taking values in X .

Additional notation. We will sometimes abuse notation by writing a function $\mathbf{u}(\xi, t)$ of space and time as $\mathbf{u}(t)$, viewing it as a function of time taking values in a particular Banach space. Similarly, we may write a function $\mathbf{u}(\xi; \lambda)$ of a spectral parameter λ as $\mathbf{u}(\lambda)$. We also let (χ_-, χ_c, χ_+) be a partition of unity on $\mathbb{R}$ such that

$$\chi_+(\xi) = \begin{cases} 1, & \xi \geq 1 \\ 0, & \xi \leq 0, \end{cases}$$

and $\chi_-(\xi) = \chi_+(-\xi)$.

Suppression of constants. Let S be a set, and let $A, B : S \rightarrow \mathbb{R}$. The expression “ $A(x) \lesssim B(x)$ for $x \in S$ ”, means that there exists a constant $C > 0$, independent of x , such that $A(x) \leq CB(x)$ holds for all $x \in S$.

2.4 Spectral assumptions

We formulate spectral stability hypotheses that precisely describe pulled uniformly translating pattern-forming fronts. We prove in [3, 4] that these conditions hold for the fronts $\mathbf{u}_{\text{fr}}$ constructed in Theorem 1.1 in open parameter ranges, and we expect these assumptions to hold in open classes of systems, generically for pulled uniformly translating pattern-forming fronts.

We decompose the spectrum $\Sigma(\mathcal{L})$ of an operator $\mathcal{L}$ as follows. We say that $\lambda \in \mathbb{C}$ is in the essential spectrum $\Sigma_{\text{ess}}(\mathcal{L})$ if either $\mathcal{L} - \lambda$ is not Fredholm, or it is Fredholm with nonzero index. We say $\lambda \in \mathbb{C}$ is in the point spectrum $\Sigma_{\text{pt}}(\mathcal{L})$ if $\mathcal{L} - \lambda$ is Fredholm with index zero, but not invertible. The essential

spectrum is invariant under compact perturbations. Thus, for the operator $\mathcal{L}_{\text{fr}}$, the essential spectrum may be determined from the linearizations about the asymptotic end states at $\pm\infty$, which are

$$\mathcal{L}_+ = D(\partial_\xi - \eta_{\text{lin}})^2 + c_{\text{lin}}(\partial_\xi - \eta_{\text{lin}}) + F'(0), \quad (2.11)$$

at $\xi = +\infty$, and

$$\mathcal{L}_{\text{wt}} = D\partial_\xi^2 + c_{\text{lin}}\partial_\xi + F'(\mathbf{u}_{\text{wt}}(\xi)), \quad (2.12)$$

at $\xi = -\infty$. The spectrum of $\mathcal{L}_+$ is marginally stable, a fact that effectively characterizes the linear spreading speed; see Lemma 3.2. We now state assumptions on the essential spectrum of $\mathcal{L}_{\text{wt}}$ and on the point spectrum of $\mathcal{L}_{\text{fr}}$.

Essential spectrum of the wave train. The limiting operator $\mathcal{L}_{\text{wt}}$ in the wake has coefficients which are periodic in ξ with period L . By Floquet-Bloch theory (see for instance [58]), $\mathcal{L}_{\text{wt}}$ is conjugate to a family of Bloch operators

$$\hat{\mathcal{L}}_{\text{wt}}(\nu) : H^2(\mathbb{R}/L\mathbb{Z}, \mathbb{C}) \times H^1(\mathbb{R}/L\mathbb{Z}, \mathbb{C}) \subset L^2(\mathbb{R}/L\mathbb{Z}, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}/L\mathbb{Z}, \mathbb{C}^2)$$

given by

$$\hat{\mathcal{L}}_{\text{wt}}(\nu) = D(\partial_\xi + \nu)^2 + c_{\text{lin}}(\partial_\xi + \nu) + F'(\mathbf{u}_{\text{wt}}(\xi))$$

for ν purely imaginary, $\nu \in [-ik_{\text{wt}}, ik_{\text{wt}})$, with $k_{\text{wt}} = \frac{\pi}{L}$. In particular, we have

$$\Sigma(\mathcal{L}_{\text{wt}}) = \bigcup_{\nu \in [-ik_{\text{wt}}, ik_{\text{wt}})} \Sigma(\hat{\mathcal{L}}_{\text{wt}}(\nu)).$$

We assume that $\mathcal{L}_{\text{wt}}$ satisfies the following standard diffusive spectral stability assumptions for periodic wave trains [46, 66, 72].

Hypothesis 1 (Diffusive spectral stability of the wave train). *We assume that the following spectral stability conditions hold for the operator $\mathcal{L}_{\text{wt}} : H^2(\mathbb{R}, \mathbb{C}) \times H^1(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}, \mathbb{C}^2)$ given by (2.12):*

1. *We have $\Sigma(\mathcal{L}_{\text{wt}}) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda < 0\} \cup \{0\}$;*
2. *There exists $\theta > 0$ such that for any $\kappa \in [-k_{\text{wt}}, k_{\text{wt}})$, we have $\text{Re } \Sigma(\hat{\mathcal{L}}_{\text{wt}}(i\kappa)) \leq -\theta\kappa^2$;*
3. *$\lambda = 0$ is an algebraically simple eigenvalue of $\hat{\mathcal{L}}_{\text{wt}}(0)$.*

We note that 0 is always in the spectrum of $\hat{\mathcal{L}}_{\text{wt}}(0)$ with associated eigenfunction $\partial_\xi \mathbf{u}_{\text{wt}}$ by translation invariance of the wave train. The spectral information encoded in Hypothesis 1 is sufficient to establish nonlinear stability of pure wave trains [46, 66], for strictly parabolic systems. For degenerate systems such as (1.2), one has to additionally control high frequency modes to obtain a spectral mapping estimate; see Theorem 2.2. Since the Bloch operators $\hat{\mathcal{L}}_{\text{wt}}(\nu)$ depend analytically on the Floquet exponent ν , Hypothesis 1 yields open neighborhoods $U, V \subset \mathbb{C}$ of the origin and an analytic function $\lambda_{\text{wt}} : U \rightarrow \mathbb{C}$ with the expansion

$$\lambda_{\text{wt}}(\nu) = -c_g\nu + D_{\text{eff}}^{\text{wt}}\nu^2 + O(\nu^3), \quad (2.13)$$

with $c_g \in \mathbb{R}$ and $D_{\text{eff}}^{\text{wt}} > 0$ such that the spectrum of $\hat{\mathcal{L}}_{\text{wt}}(\nu)$ in V is given by the simple eigenvalue $\lambda_{\text{wt}}(\nu)$ for $\nu \in U$, see [26]. Consequently, the spectrum of $\mathcal{L}_{\text{wt}}$ in V is given by the curve $\{\lambda_{\text{wt}}(i\kappa) : i\kappa \in U\} \cap V$ touching the origin in a quadratic tangency. The equation $\lambda_{\text{wt}}(\nu) = 0$ is called the *linear dispersion relation*

and the coefficient c_g is the group velocity of the wave train (in the frame moving with the speed c_{lin}). With a standard Lyapunov-Schmidt reduction procedure [26] one obtains

$$c_g = 2 \langle \mathbf{u}_{\text{ad}}, D\partial_{\xi\xi}\mathbf{u}_{\text{wt}} \rangle_{L^2(\mathbb{R}/L\mathbb{Z}, \mathbb{C}^2)} + c_{\text{lin}} = 2 \langle u_{\text{ad},1}, \partial_{\xi\xi}u_{\text{wt},1} \rangle_{L^2(\mathbb{R}/L\mathbb{Z}, \mathbb{C})} + c_{\text{lin}}, \quad (2.14)$$

where $\mathbf{u}_{\text{ad}}$ spans the kernel of the adjoint operator $\hat{\mathcal{L}}_{\text{wt}}(0)^*$ normalized such that $\langle \mathbf{u}_{\text{ad}}, \partial_{\xi}\mathbf{u}_{\text{wt}} \rangle_{L^2(\mathbb{R}/L\mathbb{Z}, \mathbb{C}^2)} = 1$. For pattern-forming fronts, it is important to characterize the group velocity of the wave train in the wake, relative to the front speed, so that the front acts as a *source* of patterns [63]. Thinking of the invasion process as a source of patterns, the group velocity of the wave train should be negative in the frame moving with the spreading speed c_{lin} . We make this precise as follows.

Hypothesis 2 (Outgoing group velocity for the wave train). *Assuming, in accordance with Hypothesis 1, that 0 is an algebraically simple eigenvalue of $\hat{\mathcal{L}}_{\text{wt}}(0)$, we require that the group velocity c_g , given by (2.14), is negative.*

Hypothesis 2 implies that the critical diffusive mode associated to the dynamics near the wave train is an outgoing diffusive mode, in the terminology of Section 2.1. The main result of [4] establishes that (1.2) satisfies Hypotheses 1 and 2 for $(3 - \sqrt{5})/6 < a < \frac{1}{2}$ and ε sufficiently small. It was shown in [18] that the group velocity of the wave trains is negative in the frame moving with the front velocity, so Hypothesis 2 follows from the combination of Hypothesis 1 and [18].

Point spectrum. Hypothesis 1, together with the calculation of the linear spreading speed in Lemma 3.2 below, imply that the essential spectrum of $\mathcal{L}_{\text{fr}}$ is marginally stable; see Proposition 3.5. We exclude unstable point spectrum as follows.

Hypothesis 3 (No unstable point spectrum). *We assume that the operator $\mathcal{L}_{\text{fr}}: H^2(\mathbb{R}, \mathbb{C}) \times H^1(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}, \mathbb{C}^2)$, given by (2.1) has no eigenvalues λ with $\text{Re } \lambda \geq 0$. Moreover, we assume that there is no nontrivial bounded solution $\mathbf{u}$ to the equation $\mathcal{L}_{\text{fr}}\mathbf{u} = 0$.*

A nontrivial bounded solution to $\mathcal{L}_{\text{fr}}\mathbf{u} = 0$ would indicate that the fronts are not strictly pulled, but rather at the boundary between pushed and pulled front propagation [6], and hence we exclude this possibility here. Such a bounded solution is sometimes referred to as a resonance and would correspond to a zero of the associated Evans function, appropriately extended into the essential spectrum via the gap lemma [35, 49]. The improved $t^{-3/2}$ decay rate in the leading edge, when compared with diffusive decay in one dimension, is associated to this lack of a resonance at $\lambda = 0$ [7]. We establish Hypothesis 3 for (1.2) in a natural open parameter range in [3].

2.5 Main result — precise statement

Having introduced the necessary notation, we are ready to formulate the precise statement of our main result.

Theorem 2.1 (Nonlinear stability — precise formulation). *Assume $\mathbf{u}_{\text{fr}}$ is a pulled front solution from Theorem 1.1 which obeys Hypotheses 1 through 3. Then, there exist constants $M, \delta > 0$ such that if we consider (1.5) with initial data $\mathbf{u}_0 = \mathbf{u}_{\text{fr}} + \mathbf{w}_0$, where $\mathbf{w}_0 \in L^1_{0,1}(\mathbb{R}, \mathbb{R}^2) \cap (Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R}))$ satisfies*

$$E_0 := \|\omega\mathbf{w}_0\|_{L^1_{0,1}} + \|\omega\mathbf{w}_0\|_{H^3 \times H^2} < \delta, \quad (2.15)$$

then there exist functions $\mathbf{w} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$ and $\psi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\omega\mathbf{w} \in C([0, \infty), Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R})) \cap C^1([0, \infty), Z_1(\mathbb{R}, \mathbb{R}^2)), \quad \psi \in C^\infty([0, \infty) \times \mathbb{R}, \mathbb{R}) \quad (2.16)$$

and

$$\partial_t^k \psi(\cdot, t) \in H^\ell(\mathbb{R}, \mathbb{R}), \quad t > 0, k, \ell \in \mathbb{N}_0,$$

with initial conditions $\mathbf{w}(0) = \mathbf{w}_0$ and $\psi(0) = 0$, such that $\mathbf{u}(t) = \mathbf{u}_{\text{fr}} + \mathbf{w}(t)$ is the unique global classical solution of (1.5) with initial data $\mathbf{u}_0$, enjoying the following estimates

$$\|\omega(\mathbf{u}(t) - \mathbf{u}_{\text{fr}})\|_{H^3 \times H^2} + \|\psi(t)\|_{L^2} \leq \frac{ME_0}{(1+t)^{\frac{1}{4}}}, \quad \|\omega(\mathbf{u}(t) - \mathbf{u}_{\text{fr}})\|_{L^\infty} + \|\psi(t)\|_{L^\infty} \leq \frac{ME_0}{(1+t)^{\frac{1}{2}}}, \quad (2.17)$$

and

$$\begin{aligned} \|\omega(\mathbf{u}(t) - \mathbf{u}_{\text{fr}}(\cdot + \psi(\cdot, t)))\|_{H^3 \times H^2} + \|(\psi_\xi(t), \partial_t \psi(t))\|_{H^3 \times H^2} &\leq \frac{ME_0}{(1+t)^{\frac{3}{4}}}, \\ \|\omega(\mathbf{u}(t) - \mathbf{u}_{\text{fr}}(\cdot + \psi(\cdot, t)))\|_{L^\infty} + \|(\psi_\xi(t), \partial_t \psi(t))\|_{L^\infty} &\leq \frac{ME_0}{1+t}, \end{aligned} \quad (2.18)$$

for all $t \geq 0$. Furthermore, the following refined estimate holds on the leading edge:

$$\sup_{\xi \geq 1} \left| \frac{\omega(\xi)}{1+\xi} (\mathbf{u}(\xi, t) - \mathbf{u}_{\text{fr}}(\xi)) \right| \leq \frac{ME_0}{(1+t)^{\frac{3}{2}}} \quad (2.19)$$

for all $t \geq 0$. Finally, the phase modulation function is supported on the wake: we have $\text{supp}(\psi(\cdot, t)) \subset (-\infty, 0]$ for all $t \geq 0$.

Rephrasing, Theorem 2.1 establishes the following:

1. The weighted, unmodulated perturbation to the front, $\omega(\mathbf{u}(t) - \mathbf{u}_{\text{fr}})$, decays in $L^\infty(\mathbb{R})$ with diffusive rate $(1+t)^{-1/2}$, provided the initial perturbation $\mathbf{w}_0$ is sufficiently small and localized.
2. The leading-order behavior of the perturbation is described by a spatio-temporal phase modulation $\psi(\xi, t)$ of the pattern-forming front $\mathbf{u}_{\text{fr}}$, where $\psi(\cdot, t)$ is supported on $(-\infty, 0]$. That is, we only need to modulate the phase in the wake.
3. In the leading edge, $\xi \geq 1$, the perturbation decays with an improved rate $(1+t)^{-3/2}$ characteristic of pulled invasion fronts [2, 7, 9].

Choice of function spaces. We briefly explain the choices of function spaces we make in Theorem 2.1. The additional algebraic localization of the initial data enforced by requiring $\omega \mathbf{w}_0 \in L^1_{0,1}$ is needed to obtain optimal linear decay rates for the pulled front dynamics in the leading edge [2, 7, 8]. A nonlinear stability argument could then be closed measuring the solution only in L^2 based spaces, but we choose to additionally measure in L^∞ in order to capture the sharp pointwise decay rate of the perturbation — notice that the L^∞ decay rates in Theorem 2.1 are faster than the L^2 decay rates. On the other hand, even if we are primarily interested in L^∞ decay rates, we control regularity in the resulting quasilinear iteration scheme through energy estimates, for which we must additionally measure the solution in L^2 -based spaces. From the structure of the equation, it is natural to expect to require $H^2 \times H^1$ regularity of the initial data, and indeed if we were only interested in L^2 decay rates we could relax the regularity requirement in (2.15) to $\omega \mathbf{w}_0 \in H^2 \times H^1$. However, in also controlling L^∞ decay rates, we lose a degree of regularity through the embedding $H^1 \hookrightarrow L^\infty$, explaining the $H^3 \times H^2$ requirement in (2.15).

2.6 Nonlinear stability of pure wave trains

The periodic wave trains constructed in [18] and references therein are interesting objects of their own right. We establish their spectral stability in the companion paper [4], but their nonlinear stability is not quite included in general results on stability of wave trains in reaction-diffusion systems under spectral assumptions [45, 46, 66], since the degenerate diffusion matrix in the FitzHugh-Nagumo system (1.2) introduces additional technical difficulties in establishing spectral mapping properties of the linear semigroup and controlling regularity in the nonlinear argument. One could attack these issues by proceeding as in [36, 59], that is, using a Bloch wave decomposition and the Gearhart-Prüss theorem to obtain high frequency resolvent bounds and associated linear decay estimates on the semigroup. These techniques are not available for the pattern-forming fronts, which are not purely periodic in space, preventing the use of the Bloch wave decomposition. A simplified version of our proof of Theorem 2.1, simply ignoring parts which have to do with dynamics in the leading edge, then gives an alternative proof of nonlinear stability of the wave trains, formulated as follows.

Theorem 2.2 (Nonlinear stability of wave trains). *Let $\mathbf{u}_{\text{wt}}$ be a stationary periodic solution to (1.5) which is spectrally stable in the sense of Hypothesis 1. Then, there exist constants $M, \delta > 0$ such that if we consider (1.5) with initial data $\mathbf{u}_0 = \mathbf{u}_{\text{wt}} + \mathbf{w}_0$, where $\mathbf{w}_0 \in L^1(\mathbb{R}, \mathbb{R}^2) \cap (Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R}))$ satisfies*

$$E_0 := \|\mathbf{w}_0\|_{L^1} + \|\mathbf{w}_0\|_{H^3 \times H^2} < \delta,$$

then there exist functions $\mathbf{w} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$ and $\psi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\mathbf{w} \in C([0, \infty), Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R})) \cap C^1([0, \infty), Z_1(\mathbb{R}, \mathbb{R}^2)), \quad \psi \in C^k([0, \infty), H^\ell(\mathbb{R}, \mathbb{R}))$$

for any $k, \ell \in \mathbb{N}$, with $\mathbf{w}(0) = \mathbf{w}_0$ and $\psi(0) = 0$ such that $\mathbf{u}(t) = \mathbf{u}_{\text{wt}} + \mathbf{w}(t)$ is the unique global solution of (1.5) with initial data $\mathbf{u}_0$, and we have the following estimates

$$\|\mathbf{u}(t) - \mathbf{u}_{\text{wt}}\|_{H^3 \times H^2} + \|\psi(t)\|_{L^2} \leq \frac{ME_0}{(1+t)^{\frac{1}{4}}}, \quad \|\mathbf{u}(t) - \mathbf{u}_{\text{wt}}\|_{L^\infty} + \|\psi(t)\|_{L^\infty} \leq \frac{ME_0}{(1+t)^{\frac{1}{2}}},$$

and

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{u}_{\text{wt}}(\cdot + \psi(\cdot, t))\|_{H^3 \times H^2} + \|(\psi_\xi(t), \partial_t \psi(t))\|_{H^3 \times H^2} &\leq \frac{ME_0}{(1+t)^{\frac{3}{4}}}, \\ \|\mathbf{u}(t) - \mathbf{u}_{\text{wt}}(\cdot + \psi(\cdot, t))\|_{L^\infty} + \|(\psi_\xi(t), \partial_t \psi(t))\|_{L^\infty} &\leq \frac{ME_0}{1+t}, \end{aligned}$$

for all $t \geq 0$.

Since the proof of Theorem 2.2 is a simplified version of the proof of Theorem 2.1, we do not present it in detail.

We note that it is possible to relax the regularity requirement on the initial data in Theorem 2.2, to requiring smallness in $L^1(\mathbb{R}, \mathbb{R}^2) \cap (W^{2,\infty}(\mathbb{R}, \mathbb{R}) \times W^{1,\infty}(\mathbb{R}, \mathbb{R}))$, only, by using an L^1 - L^∞ iteration scheme and replacing the L^2 based nonlinear damping estimates with alternative arguments developed in [23, 36] to control regularity of the modulated perturbation. It is more difficult to relax the regularity requirement on the initial data for the nonlinear stability of the pattern-forming fronts in Theorem 2.1, since the alternative argument for controlling regularity in the iteration scheme relies on obtaining estimates on the operator $\partial_x^\ell S_p(t) \partial_x^m$, where $S_p(t)$ arises in the decomposition (2.7). In the pure wave train case, the leading-order terms in the resolvent decomposition roughly have the structure of a convolution, so it is easy to integrate by parts to pass derivatives from one side of $S_p(t)$ to another, while it is not trivial to obtain estimates on $S_p(t)$ composed with derivatives on the right through the far-field/core decomposition used here in the linear analysis of the pattern-forming fronts. We expect that this technical issue could be overcome with more effort, but do not pursue it further, instead focusing on the most interesting aspects of the dynamics.

2.7 Outline of the paper

The remainder of the paper is organized as follows. In Section 3, we determine the essential spectrum of $\mathcal{L}_{\text{fr}}$ under Hypotheses 1 and 2 and explore related Fredholm properties. In Section 4, we use standard results from semigroup theory to represent the semigroup $e^{\mathcal{L}_{\text{fr}} t}$ via the inverse Laplace transform of the resolvent. In Section 5, we state and prove some general linear estimates based on different possible types of behavior of the resolvent near the essential spectrum at the origin. With these abstract estimates informing what kind of behavior we want to extract from the resolvent, we perform a detailed analysis of the resolvent in Section 6. In Section 7, we combine the resolvent analysis of Section 6 with the abstract linear estimates of Section 5 to establish a detailed decomposition of the semigroup $e^{\mathcal{L}_{\text{fr}} t}$. In Section 8, we set up our nonlinear iteration scheme and establish equivalence of the forward-modulated and inverse-modulated perturbations. Finally, we close our nonlinear stability argument in Section 9, proving Theorem 2.1. We conclude in Section 10 by discussing applications of our methods to related problems. Throughout, we relegate some technical computations to the appendices.

3 Spectral and Fredholm properties

We compute the linear spreading speed and characterize the resulting essential spectrum and Fredholm properties of $\mathcal{L}_{\text{fr}}$, under Hypotheses 1 and 2.

3.1 Essential spectrum in the wake

Recall that the limiting operator in the wake is the linearization $\mathcal{L}_{\text{wt}}$ of (1.5) about the wave train $\mathbf{u}_{\text{wt}}$, see (2.12). Hypotheses 1 and 2 yield that the spectrum of the operator $\mathcal{L}_{\text{wt}}$ (on, for instance, $L^2(\mathbb{R})$) is bounded away from the imaginary axis, except for a quadratic touching at the origin. In a neighborhood of the origin the spectrum of $\mathcal{L}_{\text{wt}}$ is given by the curve $\kappa \mapsto \lambda_{\text{wt}}(i\kappa)$, where λ_{wt} is given by (2.13) with $c_g < 0$ and $D_{\text{eff}}^{\text{wt}} > 0$. We consider the eigenvalue problem $(\mathcal{L}_{\text{wt}} - \lambda)\mathbf{u} = 0$ as a first-order system of linear differential equations with periodic coefficients. We call the Floquet exponents ν associated to this system the *spatial Floquet exponents* of $\mathcal{L}_{\text{wt}} - \lambda$. They arise by solving the linear dispersion relation $\lambda_{\text{wt}}(\nu) = 0$. An application of the implicit function theorem then readily yields the following result.

Proposition 3.1 (Configuration of spatial Floquet exponents). *Assume Hypotheses 1 and 2. For λ to the right of $\Sigma(\mathcal{L}_{\text{wt}})$, the operator $\mathcal{L}_{\text{wt}} - \lambda$ has two unstable spatial Floquet exponents and one stable spatial Floquet exponent, which are analytic in λ in a neighborhood of the origin. As λ approaches the origin from the right, one spatial Floquet exponent $\nu_{\text{wt}}(\lambda)$ crosses the imaginary axis from right to left, with expansion*

$$\nu_{\text{wt}}(\lambda) = \nu_{\text{wt}}^1 \lambda - \nu_{\text{wt}}^2 \lambda^2 + O(\lambda^3), \quad \nu_{\text{wt}}^1 = -\frac{1}{c_g} > 0, \quad \nu_{\text{wt}}^2 = -\frac{D_{\text{eff}}^{\text{wt}}}{c_g^3} > 0.$$

The other Floquet exponents remain uniformly bounded away from the imaginary axis for $|\lambda|$ small.

3.2 Essential spectrum in the leading edge

Recall that the limiting operator in the leading edge is given by the linearization $\mathcal{L}_+$ of (1.5) about the rest state 0, see (2.11). Since $\mathcal{L}_+$ has constant coefficients, its spectrum (on, for instance, $L^2(\mathbb{R})$) can be readily computed via the Fourier transform as

$$\Sigma(\mathcal{L}_+) = \{\lambda \in \mathbb{C} : d_{\text{cin}}(\lambda, ik - \eta_{\text{lin}}) = 0 \text{ for some } k \in \mathbb{R}\},$$

where, for a given speed c , $d_c: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ is the *linear dispersion relation* given by

$$d_c(\lambda, \nu) = \det \left(D\nu^2 + c\nu I + F'(0) - \lambda I \right).$$

We determine the essential spectrum of $\mathcal{L}_+$ in the following lemma, whose proof is a straightforward computation; see also [18, §2] for some computational details.

Lemma 3.2. *Fix $(3 - \sqrt{6})/6 < a < \frac{1}{2}$, $0 < \gamma < 4$, and $\varepsilon > 0$ sufficiently small. Then the spreading speed $c_{\text{lin}}(a, \gamma, \varepsilon)$ and spatial decay rate $\eta_{\text{lin}}(a, \gamma, \varepsilon) < 0$ from Theorem 1.1 are smooth in all arguments and satisfy the following.*

1. (Simple double root) For λ and ν small, we have the expansion

$$d_{c_{\text{lin}}}(\lambda, \nu - \eta_{\text{lin}}) = -d_{10}\lambda + d_{02}\nu^2 + O(\nu^3, \lambda^2, \lambda\nu)$$

for some $d_{10}, d_{02} \in \mathbb{R}$ with $d_{10}d_{02} > 0$.

2. (Minimal critical spectrum) If $d_{c_{\text{lin}}}(\mathrm{i}\omega, \mathrm{i}k - \eta_{\text{lin}}) = 0$ for some $\omega, k \in \mathbb{R}$, then $\omega = k = 0$.
3. (No unstable spectrum) $d_{c_{\text{lin}}}(\lambda, \mathrm{i}k - \eta_{\text{lin}}) \neq 0$ for any $k \in \mathbb{R}, \lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > 0$.

Remark 3.3. Lemma 3.2 together with [39] imply that the dynamics of the linearization in the leading edge at the speed c_{lin} , which we call the linear spreading speed, given by

$$\mathbf{u}_t = D\partial_\xi^2 \mathbf{u} + c_{\text{lin}}\partial_\xi \mathbf{u} + F'(0)\mathbf{u},$$

are marginally pointwise stable, that is, neither exponentially growing nor decaying in time at any fixed ξ . See [9, Section 1.2] for a concise explanation of how Lemma 3.2 relates to marginal pointwise stability, and why this leads to a prediction for a selected front speed.

Lemma 3.2 implies that the spectrum of $\mathcal{L}_+$ is stable and uniformly bounded away from the imaginary axis except for a neighborhood of the origin, where it is given by the curve

$$\lambda_+(ik) = -D_{\text{eff}}^+ k^2 + O(k^3),$$

where $D_{\text{eff}}^+ = \frac{d_{02}}{d_{10}} > 0$, as suggested in Section 2.1. In particular, the spectrum of $\mathcal{L}_+$ is marginally stable, and the critical spectral curve is associated to a branched diffusive mode, in the terminology of Section 2.1.

We will also need information on the *spatial eigenvalues* of $\mathcal{L}_+ - \lambda$, that is, the eigenvalues of the matrix $\mathcal{M}(\lambda)$ obtained from writing $(\mathcal{L}_+ - \lambda)\mathbf{u} = 0$ as a first-order system $\partial_\xi U = \mathcal{M}(\lambda)U$ in $U = (u_1, \partial_\xi u_1, u_2)^\top$.

Corollary 3.4. *Assume a, γ and ε are such that Lemma 3.2 holds. Let $\sigma = \sqrt{\lambda}$, with branch cut chosen along the negative real axis. If σ^2 is small, $\mathcal{M}(\sigma^2)$ has precisely two eigenvalues $\nu_{\text{fr}}^\pm(\sigma)$ in a neighborhood of the origin, which are analytic in σ and satisfy the expansions*

$$\nu_{\text{fr}}^\pm(\sigma) = \pm \nu_{\text{fr}}^1 \sigma + O(\sigma^2), \tag{3.1}$$

where $\nu_{\text{fr}}^1 > 0$. The spatial eigenvalues $\nu_{\text{fr}}^\pm(\sigma)$ cross the imaginary axis precisely when $\lambda = \sigma^2$ crosses $\Sigma(\mathcal{L}_+)$. The remaining third eigenvalue $\nu_3(\sigma^2)$ of $\mathcal{M}(\sigma^2)$ is analytic in σ^2 , satisfies $\operatorname{Re} \nu_3(\sigma^2) > 0$ for $|\sigma|$ small, and is uniformly bounded away from the imaginary axis for $|\sigma|$ small.

Proof. The expansion (3.1) follows from Lemma 3.2 and the Newton polygon. The fact that $\operatorname{Re} \nu_3(\sigma^2) > 0$ follows from a direct computation. $\square$

3.3 Essential spectrum and Fredholm properties of $\mathcal{L}_{\text{fr}}$

Fredholm properties of the linearization about a traveling front may be determined through properties of the asymptotic rest states through Palmer's theorem [54]; see for instance any of [33, 48, 60] for a review of spectral properties of traveling waves. Specifically, in our setting, the operator $\mathcal{L}_{\text{fr}} : W^{2,p}(\mathbb{R}, \mathbb{C}) \times W^{1,p}(\mathbb{R}, \mathbb{C}) \subset L^p(\mathbb{R}, \mathbb{C}^2) \rightarrow L^p(\mathbb{R}, \mathbb{C}^2)$ is Fredholm if and only if the limiting rest states $\mathbf{u} = 0$ and $\mathbf{u}_{\text{wt}}$ are *hyperbolic*, that is, if and only if $\mathcal{L}_+$ has no purely imaginary spatial eigenvalues and $\mathcal{L}_{\text{wt}}$ has no purely imaginary spatial Floquet exponents. If this is the case, then the Fredholm index of $\mathcal{L}_{\text{fr}}$ is

$$\text{ind}(\mathcal{L}_{\text{fr}}) = i_M(\mathcal{L}_{\text{wt}}) - i_M(\mathcal{L}_+), \quad (3.2)$$

where the asymptotic Morse indices $i_M(\mathcal{L}_{\text{wt}})$ and $i_M(\mathcal{L}_+)$ measure the number of unstable spatial Floquet exponents of $\mathcal{L}_{\text{wt}}$ and the number of unstable spatial eigenvalues of $\mathcal{L}_+$, respectively.

Proposition 3.5. *Assume a, γ , and ε are such that Hypotheses 1 and 2 and Lemma 3.2 hold. Then the essential spectrum of $\mathcal{L}_{\text{fr}}$ is marginally stable, touching the imaginary axis only at the origin. In particular, in a neighborhood of the origin the rightmost boundary of $\Sigma_{\text{ess}}(\mathcal{L}_{\text{fr}})$ coincides with $\Sigma(\mathcal{L}_{\text{wt}})$.*

Proof. This follows from a short computation using the formula (3.2), Proposition 3.1 and Lemma 3.2; see the right panel of Figure 2 for a depiction of the configuration of spatial Floquet exponents and spatial eigenvalues. $\square$

We can also use Palmer's theorem to determine Fredholm properties of $\mathcal{L}_{\text{fr}}$ acting on exponentially weighted spaces, as follows.

Proposition 3.6. *Fix $\eta > 0$ sufficiently small. Recall from Section 2.3 the notation $X_\eta^p = L_{\text{exp}, -\eta, \eta}^p(\mathbb{R}, \mathbb{C}^2)$ and $Y_\eta^p = W_{\text{exp}, -\eta, \eta}^{2,p}(\mathbb{R}, \mathbb{C}) \times W_{\text{exp}, -\eta, \eta}^{1,p}(\mathbb{R}, \mathbb{C})$. The operator*

$$\mathcal{L}_{\text{fr}} : Y_\eta^p \subset X_\eta^p \rightarrow X_\eta^p$$

is Fredholm with index -2.

Proof. Conjugating with exponential weights and using Palmer's theorem, one finds $\mathcal{L}_{\text{fr}}$ is Fredholm on these spaces, with index given by

$$\text{ind}(\mathcal{L}_{\text{fr}} : Y_\eta^p \rightarrow X_\eta^p) = i_M^{-\eta}(\mathcal{L}_{\text{wt}}) - i_M^\eta(\mathcal{L}_+),$$

where $i_M^{-\eta}(\mathcal{L}_{\text{wt}})$ is the number of spatial Floquet exponents ν of $\mathcal{L}_{\text{wt}}$ with $\text{Re } \nu > \eta$, and $i_M^\eta(\mathcal{L}_+)$ is the number of spatial eigenvalues ν of $\mathcal{L}_+$ with $\text{Re } \nu > -\eta$. Using Corollary 3.4 and Proposition 3.1, we find

$$\text{ind}(\mathcal{L}_{\text{fr}} : Y_\eta^p \rightarrow X_\eta^p) = 1 - 3 = -2,$$

provided $\eta > 0$ is sufficiently small, as desired; again, see the right panel of Figure 2 for a depiction of the configuration of spatial Floquet exponents and spatial eigenvalues. $\square$

Throughout the remainder of the paper, we assume that the parameters a, γ , and ε are such that Hypotheses 1 through 3 together with Lemma 3.2 hold.

4 Contour integral representation of semigroup

Standard results from semigroup theory (see e.g. [55]) imply that for any fixed non-negative integer k , $\mathcal{L}_{\text{fr}}$ generates a strongly continuous semigroup on $Z_k(\mathbb{R}, \mathbb{C}^2)$.

Proposition 4.1. *Fix a non-negative integer k . The operator*

$$\mathcal{L}_{\text{fr}} : Z_{k+2}(\mathbb{R}, \mathbb{C}) \times Z_{k+1}(\mathbb{R}, \mathbb{C}) \rightarrow Z_k(\mathbb{R}, \mathbb{C}^2)$$

is closed and densely defined, and generates a strongly continuous semigroup $e^{\mathcal{L}_{\text{fr}} t}$ on $Z_k(\mathbb{R}, \mathbb{C}^2)$. Furthermore, there exist constants $M, \mu_ > 0$ such that $\|e^{\mathcal{L}_{\text{fr}} t}\|_{Z_k \rightarrow Z_k} \leq M e^{\mu_* t}$ holds for $t \geq 0$.*

For completeness, we give a proof in Appendix A. From now on we consider $\mathcal{L}_{\text{fr}}$ as an operator on $Z_k(\mathbb{R}, \mathbb{C}^2)$ for a fixed nonnegative integer k . Using another standard result, we can write the semigroup $e^{\mathcal{L}_{\text{fr}} t}$ as the inverse Laplace transform of the resolvent $(\mathcal{L}_{\text{fr}} - \lambda)^{-1}$, at least for regular enough initial data.

Corollary 4.2 ([55, Chapter 1, Corollary 7.5]). *There exists $\eta > 0$ such that for all test functions $\mathbf{u}_0 \in C_c^\infty(\mathbb{R}, \mathbb{C}^2)$, we have the representation*

$$e^{\mathcal{L}_{\text{fr}} t} \mathbf{u}_0 = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\eta - iR}^{\eta + iR} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{u}_0 d\lambda, \quad t > 0. \quad (4.1)$$

We will eventually relax the restriction $\mathbf{u}_0 \in C_c^\infty(\mathbb{R}, \mathbb{C}^2)$ via an approximation argument using that test functions are dense in $Z_k(\mathbb{R}, \mathbb{C}^2)$. To extract decay from this contour integral representation, we would like to shift the contour as far left as possible. As a first step, we show that we can shift to a contour which is in the left half-plane except in a neighborhood of the origin. This allows us to extract exponential decay from the high frequency parts of the semigroup.

Proposition 4.3. *For $\mathbf{u}_0 \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$, we have the representation*

$$e^{\mathcal{L}_{\text{fr}} t} \mathbf{u}_0 = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^1} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{u}_0 d\lambda,$$

where the contour Γ_R^1 is depicted in the middle panel of Figure 3. Moreover, for each $1 \leq p \leq \infty$, there exists a constant $C > 0$ such that

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon \gamma}{4} t} \|\mathbf{g}\|_{L^p}$$

for all $t > 0$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

The main difficulty in proving Proposition 4.3 is in uniformly controlling the resolvent $(\mathcal{L}_{\text{fr}} - \lambda)^{-1}$ for $|\text{Im } \lambda| \gg 1$. We do this by computing an expansion of the resolvent in this limit, identifying the first few terms explicitly until we can truncate with an error integrable in λ , an approach similar to [52]. Since the main idea is intuitive but the details are fairly technical, we carry out this procedure in Appendix B.

Corollary 4.4. *For $\mathbf{u}_0 \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$, we have the representation*

$$e^{\mathcal{L}_{\text{fr}} t} \mathbf{u}_0 = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^2} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{u}_0 d\lambda, \quad (4.2)$$

where the contour Γ_R^2 is depicted in the right panel of Figure 3. Moreover, for each $1 \leq p \leq \infty$ there exist constants $C, r > 0$ such that

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_{\text{int}}^{2,+} \cup \Gamma_{\text{int}}^{2,-} \cup \Gamma_R^{1,-}} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda \right\|_{L^p} \leq C e^{-r t} \|\mathbf{g}\|_{L^p} \quad (4.3)$$

for all $t > 0$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

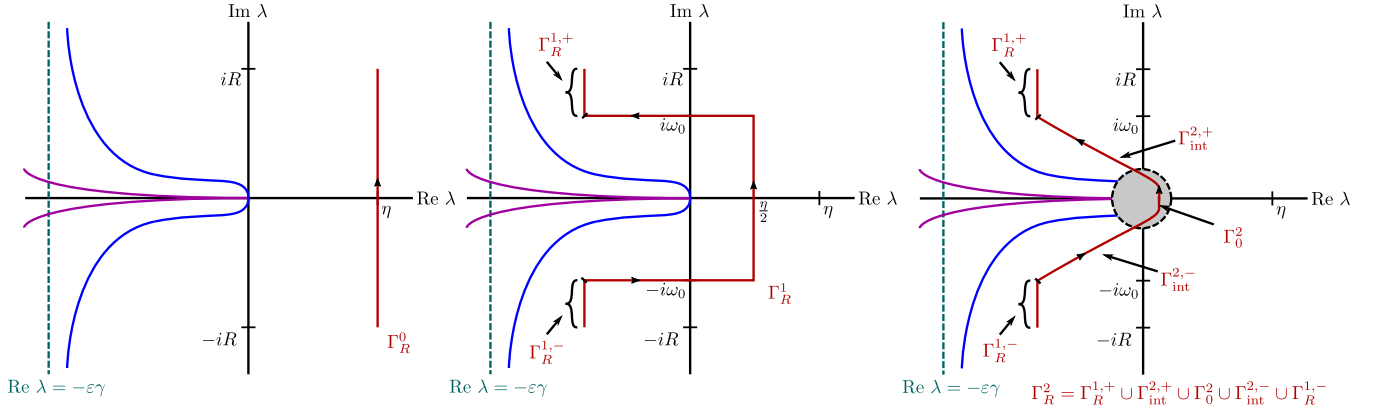


Figure 3: All three panels include the Fredholm borders of $\mathcal{L}_{\text{fr}}$, associated to essential spectrum at $-\infty$ (blue) and $+\infty$ (purple). Left: the starting contour $\{\text{Re } \lambda = \eta, |\text{Im } \lambda| \leq R\}$ from Corollary 4.2. Middle: the contour Γ_R^1 (dark red) of Proposition 4.3, with high frequency parts shifted into the left half-plane. Right: the grey ball shows a neighborhood of the origin in which the resolvent analysis of Section 6 holds; the contour Γ_R^2 (dark red) is contained in the left half-plane apart from a segment Γ_0^2 in this ball.

Proof. By the spectral stability captured in Proposition 3.5 and Hypothesis 3, we can homotope Γ_R^1 to Γ_R^2 while remaining in the resolvent set of $\mathcal{L}_{\text{fr}}$, so (4.2) follows. The estimate (4.3) follows from Proposition 4.3 together with the fact that the compact portion $\Gamma_{\text{int}}^{2,+} \cup \Gamma_{\text{int}}^{2,-}$ is contained both in the left half-plane, bounded away from the imaginary axis, and in the resolvent set of $\mathcal{L}_{\text{fr}}$. $\square$

5 Abstract linear estimates and choice of contours near the origin

One observes from Corollary 4.4 that the only part of the semigroup which does not decay exponentially is the contribution from Γ_0^2 . This is due to two curves of essential spectrum of $\mathcal{L}_{\text{fr}}$ touching the origin, one associated to the branched diffusive mode in the leading edge, and the other associated to the outgoing diffusive mode in the wake. In Section 6 we decompose the resolvent in a neighborhood of the origin, identifying terms associated to the branched mode, terms associated to the outgoing mode, and terms associated to the interaction of both modes. In this section, we choose integration contours which are adapted to the expected behavior of these three type of terms and use these contours to prove abstract linear decay estimates. We will combine these estimates with the resolvent analysis and decomposition, carried out in Section 6, to establish sharp bounds on the semigroup $e^{\mathcal{L}_{\text{fr}} t}$ in Section 7.

5.1 Regions of analyticity near the origin

We identify regions near the origin in which parts of the resolvent associated to the branched or outgoing diffusive mode, or parts associated to their interaction, can be rendered analytic in λ . Our choice of integration contours for the different type of terms arising in the resolvent decomposition will thus be determined by these regions. Some critical parts of the resolvent can be represented as a convolution with a suitable integral kernel (or spatial Green's function), allowing one to consider the analyticity properties of the integral kernel after swapping integrals.

We distinguish between regions of pointwise analyticity and regions of analyticity in a fixed function space (typically L^2 - or L^∞ -spaces). In regions where certain parts of the resolvent (or their associated integral kernels) are pointwise analytic, but perhaps not analytic in fixed function spaces due to loss of spatial localization, we can deform our integration contour pointwise and employ pointwise estimates. We explain in Section 5.3 below the need for pointwise estimates, i.e., why normed estimates are insufficient to obtain sharp bounds on certain parts of the resolvent.

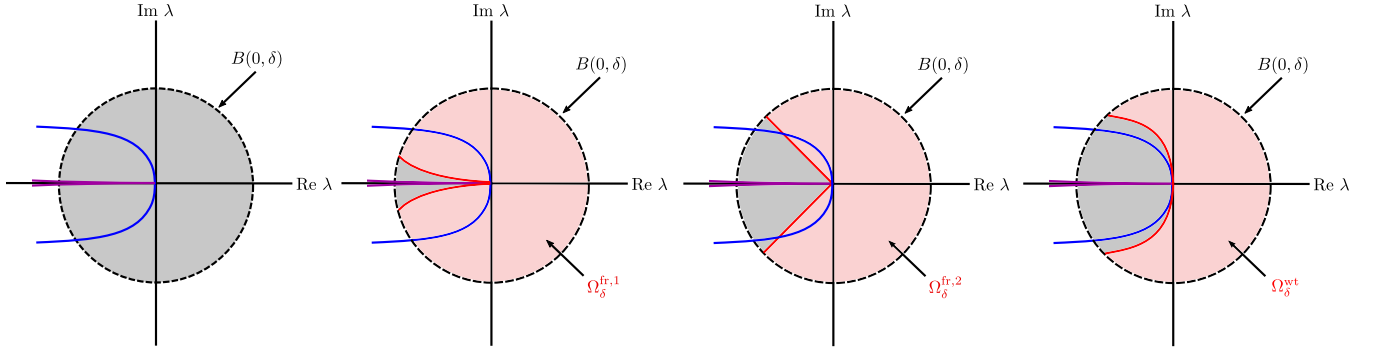


Figure 4: Far left: Fredholm borders of $\mathcal{L}_{\text{fr}}$ (blue, purple) in a neighborhood $B(0, \delta)$ (shaded, grey) of the origin. Middle left: the region $\Omega_{\delta}^{\text{fr},1}$ (shaded, red) defined in (5.3). Middle right: the region $\Omega_{\delta}^{\text{fr},2}$ (shaded, red) defined in (5.2). Far right: the region $\Omega_{\delta}^{\text{wt}}$ (shaded, red) defined in (5.7).

Terms associated to branched diffusive modes. The essential spectrum in the leading edge gives rise to branched diffusive modes, for which the associated parts of the resolvent are roughly of the form

$$\mathbf{u}_+(\xi; \sigma) \sim \alpha_+(\sigma) \chi_+(\xi) e^{\nu_{\text{fr}}^-(\sigma)\xi},$$

where we denote $\sigma = \sqrt{\lambda}$ with branch cut along the negative real axis, and $\alpha_+(\sigma)$ is some analytic function of σ defined on a neighborhood of the origin. We recall that $\chi_+(\xi)$ is a smooth positive cutoff function supported on $\{\xi \geq 0\}$, and $\nu_{\text{fr}}^-(\sigma)$ is the spatial eigenvalue from Corollary 3.4, which is analytic in σ on a neighborhood of the origin. The obstructions to analyticity of $\mathbf{u}_+$ are twofold:

1. Due to the branch point in $\alpha_+(\sigma)$ and $\nu_{\text{fr}}^-(\sigma)$ at $\lambda = 0$, $\mathbf{u}_+(\xi; \sigma)$ is only pointwise analytic in $\lambda = \sigma^2$ away from the negative real axis.
2. Since $\text{Re } \nu_{\text{fr}}^-(\sigma)$ changes sign as $\lambda = \sigma^2$ crosses $\Sigma(\mathcal{L}_+)$, $\mathbf{u}_+(\xi; \sigma)$ loses spatial localization as λ passes through $\Sigma(\mathcal{L}_+)$, and hence loses analyticity in a fixed function space.

The first obstruction prevents us, even in a pointwise sense, from using contours which pass through the branch point at the origin. The second obstruction prevents us from passing $\Sigma(\mathcal{L}_+)$ with our integration contours when measuring in a fixed function space.

What contours we may use depends on what space we want to measure in. In our nonlinear argument, we will bound the solution (and some derivatives) in both the L^2 - and L^∞ -norms. Taking the L^2 -norm, we find

$$\|\mathbf{u}_+(\cdot; \sigma)\|_{L^2} \sim \frac{1}{\sqrt{\text{Re } \nu_{\text{fr}}^-(\sigma)}} \sim \frac{1}{\sqrt{\text{Re } \sigma}}. \quad (5.1)$$

To extract temporal decay, we want to move the contours as far left as possible. Since we can't pass through the branch point, here this means we want to shrink the contour into as small of a neighborhood of the origin as possible. By (5.1), $\mathbf{u}_+(\cdot; \sigma)$ blows up in $L^2(\mathbb{R})$ as σ approaches zero. Nonetheless, if we restrict $\lambda = \sigma^2$ to a sector

$$\Omega_{\delta}^{\text{fr},2} = \left\{ r e^{i\varphi} : 0 < r < \delta, -\theta_0 < \varphi < \theta_0 \right\}, \quad (5.2)$$

for some $\delta, \theta_0 > 0$, then there exists a constant $C > 0$ such that $\text{Re } \sigma \geq C|\sigma|$ for all $\sigma \in \Omega_{\delta}^{\text{fr},2}$; see Figure 4. Then, for $\sigma \in \Omega_{\delta}^{\text{fr},2}$ we can quantify the rate of blowup of the resolvent as

$$\|\mathbf{u}_+(\cdot; \sigma)\|_{L^2} \sim \frac{1}{|\sigma|} = \frac{1}{|\lambda|^{1/2}}.$$

Hence, when estimating branched diffusive modes in $L^2(\mathbb{R})$, we will use contours which remain in $\Omega_\delta^{\text{fr},2}$, but can now be shrunk arbitrarily small, since the blowup rate $|\lambda|^{-1/2}$ is integrable in λ .

On the other hand, we find that $\|\mathbf{u}_+(\cdot; \sigma)\|_{L^\infty}$ remains bounded all the way up to $\Sigma(\mathcal{L}_+)$, at which point $\text{Re } \nu_{\text{fr}}^-(\sigma)$ becomes zero. If we measure in the even weaker $L_{0,-1}^\infty$ -norm, then by Taylor expansion, one sees that $\mathbf{u}_+$ obeys the Lipschitz estimate $\|\mathbf{u}_+(\cdot; \sigma) - \mathbf{u}_+(\cdot; 0)\|_{L_{0,-1}^\infty} \lesssim |\sigma|$ for σ to the right of $\Sigma(\mathcal{L}_+)$. We demonstrate in Proposition 5.1 that such a Lipschitz estimate leads to faster decay compared to decay estimates in norms in which $\mathbf{u}_+(\cdot, \sigma)$ is merely bounded in σ .

All in all, we will obtain the fastest decay estimates for branched modes when measuring in $L_{0,-1}^\infty(\mathbb{R})$, and when measuring in this space, we can use contours which pass into a larger region than $\Omega_\delta^{\text{fr},2}$, as long as we remain to the right of $\Sigma(\mathcal{L}_+)$. We therefore define the curve

$$K^{\text{fr},1} = \left\{ -k^2 + iC^{\text{fr},3}k^3 : k \in \mathbb{R} \right\}.$$

Choosing $C^{\text{fr},3} > 0$ sufficiently large, we can guarantee that in a small enough neighborhood of the origin, $K^{\text{fr},1}$ lies to the right of the essential spectrum of $\mathcal{L}_+$. Thus, given $\delta > 0$, we define

$$\Omega_\delta^{\text{fr},1} = \left\{ \lambda \in B(0, \delta) : \lambda \text{ lies to the right of } K^{\text{fr},1} \right\}, \quad (5.3)$$

see Figure 4. When measuring in $L_{0,-1}^\infty(\mathbb{R})$, we will use contours which remain in the larger region $\Omega_\delta^{\text{fr},1}$.

Terms associated to the outgoing diffusive mode. The parts of the resolvent that are associated to the outgoing diffusive mode only are either roughly of the form

$$\mathbf{u}_{\text{out}}(\xi; \lambda) \sim \alpha_{\text{out}}(\lambda) \chi_-(\xi) e^{\nu_{\text{wt}}(\lambda)\xi}, \quad (5.4)$$

or can be represented as a convolution with a suitable integral kernel and are then roughly of the form

$$\mathbf{u}_{\text{out}}(\xi; \lambda) \sim \int_{\mathbb{R}} \alpha_{\text{out}}(\lambda) \chi_-(\xi) \chi_-(\xi - \zeta) e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)} Q(\zeta) d\zeta, \quad (5.5)$$

where $\alpha_{\text{out}}(\lambda)$ is some function which is analytic in λ in a full neighborhood of the origin and $Q(\zeta)$ is a bounded function of ζ . We recall that $\chi_-(z)$ is a smooth cutoff function supported on $\{z \leq 0\}$ and $\nu_{\text{wt}}(\lambda)$ is the spatial Floquet exponent from Proposition 3.1, which is analytic in λ in a neighborhood of the origin. The only obstruction to analyticity is that $\chi_-(z) e^{\nu_{\text{wt}}(\lambda)z}$ loses spatial localization as λ crosses $\Sigma(\mathcal{L}_{\text{wt}})$, since $\text{Re } \nu_{\text{wt}}(\lambda)$ then changes sign by Proposition 3.1. After swapping the convolution integral with respect to ζ with the complex line integral with respect to λ in case of the representation (5.5), we may deform integration contours pointwise, i.e., we may let the choice of integration contour depend on the values of ξ and t in case of (5.4) and on ξ, ζ , and t in case of (5.5). Since this only requires pointwise analyticity, we can pass through $\Sigma(\mathcal{L}_{\text{wt}})$. Proceeding as in [52, 77], we make judicious choices for our pointwise integration contours to extract sharp spatio-temporal localization of the inverse Laplace transform of (5.4) or of the inverse Laplace transform of the integral kernel associated to (5.5), which eventually yields sharp normed estimates on the corresponding parts of the semigroup.

Interaction terms. Terms which capture the interaction between the branched and outgoing modes are roughly of the form

$$\mathbf{u}_{\text{int}}(\xi; \sigma) \sim \alpha_{\text{int}}(\sigma) \chi_-(\xi) e^{\nu_{\text{wt}}(\sigma^2)\xi}, \quad (5.6)$$

where $\alpha_{\text{int}}(\sigma)$ is some analytic function of $\sigma = \sqrt{\lambda}$ defined on a full neighborhood of the origin, which satisfies $\alpha_{\text{int}}(0) = 0$ without loss of generality.¹ We cannot estimate these terms using the pointwise contours

¹If $\alpha_{\text{int}}(\sigma)$ does not vanish at $\sigma = 0$, we can always subtract $\alpha_{\text{int}}(0) \chi_-(\xi) e^{\nu_{\text{wt}}(\sigma^2)\xi}$ from (5.6), which is a term of the form (5.4), associated to the outgoing mode only.

of [52, 77]. Indeed, these contours pass through the origin which is not allowed here due to the branch point of $\alpha_{\text{int}}(\sigma)$ at $\lambda = 0$. Pointwise contours designed to estimate terms like (5.6), which thereby avoid passing through the branch point, were developed in [41]. However, instead of using the delicate pointwise contours of [41], we obtain linear estimates sufficient to establish sharp nonlinear decay rates by measuring $\mathbf{u}_{\text{int}}$ in norm using simple contours which remain to the right of $\Sigma(\mathcal{L}_{\text{wt}})$. We therefore define a curve

$$K^{\text{wt}} = \{ik - C^{\text{wt}}k^2 : k \in \mathbb{R}\}.$$

Provided $C^{\text{wt}} > 0$ is sufficiently small, this curve lies to the right of $\Sigma(\mathcal{L}_{\text{wt}})$, in a small enough neighborhood of the origin. Given $\delta > 0$ small, we then define the region

$$\Omega_{\delta}^{\text{wt}} = \{\lambda \in B(0, \delta) : \lambda \text{ lies to the right of } K^{\text{wt}}\}. \quad (5.7)$$

We will restrict our contours to the region $\Omega_{\delta}^{\text{wt}}$; see Figure 4. A risk of our approach is that normed estimates do not always afford sharp bounds on terms associated to outgoing diffusive modes, as explained in §5.3 below. Nevertheless, the observation that $\alpha_{\text{int}}(\sigma)$ vanishes at $\sigma = 0$ yields an additional factor σ in (5.6), rendering additional temporal decay which we can exploit to compensate for the use of normed estimates.

5.2 Estimates for branched modes

Fix $\delta > 0$ small, and let $\Omega_{\delta}^{\text{fr},1}$, $\Omega_{\delta}^{\text{fr},2}$, and $\Omega_{\delta}^{\text{wt}}$ be defined by (5.3), (5.2), and (5.7), respectively. Moreover, let $\Delta_{\delta}^{\text{fr},j}$ be the image of $\Omega_{\delta}^{\text{fr},j}$ under the principal square root $\lambda \mapsto \sqrt{\lambda}$ for $j = 1, 2$. We start by establishing normed estimates which will be used for parts of the resolvent associated to branched modes.

Proposition 5.1 (Normed estimates with sharp decay rates for branched modes). *Let X be a Banach space, and suppose that for some small $\delta > 0$, we have a function $\sigma \mapsto u(\sigma) : \Delta_{\delta}^{\text{fr},1} \rightarrow X$ which is analytic in σ^2 on $\Omega_{\delta}^{\text{fr},1}$, and extends Lipschitz-continuously to $\sigma = 0$ with constant $L > 0$ such that*

$$\|u(\sigma) - u(0)\|_X \leq L|\sigma|, \quad (5.8)$$

for $\sigma \in \Delta_{\delta}^{\text{fr},1}$. Then, there exists a constant $C > 0$ such that for all $t > 0$, we have

$$\left\| \int_{\Gamma_0^2} e^{\sigma^2 t} u(\sigma) d(\sigma^2) \right\|_X \leq \frac{C}{(1+t)^{\frac{3}{2}}}.$$

Proof. The proof is identical to [7, Proof of Proposition 4.1], but we sketch it here for completeness. We will integrate in σ rather than in $\lambda = \sigma^2$. Using analyticity of $\sigma^2 \mapsto u(\sigma)$ on $\Omega_{\delta}^{\text{fr},1}$ and continuity up to $\sigma = 0$, we can shift to an integration contour which runs tangent to the imaginary axis in the σ -plane, given by

$$\tilde{\Gamma}^{\text{fr},1} = \tilde{\Gamma}_-^{\text{fr},1} \cup \tilde{\Gamma}_0^{\text{fr},1} \cup \tilde{\Gamma}_+^{\text{fr},1},$$

where

$$\tilde{\Gamma}_0^{\text{fr},1} = \{\sigma(a) : a \in [-a_*, a_*]\}, \quad \sigma(a) := ia + c_2 a^2, \quad (5.9)$$

where $c_2 > 0$ is chosen sufficiently large and $a_* > 0$ is chosen sufficiently small so that the image $\Gamma_0^{\text{fr},1}$ of $\tilde{\Gamma}_0^{\text{fr},1}$ under $\sigma \mapsto \sigma^2$ lies within $\Omega_{\delta}^{\text{fr},1}$, touching its boundary only at the origin. Let $\Gamma_{\pm}^{\text{fr},1}$ be straight line segments which connect the ends of $\Gamma_0^{\text{fr},1}$ to points on the boundary of $B(0, \delta)$ which are in the left half-plane but to the right of $\Sigma(\mathcal{L}_{\text{fr}})$. Let $\tilde{\Gamma}_{\pm}^{\text{fr},1}$ denote the images of $\Gamma_{\pm}^{\text{fr},1}$ under the map $\lambda \mapsto \sqrt{\lambda}$. See Figure 5 for a depiction of the integration contour $\Gamma^{\text{fr},1} = \Gamma_-^{\text{fr},1} \cup \Gamma_0^{\text{fr},1} \cup \Gamma_+^{\text{fr},1}$ in the λ -plane.

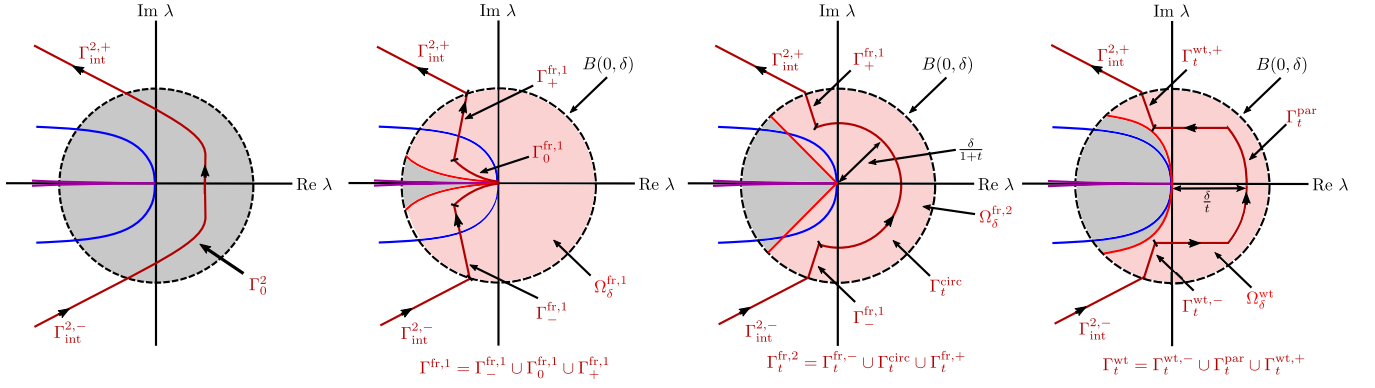


Figure 5: Far left: the original contour Γ_0^2 (dark red) in a neighborhood $B(0, \delta)$ (grey, shaded) of the origin. Middle left: the contour $\Gamma_{fr,1}^{fr,1}$ (dark red) used in the proof of Proposition 5.1. Middle right: the contour $\Gamma_t^{fr,2}$ used in the proof of Proposition 5.2. Far right: the contour Γ_t^{wt} (dark red) used in the proof of Proposition 5.3.

Integrating in σ and deforming to this new contour, we then have

$$\int_{\Gamma_0^2} e^{\sigma^2 t} u(\sigma) d(\sigma^2) = \int_{\tilde{\Gamma}_0^{fr,1}} e^{\sigma^2 t} u(\sigma) 2\sigma d\sigma + \sum_{\iota=\pm} \int_{\tilde{\Gamma}_t^{\iota}} e^{\sigma^2 t} u(\sigma) 2\sigma d\sigma.$$

Since $\Gamma_{\pm}^{fr,1}$ lie in the left half-plane, and $u(\sigma)$ is uniformly bounded along these segments, the contributions from these portions of the contour decay exponentially in time, so we focus on the first integral. We use (5.8) to expand

$$\int_{\tilde{\Gamma}_0^{fr,1}} e^{\sigma^2 t} u(\sigma) 2\sigma d\sigma = 2 \int_{\tilde{\Gamma}_0^{fr,1}} e^{\sigma^2 t} u(0) \sigma d\sigma + \int_{\tilde{\Gamma}_0^{fr,1}} e^{\sigma^2 t} (u(\sigma) - u(0)) \sigma d\sigma.$$

Since $u(0)$ does not depend on σ , we can change variables in the first integral to integrate again over σ^2 , with integrand analytic in σ^2 in a full neighborhood of the origin, so that we may shift the integration contour fully into the left half-plane and thereby extract exponential decay. For the second integral, we use (5.9) to explicitly parameterize the integral as

$$\int_{\tilde{\Gamma}_0^{fr,1}} e^{\sigma^2 t} (u(\sigma) - u(0)) \sigma d\sigma = \int_{-a_*}^{a_*} e^{(-a^2 + c_2^2 a^4)t} e^{2ic_2 a^3 t} (u(ia + c_2 a^2) - u(0)) (ia + c_2 a^2) (i + 2c_2 a) da.$$

Using the estimate (5.8) and choosing a_* sufficiently small, we arrive at

$$\left\| \int_{\tilde{\Gamma}_0^{fr,1}} e^{\sigma^2 t} (u(\sigma) - u(0)) \sigma d\sigma \right\| \lesssim \int_{-a_*}^{a_*} e^{-a^2 t/2} |a|^2 da \lesssim \frac{1}{(1+t)^{\frac{3}{2}}},$$

for $t > 0$, where the last inequality follows from applying the change of variables $z = a\sqrt{t}$ for $t > 1$, whereas for $t \in [0, 1]$ we use that $|a|^2 e^{-a^2 t/2} \leq a_*^2$ for $a \in (-a_*, a_*)$. $\square$

We will only be able to obtain the estimate (5.8) and the associated $t^{-3/2}$ decay rate when measuring in fairly weak norms. When measuring in $L^2(\mathbb{R})$, as needed for our nonlinear argument, the resolvent will instead blowup at $\sigma = 0$ as explained in §5.1, but with a quantified rate in the region $\Omega_\delta^{fr,2}$. The next estimate is adapted to precisely this scenario.

Proposition 5.2 (Estimates for branched modes in stronger norms). *Let X be a Banach space, and suppose that for some $\delta > 0$ small, we have a map $\sigma \mapsto u(\sigma) : \Delta_\delta^{fr,2} \rightarrow X$, which is analytic in σ^2 on $\Omega_\delta^{fr,2}$, and there exist $r \in [0, 2)$ and $K > 0$ such that the blowup estimate*

$$\|u(\sigma)\| \leq \frac{K}{|\sigma|^r} \quad (5.10)$$

holds for $\sigma \in \Delta_{\delta}^{\text{fr},2}$. Then, there exists a constant $C > 0$ such that for all $t > 0$, we have

$$\left\| \int_{\Gamma_0^2} e^{\sigma^2 t} u(\sigma) d(\sigma^2) \right\|_X \leq \frac{C}{(1+t)^{1-\frac{r}{2}}}.$$

Proof. This is a standard argument, but we sketch a proof here for completeness. Using analyticity of $\sigma^2 \mapsto u(\sigma)$ on $\Omega_{\delta}^{\text{fr},2}$, for any fixed $t > 0$, we can deform the integration contour from Γ_0^2 to the t -dependent integration contour $\Gamma_t^{\text{fr},2} = \Gamma_t^{\text{fr},+} \cup \Gamma_t^{\text{circ}} \cup \Gamma_t^{\text{fr},-}$ lying in $\Omega_{\delta}^{\text{fr},2}$, where $\Gamma_t^{\text{fr},\pm}$ are line segments connecting the ends of $\Gamma_{\text{int}}^{2,\pm}$ to points $\delta e^{\pm i(\theta_0 - \tilde{\delta})} / (1+t)$, where $\tilde{\delta} > 0$ is chosen such that the extension of $\Gamma_t^{\text{fr},\pm}$ passes through the origin. We note here that $\tilde{\delta}$ only depends on $\Gamma_{\text{int}}^{2,\pm}$ and is thus independent of t . Finally, Γ_t^{circ} is the circle segment

$$\Gamma_t^{\text{circ}} = \left\{ \frac{\delta}{1+t} e^{i\theta} : \theta \in [-\theta_0 + \tilde{\delta}, \theta_0 - \tilde{\delta}] \right\};$$

see Figure 5. Note that if $\sigma^2 \in \Gamma_t^{\text{circ}}$, then we have $|\sigma|^2 = \frac{\delta}{1+t}$ and $\text{Re } \sigma^2 \leq \frac{\delta}{1+t}$. Hence, using (5.10) and the fact that Γ_t^{circ} has length bounded by $2\pi\delta/(1+t)$, we establish

$$\left\| \int_{\Gamma_t^{\text{circ}}} e^{\sigma^2 t} u(\sigma) d(\sigma^2) \right\|_X \lesssim \exp\left(\frac{\delta t}{1+t}\right) \left(\frac{\delta}{1+t}\right)^{1-\frac{r}{2}} \lesssim \frac{C_2}{(1+t)^{1-\frac{r}{2}}},$$

for $t > 0$. On the other hand, explicitly parameterizing the line segment $\Gamma_t^{\text{fr},+}$ which lies in the open left half-plane and using (5.10) we obtain

$$\left\| \int_{\Gamma_t^{\text{fr},+}} e^{\sigma^2 t} u(\sigma) d(\sigma^2) \right\|_X \lesssim \int_{\frac{\delta}{1+t}}^{\delta} \exp\left(\cos(\theta_0 - \tilde{\delta})st\right) |s|^{-\frac{r}{2}} ds \lesssim \frac{1}{(1+t)^{1-\frac{r}{2}}},$$

for $t > 0$, where the last inequality follows for $t > 1$ from the change of variables $z = st$ and the fact that the integral $\int_0^{\infty} e^{-az} z^{-r/2} dz$ is bounded for $a > 0$ and $r \in [0, 2)$, whereas for $t \in [0, 1]$ we use $\exp(\cos(\theta_0 - \tilde{\delta})st) \leq 1$. The integral over $\Gamma_t^{\text{fr},-}$ is estimated analogously. $\square$

5.3 Estimates for the outgoing mode

For estimating those parts of the resolvent associated to outgoing diffusive modes only, we may use either pointwise contours which pass fully into the essential spectrum, or contours which remain in the region $\Omega_{\delta}^{\text{wt}}$. The latter turn out to give sharp estimates only in certain norms; see the discussion below. Nonetheless, for terms in the resolvent decomposition which capture the interaction between outgoing and branched modes, we must use contours which remain in $\Omega_{\delta}^{\text{wt}}$ as pointed out in §5.1. We capture decay using contours of this type in the following result.

Proposition 5.3 (Normed estimates for outgoing modes). *Let X be a Banach space, and suppose that for some $\delta > 0$ small, we have an analytic function $\lambda \mapsto u(\lambda) : \Omega_{\delta}^{\text{wt}} \rightarrow X$, and there exists $r > -1$ such that*

$$\|u(\lambda)\|_X \leq C|\lambda|^r$$

for all $\lambda \in \Omega_{\delta}^{\text{wt}}$. Then, there exists a constant $C > 0$ such that for all $t > 0$,

$$\left\| \int_{\Gamma_0^2} e^{\lambda t} u(\lambda) d\lambda \right\|_X \leq \frac{C}{(1+t)^{\frac{1}{2}+\frac{r}{2}}}.$$

Proof. Estimates of this type have been established and used in [8, 41, 47], but we sketch a proof for completeness. Estimates for small times may be obtained using the fixed contour Γ_0^2 which remains to the right of the spectrum, so we focus on $t > 1$. Using analyticity of $\lambda \mapsto u(\lambda)$ on $\Omega_\delta^{\text{fr},2}$, for any fixed $t > 1$, we can deform the integration contour from Γ_0^2 to the t -dependent integration contour $\Gamma_t^{\text{wt}} = \Gamma_t^{\text{wt},-} \cup \Gamma_t^{\text{par}} \cup \Gamma_t^{\text{wt},+}$ lying in $\Omega_\delta^{\text{wt}}$ with

$$\Gamma_t^{\text{par}} = \left\{ \frac{\delta}{t} + ia - da^2 : a \in [-a_*, a_*] \right\},$$

for some t -independent constants $a_*, d > 0$. Here, $a_* > 0$ is chosen sufficiently small and $d > 0$ sufficiently large such that the contours Γ_t^{par} lie in $B(0, \delta)$ and their end-points are in the open left half-plane for any $t \geq 1$. Moreover, we let $\Gamma_t^{\text{wt},\pm}$ be the line segments connecting the ends of Γ_t^{par} to $\Gamma_{\text{int}}^{2,\pm}$, see Figure 5. Since the contours $\Gamma_t^{\text{wt},\pm}$ are contained in the left half-plane and are t -uniformly bounded away from the imaginary axis, their contributions are exponentially decaying in time. So, we focus on the integral over Γ_t^{par} . For a point $\lambda = \frac{\delta}{t} + ia - da^2 \in \Gamma_t^{\text{par}}$ with $a \in [-a_*, a_*]$, we have $|\lambda|^r \leq |\text{Im } \lambda|^r = |a|^r$ in case $r \in (-1, 0)$. On the other hand, in case $r \geq 0$, there exists a t - and a -independent constant $C > 0$ such that we have $|\lambda|^r \leq ((\delta/t - da^2)^2 + a^2)^{r/2} \leq C(t^{-r} + |a|^r)$ for $a \in [-a_*, a_*]$ and $t \geq 1$. Thus, we arrive at

$$\left\| \int_{\Gamma_t^{\text{par}}} e^{\lambda t} u(\lambda) d\lambda \right\|_X \lesssim \int_{-a_*}^{a_*} e^{\delta - da^2 t} \left(t^{-\max\{r, 0\}} + |a|^r \right) da \lesssim \frac{1}{t^{\frac{1}{2} + \frac{r}{2}}},$$

for $t > 1$, where the last inequality follows from applying the change of variables $z = a\sqrt{t}$. $\square$

Need for pointwise estimates. We claim that Proposition 5.3 gives estimates with sharp decay rates when we measure the solution in $L^\infty(\mathbb{R})$, but not when we measure it, for instance, in $L^2(\mathbb{R})$. To see this heuristically, recall from §2.1 that a good model for the dynamics in the wake is the advection-diffusion equation $u_t = D_{\text{eff}} u_{\xi\xi} + c_g u_\xi$ with $D_{\text{eff}} > 0$ and $c_g < 0$. Since advection does not affect decay rates in translation invariant norms, we have for this equation the well-known diffusive decay estimates, i.e., for $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ the solution $u(t)$ with initial condition $u(0) = u_0$ obeys the bound

$$\|\partial_\xi^k u(t)\|_{L^p} \lesssim \frac{1}{t^{\frac{1}{2} - \frac{1}{2p} + \frac{k}{2}}} \|u_0\|_{L^1}, \quad (5.11)$$

for $t > 0$. On the other hand, the resolvent kernel (or spatial Green's function) G_λ for this equation, which solves $(D_{\text{eff}} \partial_\xi^2 + c_g \partial_\xi - \lambda) G_\lambda = -\delta_0$, resembles

$$G_\lambda(\xi) \sim \chi_-(\xi) e^{\nu_{\text{wt}}(\lambda)\xi},$$

cf. §3.1. By simple scaling, we obtain, at best,²

$$\|\partial_\xi^k G_\lambda\|_{L^p} \leq C \frac{|\nu_{\text{wt}}(\lambda)|^k}{|\text{Re } \nu_{\text{wt}}(\lambda)|^{1/p}},$$

for some λ -independent constant $C > 0$. Along the parabolic contours used in Proposition 5.3, we find $|\nu_{\text{wt}}(\lambda)| \sim |\lambda|$ and $|\text{Re } \nu_{\text{wt}}(\lambda)| \sim |\lambda|^2$; see Proposition 3.1. Using these bounds one arrives at the estimate

$$\|\partial_\xi^k G_\lambda\|_{L^p} \lesssim |\lambda|^{k - \frac{2}{p}},$$

for points λ on these contours. Hence, if $\tilde{u}(\lambda)$ solves the resolvent equation $(D_{\text{eff}} \partial_\xi^2 + c_g \partial_\xi - \lambda) \tilde{u} = g$, then using Young's convolution inequality, we find the estimate

$$\|\partial_\xi^k \tilde{u}(\lambda)\|_{L^p} \lesssim |\lambda|^{k - \frac{2}{p}} \|g\|_{L^1},$$

²The ξ -derivatives may fall on $\chi_-(\xi)$ instead of on $e^{\nu_{\text{wt}}(\lambda)\xi}$, but terms where this happens are better behaved since χ'_- is compactly supported.

for points λ on these contours. Using Proposition 5.3, this then yields, for each $k \in \mathbb{N}_0$ and $p \in [1, \infty]$,

$$\|\partial_\xi^k u(\cdot, t)\|_{L^p} \lesssim \frac{1}{t^{\frac{1}{2} + \frac{k}{2} - \frac{1}{p}}}, \quad (5.12)$$

for $t > 0$. When $p = \infty$, these estimates agree with the sharp estimates of (5.11), but when $p < \infty$ the decay rates are worse by a factor of $t^{\frac{1}{p} - \frac{1}{2}}$. The reason is that when we measure in $L^p(\mathbb{R})$ for $p < \infty$, we must control spatial localization of the resolvent kernel G_λ . This spatial localization is controlled by $\operatorname{Re} \nu_{\text{wt}}(\lambda) \sim \operatorname{Re} \lambda$, but along the parabolic contours used in Proposition 5.3 we have $\operatorname{Re} \lambda \sim (\operatorname{Im} \lambda)^2$, which ultimately leads to the discrepancy in decay rates. To close a nonlinear argument, we will need sharp bounds on the solution in the wake at least in $L^p(\mathbb{R})$ for some $p < \infty$, in order to leverage spatial localization of quadratic terms in the nonlinearity (in our nonlinear argument we choose $p = 2$). We will therefore rely on pointwise estimates for this particular aspect to prove sharp bounds on the solution in $L^2(\mathbb{R})$.

Pointwise estimates. The estimates we need were essentially proven in [77, Section 8], with an updated statement closer to our exact needs given in [52, Section 7, example 2]. We give an adapted exposition of the proof from [52] in Appendix D for completeness.

Proposition 5.4 (Pointwise estimates for outgoing diffusive modes). *Suppose $\delta > 0$ is small. Let λ_0 denote the point at which $\Gamma_{\text{int}}^{2,+}$ intersects $B(0, \delta)$, and let λ_0^* denote its complex conjugate; see Figure 5. Let $g: \mathbb{R}^2 \times B(0, \delta) \rightarrow \mathbb{C}^2$ be a function such that*

$$\lambda \mapsto g(\cdot, \cdot, \lambda) : B(0, \delta) \rightarrow L^\infty(\mathbb{R}^2, \mathbb{C}^2)$$

is analytic and there exist a constant $C > 0$ and an integer $m \geq 0$ such that $\|g(\cdot, \cdot, \lambda)\|_{L^\infty} \leq C|\lambda|^m$. Then, for each $j, \ell \in \mathbb{N}_0$, there exist a function $G^{j, \ell, m}: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ and a constant $C_{j, \ell, m} > 0$ such that, for each $(\xi, \zeta, t) \in \mathbb{R}^2 \times [0, \infty)$, there exists a contour $\Gamma_{\xi, \zeta, t}$ lying in $B(0, \delta)$, which connects λ_0^ to λ_0 , satisfying*

$$\int_{\Gamma_{\xi, \zeta, t}} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\operatorname{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_{-(\xi - \zeta)} |\mathrm{d}\lambda| \leq G^{j, \ell, m}(t, \xi - \zeta), \quad (5.13)$$

and

$$\|G^{j, \ell, m}(t, \cdot)\|_{L^1} \leq \frac{C_{j, \ell, m}}{(1+t)^{\frac{\ell+j+m}{2}}}, \quad \|G^{j, \ell, m}(t, \cdot)\|_{L^\infty} \leq \frac{C_{j, \ell, m}}{(1+t)^{\frac{1+\ell+j+m}{2}}},$$

which immediately yields the interpolated estimate

$$\|G^{j, \ell, m}(t, \cdot)\|_{L^p} \leq \frac{C_{j, \ell, m}}{(1+t)^{\frac{1}{2} - \frac{1}{2p} + \frac{j+\ell+m}{2}}},$$

for $p \in [1, \infty]$.

We note that the pointwise estimates in Proposition 5.4 are tailored to pointwise analytic terms in the resolvent decomposition of the form (5.5) which can be represented as a convolution with a suitable integral kernel. However, fixing $\zeta = 0$ and working with a function g that does not depend on ζ in Proposition 5.4, the result can also be applied to those terms in the resolvent decomposition of the form (5.4) which are pointwise analytic in λ , but are not necessarily of convolution type.

Returning to the simple example $u_t = D_{\text{eff}} u_{\xi\xi} + c_g u_\xi$, we find for large times that the solution $u(t)$ with initial condition $u(0) = u_0$ behaves as

$$\partial_\xi^k u(\xi, t) \sim \int_{\mathbb{R}} \int_{\Gamma_0^2} (\nu_{\text{wt}}(\lambda))^k e^{\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta)} u_0(\zeta) \mathrm{d}\lambda \mathrm{d}\zeta,$$

for $k \in \mathbb{N}_0$, $\xi \in \mathbb{R}$ and $t > 0$. Using Proposition 5.4 and Young's convolution inequality, we find, for each $p \in [1, \infty]$, that the sharp decay estimate

$$\|\partial_\xi^k u(\cdot, t)\|_{L^p} \lesssim \|G^{0,k,0}\|_{L^p} \|u_0\|_{L^1} \lesssim \frac{1}{(1+t)^{\frac{1}{2}-\frac{1}{2p}+\frac{k}{2}}} \|u_0\|_{L^1},$$

holds for $t > 0$, thus recovering (5.11).

Estimates for interaction terms. Recall from our discussion in §5.1 that we cannot use the contours of Proposition 5.4 for estimating terms involving the interaction between branched and outgoing modes, since these contours pass through the branch point. To handle these terms, which are roughly of the form

$$\mathbf{u}_{\text{int}}(\xi; \sigma) \sim \alpha_{\text{int}}(\sigma) \chi_{-}(\xi) e^{\nu_{\text{wt}}(\sigma^2)\xi},$$

where $\alpha_{\text{int}}(\sigma)$ is analytic in $\sigma = \sqrt{\lambda}$ in a full neighborhood of the origin and satisfies $\alpha_{\text{int}}(0) = 0$, we turn to Proposition 5.3. The fact that $\alpha_{\text{int}}(\sigma) = O(\sigma)$, yields an extra decaying factor of $t^{-1/4}$. Returning to the model problem $u_t = D_{\text{eff}} u_{\xi\xi} + c_g u_\xi$, and (heuristically) giving this extra factor of decay to the model estimate (5.12), we find

$$\|\partial_\xi^k u(\cdot, t)\|_{L^p} \lesssim \frac{1}{t^{\frac{3}{4}+\frac{k}{2}-\frac{1}{p}}},$$

for $t > 0$. This agrees with the sharp model estimate (5.11) when $p = 2$. These estimates are sufficient to close a nonlinear stability argument.

6 Resolvent analysis near the origin

We analyze the resolvent near the origin with a far-field/core decomposition, aiming to identify terms associated to the branched diffusive modes, terms associated to the outgoing diffusive mode, and interaction terms. We start by considering the asymptotic contributions from $\xi = \pm\infty$.

6.1 Resolvent analysis for the wave train in the wake

In this section, we will use Floquet theory to study the resolvent $(\mathcal{L}_{\text{wt}} - \lambda)^{-1}$ of the linearization about the wave train $\mathbf{u}_{\text{wt}}$, near $\lambda = 0$. To begin, we transform the resolvent equation

$$(\mathcal{L}_{\text{wt}} - \lambda) \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{g}, \tag{6.1}$$

for $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{C}^2)$ to the first-order system

$$(\partial_\xi - A(\xi, \lambda))U = G, \tag{6.2}$$

where $U = (u, u_\xi, v)^\top$ and $G = (0, \mathbf{g})^\top$. For translating between the original formulation and the first-order formulation, we define the linear operators $\Pi_{13} : \mathbb{C}^3 \rightarrow \mathbb{C}^2$ and $\Lambda_1 : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ by

$$\Pi_{13} \begin{pmatrix} u \\ u_\xi \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \Lambda_1 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ g_1 \\ g_2 \end{pmatrix}.$$

The matrix $A(\xi, \lambda)$ is periodic in ξ and analytic in λ , and so by standard Floquet theory we have the following result.

Lemma 6.1 (Floquet theory). *Let $1 \leq p \leq \infty$. For $\delta > 0$ sufficiently small, there exists a change of coordinates $Q: \mathbb{R} \times B(0, \delta) \rightarrow \mathbb{C}^{3 \times 3}$ which is smooth and periodic in its first component and analytic in its second component and an analytic map $B: B(0, \delta) \rightarrow \mathbb{C}^{3 \times 3}$ such that, if we have $G \in L^p(\mathbb{R}, \mathbb{C}^3)$ and $U(\cdot, \lambda) \in W^{1,p}(\mathbb{R}, \mathbb{C}^3)$ is a solution of (6.2), then $V(\cdot, \lambda) = Q(\cdot, \lambda)U(\cdot, \lambda) \in W^{1,p}(\mathbb{R}, \mathbb{C}^3)$ solves*

$$(\partial_\xi - B(\lambda))V = QG. \quad (6.3)$$

For $\lambda \in B(0, \delta) \setminus \Sigma(\mathcal{L}_{\text{wt}})$ we solve (6.3) by constructing the matrix Green's function T_λ^{wt} which solves

$$(\partial_\xi - B(\lambda))T_\lambda^{\text{wt}} = \delta_0 I,$$

and is given by

$$T_\lambda^{\text{wt}}(\xi) = \begin{cases} e^{B(\lambda)\xi} P_{\text{wt}}^s(\lambda), & \xi \geq 0, \\ -e^{B(\lambda)\xi} P_{\text{wt}}^u(\lambda), & \xi < 0, \end{cases}$$

where $P_{\text{wt}}^s(\lambda)$ and $P_{\text{wt}}^u(\lambda)$ are the spectral projections onto the stable and unstable subspaces of $B(\lambda)$, respectively. Eigenvalues of $B(\lambda)$ are spatial Floquet exponents of $\mathcal{L}_{\text{wt}}$. It follows from Proposition 3.1 that for $|\lambda|$ small, the only eigenvalue of $B(\lambda)$ near the imaginary axis is

$$\nu_{\text{wt}}(\lambda) = \nu_{\text{wt}}^1 \lambda - \nu_{\text{wt}}^2 \lambda^2 + O(\lambda^3), \quad (6.4)$$

where $\nu_{\text{wt}}^1, \nu_{\text{wt}}^2 > 0$. Thus, for $|\lambda|$ small, $\nu_{\text{wt}}(\lambda)$ stays bounded away from the other eigenvalues of $B(\lambda)$ and, if $\text{Re } \lambda > 0$, this eigenvalue has positive real part. We can therefore separate $P_{\text{wt}}^u(\lambda)$ as

$$P_{\text{wt}}^u(\lambda) = P_{\text{wt}}^{\text{cu}}(\lambda) + P_{\text{wt}}^{\text{uu}}(\lambda),$$

for $\lambda \in B(0, \delta)$ with $\text{Re } \lambda > 0$, where $P_{\text{wt}}^{\text{cu}}(\lambda)$ is the spectral projection onto the eigenspace associated to $\nu_{\text{wt}}(\lambda)$. Note that, since the eigenvalues of $B(\lambda)$ stay separated, the spectral projections $P_{\text{wt}}^s(\lambda)$, $P_{\text{wt}}^{\text{uu}}(\lambda)$, and $P_{\text{wt}}^{\text{cu}}(\lambda)$ are defined for all λ in a full neighborhood of the origin. We may write

$$T_\lambda^{\text{wt}}(\xi) = -e^{\nu_{\text{wt}}(\lambda)\xi} P_{\text{wt}}^{\text{cu}}(\lambda) 1_{\{\xi < 0\}} - e^{B(\lambda)\xi} P_{\text{wt}}^{\text{uu}}(\lambda) 1_{\{\xi < 0\}} + e^{B(\lambda)\xi} P_{\text{wt}}^s(\lambda) 1_{\{\xi \geq 0\}},$$

for $\xi \in \mathbb{R}$ and $\lambda \in B(0, \delta)$ with $\text{Re } \lambda > 0$. We now modify this decomposition slightly, using the cutoff function χ_- , which is supported on $(-\infty, 0]$, to write

$$\begin{aligned} T_\lambda^{\text{wt}}(\xi) &= -e^{\nu_{\text{wt}}(\lambda)\xi} P_{\text{wt}}^{\text{cu}}(\lambda) \chi_-(\xi) + e^{\nu_{\text{wt}}(\lambda)\xi} P_{\text{wt}}^{\text{cu}}(\lambda) (\chi_-(\xi) - 1_{\{\xi < 0\}}) - e^{B(\lambda)\xi} P_{\text{wt}}^{\text{uu}}(\lambda) 1_{\{\xi < 0\}} \\ &\quad + e^{B(\lambda)\xi} P_{\text{wt}}^s(\lambda) 1_{\{\xi \geq 0\}} \\ &=: -e^{\nu_{\text{wt}}(\lambda)\xi} P_{\text{wt}}^{\text{cu}}(\lambda) \chi_-(\xi) + T_\lambda^e(\xi). \end{aligned}$$

Note that $T_\lambda^e \in L^1(\mathbb{R}, \mathbb{C}^3)$ is in fact analytic in λ in a full neighborhood of the origin due to its uniform exponential localization. We then decompose the solution to the resolvent equation (6.1) as

$$\begin{pmatrix} u(\xi, \lambda) \\ v(\xi, \lambda) \end{pmatrix} = -\Pi_{13} Q(\xi, \lambda)^{-1} \int_{\mathbb{R}} e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)} \chi_-(\xi - \zeta) P_{\text{wt}}^{\text{cu}}(\lambda) Q(\zeta, \lambda) \Lambda_1 \mathbf{g}(\zeta) d\zeta + \bar{s}_e^{\text{wt}}(\lambda) \mathbf{g}, \quad (6.5)$$

where

$$\bar{s}_e^{\text{wt}}(\lambda) \mathbf{g}(\xi) = \Pi_{13} Q(\xi, \lambda)^{-1} \int_{\mathbb{R}} T_\lambda^e(\xi - \zeta) Q(\zeta, \lambda) \Lambda_1 \mathbf{g}(\zeta) d\zeta. \quad (6.6)$$

We further modify the decomposition by expanding $P_{\text{wt}}^{\text{cu}}(\lambda)$, $Q(\xi, \lambda)$, and $Q(\xi, \lambda)^{-1}$ in λ in order to separate the most critical modes from those with improved algebraic decay in time, so that we may finally write

$$\begin{pmatrix} u(\xi, \lambda) \\ v(\xi, \lambda) \end{pmatrix} = -\Pi_{13} Q(\xi, 0)^{-1} \int_{\mathbb{R}} e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)} \chi_-(\xi - \zeta) P_{\text{wt}}^{\text{cu}}(0) Q(\zeta, 0) \Lambda_1 \mathbf{g}(\zeta) d\zeta + \bar{s}_e^{\text{wt}}(\lambda) \mathbf{g} + \bar{s}_e^{\text{wt}}(\lambda) \mathbf{g},$$

with

$$\begin{aligned} \bar{s}_e^{\text{wt}}(\lambda)\mathbf{g}(\xi) = \Pi_{13} \left[Q(\xi, 0)^{-1} \int_{\mathbb{R}} -e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} \chi_{-}(\xi-\zeta) P_{\text{wt}}^{\text{cu}}(0) Q(\zeta, 0) \Lambda_1 \mathbf{g}(\zeta) d\zeta \right. \\ \left. - Q(\xi, \lambda)^{-1} \int_{\mathbb{R}} e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} \chi_{-}(\xi-\zeta) P_{\text{wt}}^{\text{cu}}(\lambda) Q(\zeta, \lambda) \Lambda_1 \mathbf{g}(\zeta) d\zeta \right], \end{aligned} \quad (6.7)$$

and $\bar{s}_e^{\text{wt}}(\lambda)$ given by (6.6).

Lemma 6.2. *Fix $1 \leq p \leq \infty$. There exists $\delta > 0$ so that the map $\bar{s}_e^{\text{wt}}: B(0, \delta) \rightarrow B(L^p(\mathbb{R}, \mathbb{C}^2), L^p(\mathbb{R}, \mathbb{C}^2))$ is analytic. Moreover, there exists a constant $C > 0$ so that*

$$\|\bar{s}_e^{\text{wt}}(\lambda)\mathbf{g}\|_{L^p} \leq C\|\mathbf{g}\|_{L^p}$$

for $\lambda \in B(0, \delta)$ and $\mathbf{g} \in L^p(\mathbb{R}, \mathbb{C}^2)$.

Proof. Analyticity follows readily from pointwise analyticity of $Q(\xi, \lambda)$ in λ and analyticity of $T_{\lambda}^e \in L^1(\mathbb{R}, \mathbb{C}^3)$ in λ in a full neighborhood of the origin. $\square$

Undoing the Floquet change of coordinates at $\lambda = 0$, it follows that

$$-\Pi_{13} Q(\xi, 0)^{-1} \int_{\mathbb{R}} e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} \chi_{-}(\xi-\zeta) P_{\text{wt}}^{\text{cu}}(0) Q(\zeta, 0) \Lambda_1 \mathbf{g}(\zeta) d\zeta = \mathbf{u}'_{\text{wt}}(\xi) \bar{s}_p^{\text{wt}}(\lambda) \mathbf{g}(\xi),$$

where

$$\bar{s}_p^{\text{wt}}(\lambda)\mathbf{g}(\xi) = \varphi \left(\int_{\mathbb{R}} e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} \chi_{-}(\xi-\zeta) P_{\text{wt}}^{\text{cu}}(0) Q(\zeta, 0) \Lambda_1 \mathbf{g}(\zeta) d\zeta \right) \quad (6.8)$$

and $\varphi: \mathbb{C}^3 \rightarrow \mathbb{C}$ is some linear map.

We will use the following estimate in analyzing the center piece of the resolvent, which captures dynamics near the front interface.

Lemma 6.3. *For any $\eta > 0$ sufficiently small, there exist positive constants C and δ and a bounded limiting operator $R_0^{\text{wt}}: L^1(\mathbb{R}, \mathbb{C}^2) \rightarrow W^{1,\infty}(\mathbb{R}, \mathbb{C}) \times L^\infty(\mathbb{R}, \mathbb{C})$ such that the resolvent $(\mathcal{L}_{\text{wt}} - \lambda)^{-1}$ is a well-defined map on the exponentially weighted space $L_{\text{exp}, \eta, \eta}^1(\mathbb{R}, \mathbb{C}^2)$ for any $\lambda \in B(0, \delta)$. Moreover, for any $\mathbf{g} \in L^1(\mathbb{R}, \mathbb{C}^2)$ with $\text{supp}(\mathbf{g}) \subseteq (-\infty, 0]$, we have*

$$\|e^{\eta \cdot} \chi_{-} \left((\mathcal{L}_{\text{wt}} - \lambda)^{-1} - R_0^{\text{wt}} \right) \mathbf{g}\|_{W^{1,\infty} \times L^\infty} \leq C|\lambda| \|\mathbf{g}\|_{L^1} \quad (6.9)$$

for all $\lambda \in B(0, \delta)$.

Proof. First, we observe that Proposition 3.1 yields that, for $\eta > 0$ and $\delta > 0$ sufficient small, the eigenvalues ν of $B(\lambda)$ stay bounded away from the line $\text{Re}(\nu) = -\eta$ for any $\lambda \in B(0, \delta)$. Hence, the $L_{\text{exp}, \eta, \eta}^1$ -spectrum of the operator $\mathcal{L}_{\text{wt}}$ does not intersect the ball $B(0, \delta)$, implying that the resolvent $(\mathcal{L}_{\text{wt}} - \lambda)^{-1}$ is well-defined on the exponentially weighted space $L_{\text{exp}, \eta, \eta}^1(\mathbb{R}, \mathbb{C}^2)$ for any $\lambda \in B(0, \delta)$.

Upon defining $R_0^{\text{wt}}: L^1(\mathbb{R}, \mathbb{C}^2) \rightarrow W^{1,\infty}(\mathbb{R}, \mathbb{C}) \times L^\infty(\mathbb{R}, \mathbb{C})$ by $R_0^{\text{wt}} \mathbf{g}(\xi) = \mathbf{u}'_{\text{wt}}(\xi) \bar{s}_p^{\text{wt}}(0) \mathbf{g}(\xi) + \bar{s}_e^{\text{wt}}(0) \mathbf{g}(\xi)$, the proof of estimate (6.9) is similar to [8, Lemma 3.6], but we sketch it for completeness. We focus only on the most critical term in the resolvent, i.e. the one involving $\bar{s}_p^{\text{wt}}(\lambda) \mathbf{g}$, since obtaining the corresponding estimate for the other terms is strictly easier. Since we are assuming $\mathbf{g}$ is supported on $\{\zeta \leq 0\}$, the integrand in (6.8) is nonzero only for $\xi \leq \zeta \leq 0$.

Now, suppose $|\nu_{\text{wt}}(\lambda)(\xi - \zeta)| \leq 1$. Then, using the mean value theorem, Proposition 3.1 and the fact that $|\zeta| \leq |\xi|$, we establish

$$e^{\eta \xi} |e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} - 1| \lesssim e^{\eta \xi} |\nu_{\text{wt}}(\lambda)| |\xi - \zeta| \lesssim e^{\eta \xi} |\lambda| |\xi| \lesssim |\lambda|.$$

for $\xi \leq \zeta \leq 0$ and $\lambda \in B(0, \delta)$. Now, instead suppose $|\nu_{\text{wt}}(\lambda)(\xi - \zeta)| \geq 1$. Then, using that $|\zeta| \leq |\xi|$, we establish, if $|\lambda|$ is sufficiently small relative to η , that

$$e^{\eta\xi}|e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} - 1| \lesssim e^{\frac{\eta}{2}\xi} \lesssim e^{\frac{\eta}{2}\xi}|\nu_{\text{wt}}(\lambda)(\xi - \zeta)| \lesssim |\lambda|e^{\frac{\eta}{2}\xi}|\xi| \lesssim |\lambda|.$$

for $\xi \leq \zeta \leq 0$ and $\lambda \in B(0, \delta)$. Hence, in either case we have

$$e^{\eta\xi}|e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} - 1| \lesssim |\lambda|,$$

for $\xi \leq \zeta \leq 0$ and $|\lambda|$ sufficiently small, from which the result readily follows. $\square$

6.2 Resolvent analysis in the leading edge

We analyze the behavior of the limiting resolvent in the leading edge $(\mathcal{L}_+ - \sigma^2)^{-1}$ near $\sigma = 0$. A general analysis in reaction-diffusion systems with non-degenerate diffusion was carried out in [2, Section 5], based on the analysis for scalar parabolic equations in [7, 9]. So our main emphasis here is in explaining the necessary modifications to handle the degenerate diffusion in (1.2).

As in the preceding section, we recast the resolvent equation

$$(\mathcal{L}_+ - \sigma^2) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad (6.10)$$

as a first-order system in the variable $U = (u, u_\xi, v)^\top$ with $G = (0, \mathbf{g})^\top$ and $\mathbf{g} = (g_1, g_2)^\top$. This system takes the form

$$(\partial_\xi - \mathcal{M}(\sigma^2))U = G, \quad (6.11)$$

where the matrix $\mathcal{M}(\sigma^2) \in \mathbb{C}^{3 \times 3}$ is analytic in σ^2 . From a short calculation, one sees that the eigenvalues of $\mathcal{M}(\sigma^2)$ correspond precisely to roots $\nu(\sigma)$ of the dispersion relation $d_{\text{clin}}(\sigma^2, \nu)$, which we refer to as spatial eigenvalues. Recall from Corollary 3.4 that when $|\sigma|$ is small, there are precisely two spatial eigenvalues $\nu_{\text{fr}}^\pm(\sigma)$ in a neighborhood of the origin, with expansion

$$\nu_{\text{fr}}^\pm(\sigma) = \pm \nu_{\text{fr}}^1 \sigma + O(\sigma^2), \quad (6.12)$$

where $\nu_{\text{fr}}^1 > 0$. In particular, $\nu_{\text{fr}}^+(\sigma)$ has strictly positive real part and $\nu_{\text{fr}}^-(\sigma)$ has strictly negative real part for $\sigma^2 \in B(0, \delta)$ lying to the right of $\Sigma(\mathcal{L}_+)$. We record the following result, adapted from [2, Lemma 5.1].

Lemma 6.4. *Let $P_{\text{fr}}^{\text{cu}}(\sigma)$ and $P_{\text{fr}}^{\text{cs}}(\sigma)$ denote the spectral projections onto the eigenspaces of $\mathcal{M}(\sigma^2)$ associated with $\nu_{\text{fr}}^+(\sigma)$ and $\nu_{\text{fr}}^-(\sigma)$, respectively, which are one-dimensional provided $\sigma \neq 0$. Then, $P_{\text{fr}}^{\text{cu}}(\sigma)$ and $P_{\text{fr}}^{\text{cs}}(\sigma)$ are meromorphic in σ in a neighborhood of the origin, with expansions*

$$P_{\text{fr}}^{\text{cu}}(\sigma) = -\frac{1}{\sigma} P_{\text{pole}} + O(1), \quad P_{\text{fr}}^{\text{cs}}(\sigma) = \frac{1}{\sigma} P_{\text{pole}} + O(1)$$

for some matrix $P_{\text{pole}} \in \mathbb{C}^{3 \times 3}$.

Similar to Section 6.1, our strategy is to recover the solution to (6.10) from the first-order formulation, which we solve via a (somewhat) explicit Green's function. For $\sigma^2 \in B(0, \delta) \setminus \Sigma(\mathcal{L}_+)$ the matrix Green's function T_σ^{fr} associated to the first-order formulation (6.11) solves

$$(\partial_\xi - \mathcal{M}(\sigma^2))T_\sigma^{\text{fr}} = \delta_0 I,$$

so that the solution to (6.10) is given by

$$\begin{pmatrix} u(\xi; \sigma) \\ v(\xi; \sigma) \end{pmatrix} = \int_{\mathbb{R}} \Pi_{13} T_{\sigma}^{\text{fr}}(\xi - \zeta) \Lambda_1 \mathbf{g}(\zeta) d\zeta, \quad (6.13)$$

where we recall the definition (6.5) of the linear operators Π_{13} and Λ_1 . Following the reasoning in [7, Section 2] and using Corollary 3.4, we may decompose T_{σ}^{fr} as

$$T_{\sigma}^{\text{fr}}(\zeta) = e^{\nu_{\text{fr}}^{-}(\sigma)\zeta} P_{\text{fr}}^{\text{cs}}(\sigma) 1_{\{\zeta \geq 0\}} - \left(e^{\nu_{\text{fr}}^{+}(\sigma)\zeta} P_{\text{fr}}^{\text{cu}}(\sigma) + e^{\nu_3(\sigma^2)\zeta} P_{\text{fr}}^{\text{uu}}(\sigma^2) \right) 1_{\{\zeta < 0\}}, \quad (6.14)$$

for $\sigma^2 \in B(0, \delta)$ to the right of $\Sigma(\mathcal{L}_+)$, where $P_{\text{fr}}^{\text{uu}}(\sigma^2)$ is the spectral projection onto the strong unstable subspace of $\mathcal{M}(\sigma^2)$ corresponding to the remaining third eigenvalue $\nu_3(\sigma^2)$ of $M(\sigma^2)$, which resides in the open right half-plane and remains bounded away from the imaginary axis for $|\sigma|$ small. The leading-order temporal behavior of the solution to the linearization $\mathbf{u}_t = \mathcal{L}_+ \mathbf{u}$ will be governed by those terms in (6.14) which are most singular in σ . Standard spectral theory implies that $P_{\text{fr}}^{\text{uu}}(\sigma^2)$ is analytic in σ^2 in a full neighborhood of the origin. The only singularities then arise from the terms involving P_{pole} , and so from the point of view of the resulting temporal dynamics, we have

$$T_{\sigma}^{\text{fr}}(\zeta) \approx \frac{1}{\sigma} P_{\text{pole}} \left(e^{\nu_{\text{fr}}^{+}(\sigma)\zeta} 1_{\{\zeta < 0\}} + e^{\nu_{\text{fr}}^{-}(\sigma)\zeta} 1_{\{\zeta \geq 0\}} \right).$$

Using the expansion (6.12), we see that $-\nu_{\text{fr}}^{+}(\sigma) \approx \nu_{\text{fr}}^{-}(\sigma) \approx -\nu_{\text{fr}}^1 \sigma$, and hence

$$T_{\sigma}^{\text{fr}}(\zeta) \approx \frac{1}{\sigma} e^{-\nu_{\text{fr}}^1 \sigma |\zeta|} P_{\text{pole}} =: G_{\sigma}^{\text{heat}}(\zeta) P_{\text{pole}}.$$

Note that, up to scaling, G_{σ}^{heat} is the Laplace transform of the fundamental solution of the heat equation. The solution to the resolvent equation (6.10) is then given to leading order by

$$\begin{pmatrix} u(\xi; \sigma) \\ v(\xi; \sigma) \end{pmatrix} \approx \Pi_{13} P_{\text{pole}} \Lambda_1 \int_{\mathbb{R}} G_{\sigma}^{\text{heat}}(\xi - \zeta) \mathbf{g}(\zeta) d\zeta. \quad (6.15)$$

To make this reasoning rigorous, we can follow the analysis of [7, Section 2], but for now ignoring estimates on derivatives of the solution, to obtain the following description of the far-field resolvent for $\sigma \in \Delta_{\delta}^{\text{fr},1}$, where we recall that $\Delta_{\delta}^{\text{fr},1}$ is the image of the set $\Omega_{\delta}^{\text{fr},1}$ under the principal square root, see Figure 4 and (5.3).

Lemma 6.5. *Fix $r > 2$. There exist positive constants C and δ and a bounded limiting operator $R_0^{+} : L_{1,1}^1(\mathbb{R}, \mathbb{C}^2) \rightarrow L_{-1,-1}^{\infty}(\mathbb{R}, \mathbb{C}^2)$ such that for all odd functions $\mathbf{g} \in L_{1,1}^1(\mathbb{R}, \mathbb{C}^2)$, we have*

$$\|(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g} - R_0^{+} \mathbf{g}\|_{L_{-1,-1}^{\infty}} \leq C|\sigma| \|\mathbf{g}\|_{L_{1,1}^1}, \quad \|(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g} - R_0^{+} \mathbf{g}\|_{L_{-r,-r}^1} \leq C|\sigma| \|\mathbf{g}\|_{L_{1,1}^1},$$

for all $\sigma \in \Delta_{\delta}^{\text{fr},1}$.

The leading-order description (6.15) corresponds to the resolvent problem for the heat equation. For the heat equation, restricting to odd initial data is equivalent to imposing the Dirichlet boundary condition $u = 0$ at $\xi = 0$. Restricting to odd initial data $\mathbf{g}$ here models an effective absorption mechanism behind the leading edge of the front, due to stability of the state in the wake. We will enforce this oddness for the leading edge resolvent in our far-field/core decomposition in the next section.

In passing estimates to the full resolvent $(\mathcal{L}_{\text{fr}} - \sigma^2)^{-1}$, we will need to control commutators $[\mathcal{L}_+, \chi_+]$ applied to the solution $\mathbf{u}^{+}$ of the leading edge resolvent equation $(\mathcal{L}_+ - \sigma^2) \mathbf{u}^{+} = \mathbf{g}_+$. These commutator terms will involve derivatives applied to the first component of $\mathbf{u}^{+}$, but not the second component, and so we need to upgrade Lemma 6.5 to control regularity in the first component.

Proposition 6.6. *Fix $r > 2$. There exist positive constants C and δ and a bounded limiting operator $R_0^+ : L_{1,1}^1(\mathbb{R}, \mathbb{C}^2) \rightarrow W_{-1,-1}^{1,\infty}(\mathbb{R}, \mathbb{C}) \times L_{-1,-1}^\infty(\mathbb{R}, \mathbb{C})$ such that for all odd functions $\mathbf{g} = (g_1, g_2)^\top \in L_{1,1}^1(\mathbb{R}, \mathbb{C}^2)$, we have*

$$\begin{aligned} \|(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g} - R_0^+ \mathbf{g}\|_{W_{-1,-1}^{1,\infty} \times L_{-1,-1}^\infty} &\leq C|\sigma| \|\mathbf{g}\|_{L_{1,1}^1}, \\ \|(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g} - R_0^+ \mathbf{g}\|_{W_{-r,-r}^{1,1} \times L_{-r,-r}^1} &\leq C|\sigma| \|\mathbf{g}\|_{L_{1,1}^1}, \end{aligned} \quad (6.16)$$

for all $\sigma \in \Delta_\delta^{\text{fr},1}$.

Proof. Using the first-order formulation in $U = (u, u_\xi, v)^\top$, we see that, if $(\mathcal{L}_+ - \sigma^2)u = \mathbf{g}$, then we have

$$u_\xi(\xi; \sigma) = \int_{\mathbb{R}} \Pi_2 T_\sigma^{\text{fr}}(\xi - \zeta) \Lambda_1 \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix} d\zeta, \quad \text{where} \quad \Pi_2 \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = f_2.$$

Comparing with (6.13), the only difference between the expressions for $(u, v)^\top$ and u_ξ is that Π_{13} is replaced by Π_2 , both of which are independent of σ . Therefore, following the same reasoning, we readily find the same estimates for $u_\xi(\xi; \sigma)$ as for $u(\xi; \sigma)$, which implies the desired result. $\square$

We will use Proposition 6.6 to obtain the sharp $t^{-3/2}$ decay when measuring the solution in $L_{0,-1}^\infty$. Indeed, we saw in Proposition 5.1 how the Lipschitz estimate (6.16) leads to the $t^{-3/2}$ decay rate. In closing a nonlinear argument, however, we will also measure the solution in weaker norms, such as $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$. In these spaces, we lose the Lipschitz expansion of the resolvent, but retain some boundedness or quantifiable blowup for $\sigma \in \Delta_\delta^{\text{fr},2}$, where we recall that $\Delta_\delta^{\text{fr},2}$ is the image of the set $\Omega_\delta^{\text{fr},2}$ under the principal square root; see Figure 4 and (5.2).

Lemma 6.7 (L^∞ estimate in the leading edge). *There exist positive constants C and δ such that for $\sigma \in \Delta_\delta^{\text{fr},2}$ the following estimates hold.*

- For all odd functions $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$, we have

$$\|(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g}\|_{W^{2,\infty} \times W^{1,\infty}} \leq C \|\mathbf{g}\|_{L_{0,1}^1}. \quad (6.17)$$

- For all odd functions $\mathbf{g} \in L^1(\mathbb{R}, \mathbb{C}^2)$, we have

$$\|\chi_+(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g}\|_{L_{0,-1}^\infty} \leq C \|\mathbf{g}\|_{L^1}. \quad (6.18)$$

The estimate (6.17) will be used to estimate decay in the leading edge in $L^\infty(\mathbb{R})$. The estimate (6.18) will be used in controlling contributions to the center dynamics near the front interface. Finally, to estimate decay in the leading edge in $L^2(\mathbb{R})$ we will rely on the following result.

Lemma 6.8 (L^2 estimate in the leading edge). *There exist positive constants C and δ such that for all odd functions $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$, we have*

$$\|(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g}\|_{L^2} \leq \frac{C}{|\sigma|^{1/2}} \|\mathbf{g}\|_{L_{0,1}^1}$$

for all $\sigma \in \Delta_\delta^{\text{fr},2}$.

We relegate the proofs of Lemmas 6.7 and 6.8 to Appendix C.

6.3 Far-field/core decomposition

Our goal is to solve $(\mathcal{L}_{\text{fr}} - \sigma^2)\mathbf{u} = \mathbf{g}$ by using a far-field/core decomposition which takes advantage of Fredholm properties of $\mathcal{L}_{\text{fr}}$ on spaces of exponentially localized functions, captured in Proposition 3.6. To do this, we must first reduce to a problem where the data $\mathbf{g}$ is exponentially localized. Thus, we first decompose data $\mathbf{g} \in L^1(\mathbb{R}, \mathbb{C}^2) \cap L^\infty(\mathbb{R}, \mathbb{C}^2)$ as

$$\mathbf{g} = \chi_- \mathbf{g} + \chi_c \mathbf{g} + \chi_+ \mathbf{g} =: \mathbf{g}_- + \mathbf{g}_c + \mathbf{g}_+,$$

where (χ_-, χ_c, χ_+) is the partition of unity defined in Section 2.3. We then let $\mathbf{g}_+^{\text{odd}}(\xi) = \mathbf{g}_+(\xi) - \mathbf{g}_+(-\xi)$ be the odd extension of $\mathbf{g}_+$, in order to take advantage of the improved behavior of $(\mathcal{L}_+ - \sigma^2)^{-1}$ when acting on odd functions. We let $\mathbf{u}_+$ solve

$$(\mathcal{L}_+ - \sigma^2)\mathbf{u}_+ = \mathbf{g}_+^{\text{odd}}.$$

By Proposition 6.6 we can solve this equation in $L_{-1,-1}^\infty(\mathbb{R})$, thus allowing for algebraic growth, by requiring that $\mathbf{g}_+^{\text{odd}}$ is algebraically localized, i.e. $\mathbf{g}_+^{\text{odd}} \in L_{0,1}^1(\mathbb{R})$, and σ^2 lies in the set $\Omega_\delta^{\text{fr},1}$, which lies to the right of $\Sigma_{\text{ess}}(\mathcal{L}_+)$. In addition, if σ^2 lies in the smaller set $\Omega_\delta^{\text{fr},2}$, we obtain by Proposition 6.7 a solution $\mathbf{u}_+$ in $L_{-1,-1}^\infty(\mathbb{R})$ without requiring extra conditions on $\mathbf{g}_+^{\text{odd}}$.

We let $\mathbf{u}_-$ solve

$$(\mathcal{L}_{\text{wt}} - \sigma^2)\mathbf{u}_- = \mathbf{g}_-.$$

By Proposition 6.3, there exists $\delta > 0$ and $\eta > 0$ small, such that we can solve this equation uniquely for any $\sigma^2 \in B(0, \delta)$ by allowing the solution $\mathbf{u}_-$ to grow exponentially on $(-\infty, 0]$ with rate η .

We then decompose $\mathbf{u}$ as

$$\mathbf{u} = \chi_- \mathbf{u}_- + \mathbf{u}_c + \chi_+ \mathbf{u}_+,$$

and see that in order for $\mathbf{u}$ to solve $(\mathcal{L}_{\text{fr}} - \sigma^2)\mathbf{u} = \mathbf{g}$, the center correction $\mathbf{u}_c$ must solve

$$(\mathcal{L}_{\text{fr}} - \sigma^2)\mathbf{u}_c = \tilde{\mathbf{g}}(\sigma), \tag{6.19}$$

where

$$\tilde{\mathbf{g}}(\sigma) = \mathbf{g} - (\mathcal{L}_{\text{fr}} - \sigma^2)(\chi_- \mathbf{u}_-) - (\mathcal{L}_{\text{fr}} - \sigma^2)(\chi_+ \mathbf{u}_+). \tag{6.20}$$

The next result confirms that the above procedure has led to a reduced problem (6.19) in which the data $\tilde{\mathbf{g}}$ is indeed exponentially localized. In addition, it provides control on the data $\tilde{\mathbf{g}}$ in terms of the original data $\mathbf{g}$ and the spectral parameter σ .

Lemma 6.9 (Control on center data). *Fix $1 \leq p \leq \infty$. There exist $\delta > 0$ and $\eta > 0$ small such that, for any $\mathbf{g} \in L^1(\mathbb{R}, \mathbb{C}^2)$, the map $\tilde{\mathbf{g}}: \Delta_\delta^{\text{fr},1} \rightarrow X_\eta^p$ given by (6.20) is well-defined and analytic. Moreover, there exists a constant $C > 0$ such that*

1. *for all any $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$ the map $\tilde{\mathbf{g}}$ extends Lipschitz continuously to $\sigma = 0$, i.e., for all $\sigma \in \Delta_\delta^{\text{fr},1}$ we have*

$$\|\tilde{\mathbf{g}}(\sigma) - \tilde{\mathbf{g}}(0)\|_{X_\eta^p} \leq C|\sigma| \|\mathbf{g}\|_{L_{0,1}^1}, \quad \|\tilde{\mathbf{g}}(0)\| \leq C\|\mathbf{g}\|_{L_{0,1}^1}; \tag{6.21}$$

2. *for all $\sigma \in \Delta_\delta^{\text{fr},2}$ and any $\mathbf{g} \in L^1(\mathbb{R}, \mathbb{C}^2)$ we have*

$$\|\tilde{\mathbf{g}}(\sigma)\|_{X_\eta^1} \leq C\|\mathbf{g}\|_{L^1}; \tag{6.22}$$

3. for all $\sigma \in \Delta_\delta^{\text{fr},2}$ and for any $\mathbf{g} \in L^1(\mathbb{R}, \mathbb{C}^2) \cap L^\infty(\mathbb{R}, \mathbb{C}^2)$ we have

$$\|\tilde{\mathbf{g}}(\sigma)\|_{X_\eta^p} \leq C(\|\mathbf{g}\|_{L^1} + \|\mathbf{g}\|_{L^\infty}) \quad (6.23)$$

for all $1 \leq p \leq \infty$.

Proof. We rewrite $\tilde{\mathbf{g}}(\sigma)$ as

$$\begin{aligned} \tilde{\mathbf{g}}(\sigma) &= \mathbf{g} - (\mathcal{L}_{\text{wt}} - \sigma^2)(\chi_- \mathbf{u}_-) - (\mathcal{L}_{\text{fr}} - \mathcal{L}_{\text{wt}})(\chi_- \mathbf{u}_-) - (\mathcal{L}_+ - \sigma^2)(\chi_+ \mathbf{u}_+) - (\mathcal{L}_{\text{fr}} - \mathcal{L}_+)(\chi_+ \mathbf{u}_+) \\ &= \mathbf{g} - \chi_- \mathbf{g}_- - \chi_+ \mathbf{g}_+ - [\mathcal{L}_{\text{wt}}, \chi_-] \mathbf{u}_- - (\mathcal{L}_{\text{fr}} - \mathcal{L}_{\text{wt}})(\chi_- \mathbf{u}_-) - [\mathcal{L}_+, \chi_+] \mathbf{u}_+ \\ &\quad - (\mathcal{L}_{\text{fr}} - \mathcal{L}_+)(\chi_+ \mathbf{u}_+), \end{aligned} \quad (6.24)$$

where we have used the facts that $\chi_+(\mathcal{L}_+ - \sigma^2)\mathbf{u}_+ = \chi_+ \mathbf{g}_+^{\text{odd}} = \chi_+ \mathbf{g}_+$, and $\chi_-(\mathcal{L}_{\text{wt}} - \sigma^2)\mathbf{u}_- = \chi_- \mathbf{g}_-$ by construction.

We first prove (6.21). The dependence of $\tilde{\mathbf{g}}$ on σ enters through the terms involving $\mathbf{u}_-$ and $\mathbf{u}_+$. Using that the derivatives $\chi'_\pm$ are compactly supported and the coefficients of $\mathcal{L}_{\text{fr}}$ converge to their limits exponentially quickly as $\xi \rightarrow \pm\infty$, we see that all coefficients of $\mathbf{u}_\pm$ in (6.24) are exponentially localized with rate $\eta_0 > 0$ independent of σ . Note also that all commutator terms only involve derivatives in the first component. Hence, choosing $\eta \leq \frac{\eta_0}{2}$, we have by Proposition 6.6 and Hölder's inequality,

$$\begin{aligned} \|[\mathcal{L}_+, \chi_+](\mathbf{u}_+(\sigma) - \mathbf{u}_+(0))\|_{X_\eta^p} &\lesssim \left\| e^{-\frac{\eta_0}{2}|\cdot|}(\mathbf{u}_+(\sigma) - \mathbf{u}_+(0)) \right\|_{L^p} + \left\| e^{-\frac{\eta_0}{2}|\cdot|} \Pi_1 \partial_x(\mathbf{u}_+(\sigma) - \mathbf{u}_+(0)) \right\|_{L^p} \\ &\lesssim |\sigma| \|\mathbf{g}_+\|_{L_{1,1}^1} \lesssim |\sigma| \|\mathbf{g}\|_{L_{0,1}^1}, \end{aligned}$$

for $\sigma \in \Delta_\delta^{\text{fr},1}$, where $\Pi_1(u, v)^\top = u$. By the same argument relying on exponential localization, we obtain

$$\|(\mathcal{L}_{\text{fr}} - \mathcal{L}_+)[\chi_+(\mathbf{u}_+(\sigma) - \mathbf{u}_+(0))]\|_{X_\eta^1} \lesssim |\sigma| \|\mathbf{g}\|_{L_{0,1}^1},$$

for $\sigma \in \Delta_\delta^{\text{fr},1}$. The argument controlling the terms involving $\mathbf{u}_-$ is analogous, with Lemma 6.3 replacing Proposition 6.6. Note that Lemma 6.3 only controls a localized L^∞ norm, not the L^1 norm, but here the integral can be absorbed by the uniform exponential factor so that the L^∞ estimate is sufficient even in the case $p = 1$.

The proof of the estimate (6.22) is similar: after exploiting uniform exponential localization and the structure of the commutators, control on the terms involving $\mathbf{u}_+$ is obtained using Lemma 6.7, while Lemma 6.3 still suffices for control on the terms involving $\mathbf{u}_-$. The proof of (6.23) is again similar, the only difference being that the term $\|\mathbf{g}\|_{L^\infty}$ is needed on the right hand side to control $\|\mathbf{g} - \chi_- \mathbf{g}_- - \chi_+ \mathbf{g}_+\|_{L^p}$ from the point of view of spatial regularity. $\square$

With control on the exponentially localized right-hand side $\tilde{\mathbf{g}}$ in hand, we now aim to solve (6.19) for $\mathbf{u}_c$ using a far-field/core decomposition. To understand the solvability properties of (6.19) on the space X_η^p of exponentially localized functions we can look at the Fredholm index of $\mathcal{L}_{\text{fr}}$, which is -2 by Proposition 3.6. Thus, the problem (6.19) cannot be solved for exponentially localized $\mathbf{u}_c$ as we need to allow for two additional degrees of freedom. This leads us to consider the following two neutral modes.

Lemma 6.10 (Neutral mode on the left). *For $\delta > 0$ sufficiently small, there exists a solution $\mathbf{e}_-(\xi, \lambda)$ to $(\mathcal{L}_{\text{wt}} - \lambda)\mathbf{u} = 0$ given by*

$$\mathbf{e}_-(\xi, \lambda) = \mathbf{q}(\xi, \lambda) e^{\nu_{\text{wt}}(\lambda)\xi}$$

where $\mathbf{q}: \mathbb{R} \times B(0, \delta) \rightarrow \mathbb{C}^2$ is smooth and periodic in its first argument and analytic in its second argument. Moreover, we have $\mathbf{q}(\cdot, 0) = \mathbf{u}_{\text{wt}}$.

Proof. This follows readily from Lemma 6.1 and Proposition 3.1. $\square$

Lemma 6.11 (Neutral mode on the right). *For $\delta > 0$ sufficiently small, there exists a solution $\mathbf{e}_+(\xi, \sigma)$ to $(\mathcal{L}_+ - \sigma^2)\mathbf{u} = 0$ given by*

$$\mathbf{e}_+(\xi, \sigma) = \mathbf{v}_+(\sigma) e^{\nu_{\text{fr}}^-(\sigma)\xi},$$

where $\mathbf{v}_+ : B(0, \sqrt{\delta}) \rightarrow \mathbb{C}^2$ and $\nu_{\text{fr}}^- : B(0, \sqrt{\delta}) \rightarrow \mathbb{C}$ are analytic, with $\nu_{\text{fr}}^-(\sigma)$ given in Corollary 3.4.

Proof. This follows readily from Corollary 3.4. $\square$

Motivated by the previous Fredholm analysis, we make the ansatz

$$\mathbf{u}_c(\xi; \sigma) = \alpha_- \chi_-(\xi) \mathbf{e}_-(\xi, \sigma^2) + \mathbf{w}(\xi) + \alpha_+ \chi_+(\xi) \mathbf{e}_+(\xi, \sigma).$$

Inserting this ansatz into (6.19) leads to an equation

$$G(\mathbf{w}, \alpha_-, \alpha_+; \sigma) = \tilde{\mathbf{g}}(\sigma),$$

where

$$G(\mathbf{w}, \alpha_-, \alpha_+; \sigma) = (\mathcal{L}_{\text{fr}} - \sigma^2) \left[\alpha_- \chi_- \mathbf{e}_-(\cdot, \sigma^2) + \mathbf{w} + \alpha_+ \chi_+ \mathbf{e}_+(\cdot, \sigma) \right].$$

Lemma 6.12. *Fix $1 \leq p \leq \infty$. There exist constants $\eta > 0$ and $\delta > 0$ sufficiently small so that the map*

$$G : Y_\eta^p \times \mathbb{C} \times \mathbb{C} \times B(0, \sqrt{\delta}) \rightarrow X_\eta^p$$

is well defined and analytic in σ .

Proof. This follows from the fact that $\mathbf{e}_\pm$ solve the equations in the far-field (see Lemmas 6.10 and 6.11), together with exponential convergence of the coefficients of $\mathcal{L}_{\text{fr}}$ and localization of commutator terms. Analyticity in σ follows as in [57, Proposition 5.11] or [9, Lemma 3.9]. $\square$

Note that G is linear in w, α_- , and α_+ . Although $D_w G = \mathcal{L}_{\text{fr}}$ is Fredholm with index -2 by Proposition 3.6, provided $\eta > 0$ is sufficiently small, the Fredholm bordering lemma and continuity of the Fredholm index imply that, for each fixed $|\sigma|$ small, the map $(w, \alpha_-, \alpha_+) \mapsto G(w, \alpha_-, \alpha_+; \sigma)$ is Fredholm with index 0. Hence, the two additional degrees of freedom in the ansatz for $\mathbf{u}_c$, represented by the coefficients $\alpha_\pm$ of the neutral modes, led to a problem which can potentially be inverted. The next result shows that this is in fact the case.

Proposition 6.13. *Fix $1 \leq p \leq \infty$. There exist constants $\eta > 0$ and $\delta > 0$ sufficiently small so that for each $\sigma \in B(0, \sqrt{\delta})$, the map*

$$(\mathbf{w}, \alpha_-, \alpha_+) \mapsto G(w, \alpha_-, \alpha_+; \sigma) : Y_\eta^p \times \mathbb{C} \times \mathbb{C} \rightarrow X_\eta^p$$

is invertible. We denote the solution to $G(\mathbf{w}, \alpha_-, \alpha_+; \sigma) = \mathbf{g}$ by

$$(\mathbf{w}, \alpha_-, \alpha_+) = (T(\sigma)\mathbf{g}, A_-(\sigma)\mathbf{g}, A_+(\sigma)\mathbf{g}).$$

Moreover, the inverse maps

$$T(\sigma) : X_\eta^p \rightarrow Y_\eta^p, \quad A_\pm(\sigma) : X_\eta^p \rightarrow \mathbb{C}$$

are analytic in σ in $B(0, \sqrt{\delta})$.

Proof. The result will follow from the implicit function theorem provided we can verify invertibility at $\sigma = 0$. Since $(\mathbf{w}, \alpha_-, \alpha_+) \mapsto G(\mathbf{w}, \alpha_-, \alpha_+; 0)$ is Fredholm with index 0, this map is invertible if and only if it has trivial kernel. Since $\mathbf{e}_-(\xi, 0)$ and $\mathbf{e}_+(\xi, 0)$ are bounded on the real line, a nontrivial kernel of this map would correspond to a nontrivial bounded solution to $\mathcal{L}_{\text{fr}} \mathbf{u} = 0$, which is excluded by Hypothesis 3. $\square$

We have constructed a solution to $(\mathcal{L}_{\text{fr}} - \sigma^2) \mathbf{u} = \mathbf{g}$ given by

$$\mathbf{u}(\xi; \sigma) = \chi_-(\xi) \mathbf{u}_-(\xi; \sigma^2) + \alpha_-(\sigma) \chi_-(\xi) \mathbf{e}_-(\xi; \sigma) + \mathbf{w}(\xi; \sigma) + \alpha_+(\sigma) \chi_+(\xi) \mathbf{e}_+(\xi; \sigma) + \chi_+(\xi) \mathbf{u}_+(\xi; \sigma),$$

where

$$\alpha_-(\sigma) = A_-(\sigma) \tilde{\mathbf{g}}(\sigma), \quad \mathbf{w}(\cdot; \sigma) = T(\sigma) \tilde{\mathbf{g}}(\sigma), \quad \alpha_+(\sigma) = A_+(\sigma) \tilde{\mathbf{g}}(\sigma).$$

Our goal is now to isolate the terms which correspond to the leading-order temporal dynamics. Thus, we decompose the solution as

$$\mathbf{u}(\xi) = \chi_-(\xi) \mathbf{u}_-(\xi; \sigma^2) + \alpha_-(\sigma) \chi_-(\xi) \mathbf{e}_-(\xi; \sigma^2) + R_{\text{fr}}(\xi; \sigma), \quad (6.25)$$

where

$$R_{\text{fr}}(\xi; \sigma) = \chi_+(\xi) \mathbf{u}_+(\xi; \sigma) + T(\sigma) \tilde{\mathbf{g}}(\sigma) + \alpha_+(\sigma) \chi_+(\xi) \mathbf{e}_+(\xi; \sigma)$$

captures contributions from the branched mode in the leading edge. To estimate time decay rates of the branched mode in various norms, we use the following estimates on $R_{\text{fr}}(\xi; \sigma)$. The first corresponds to the sharp $t^{-3/2}$ pointwise decay rate, which is observed when measuring in sufficiently weak norms. The next two correspond to slower decay rates when measuring in stronger norms.

Lemma 6.14 (Expansion for branched mode in $L_{0,-1}^\infty$). *There exist positive constants C and δ such that for any $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$, we have the estimate*

$$\|R_{\text{fr}}(\cdot; \sigma) - R_{\text{fr}}(\cdot; 0)\|_{L_{0,-1}^\infty} \leq C|\sigma| \|\mathbf{g}\|_{L_{0,1}^1},$$

for all $\sigma \in \Delta_\delta^{\text{fr},1}$.

Proof. This follows by combining Corollary 3.4, Proposition 6.6, Lemma 6.9, and Proposition 6.13, and Taylor expanding the exponential in $\mathbf{e}_+(x; \sigma)$. $\square$

Lemma 6.15 (Blowup of the branched mode in L^2). *There exist positive constants C and δ such that for any $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2) \cap L^\infty(\mathbb{R}, \mathbb{C}^2)$, we have the estimate*

$$\|R_{\text{fr}}(\cdot; \sigma)\|_{L^2} \leq \frac{C}{|\sigma|^{1/2}} \|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty}$$

for all σ such that $\sigma \in \Delta_\delta^{\text{fr},2}$.

Lemma 6.16 (Boundedness of the branched mode in L^∞). *There exist positive constants C and δ such that for any $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2) \cap L^\infty(\mathbb{R}, \mathbb{C}^2)$, we have the estimate*

$$\|R_{\text{fr}}(\cdot; \sigma)\|_{L^\infty} \leq C \|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty}$$

for all σ such that $\sigma \in \Delta_\delta^{\text{fr},2}$.

Lemmas 6.15 and 6.16 readily follow from the control on the leading edge resolvent $(\mathcal{L}_+ - \sigma^2)^{-1}$ in Lemmas 6.8 and 6.7, control on $\nu_{\text{fr}}^-(\sigma)$ and $\mathbf{e}_+(\cdot, \sigma)$ from Corollary 3.4 and Lemma 6.11, and control on $T(\sigma)\tilde{\mathbf{g}}(\sigma)$ and $A_+(\sigma)\tilde{\mathbf{g}}(\sigma)$ from Proposition 6.13 and Lemma 6.9.

We now analyze the terms $\chi_-(\xi)\mathbf{u}_-(\xi; \sigma^2)$ and $\alpha_-(\sigma)\chi_-(\xi)\mathbf{e}_-(\xi; \sigma^2)$ in (6.25), which correspond to the leading-order temporal dynamics. We begin with the term $\chi_-(\xi)\mathbf{u}_-(\xi; \sigma^2)$. By the resolvent decomposition for the wave train in the wake, performed in Section 6.1, this contribution can be expressed as

$$\chi_-(\xi)\mathbf{u}_-(\xi; \sigma^2) = \chi_-(\xi)\mathbf{u}'_{\text{wt}}(\xi)[\bar{s}_p^{\text{wt}}(\sigma^2)\mathbf{g}_-](\xi) + \chi_-(\xi)[\bar{s}_c^{\text{wt}}(\sigma^2)\mathbf{g}_-](\xi)_- + \chi_-(\xi)[\bar{s}_e^{\text{wt}}(\sigma^2)\mathbf{g}_-](\xi),$$

where $\bar{s}_p^{\text{wt}}$, $\bar{s}_c^{\text{wt}}$ and $\bar{s}_e^{\text{wt}}$ are defined in (6.8), (6.7) and (6.6), respectively. Since we are interested in the stability of the pattern-forming front $\mathbf{u}_{\text{fr}}$, it is more natural to identify the leading-order dynamics with $\mathbf{u}'_{\text{fr}}$ rather than $\mathbf{u}'_{\text{wt}}$. We can make this replacement up to a manageable error, writing

$$\chi_-(\xi)\mathbf{u}'_{\text{wt}}(\xi)[\bar{s}_p^{\text{wt}}(\sigma^2)\mathbf{g}_-](\xi) = \chi_-(\xi)\mathbf{u}'_{\text{fr}}(\xi)[\bar{s}_p^{\text{wt}}(\sigma^2)\mathbf{g}_-](\xi) + \chi_-(\xi)(\mathbf{u}'_{\text{wt}}(\xi) - \mathbf{u}'_{\text{fr}}(\xi))[\bar{s}_p^{\text{wt}}(\sigma^2)\mathbf{g}_-](\xi).$$

Since $\chi_-(\mathbf{u}'_{\text{wt}} - \mathbf{u}'_{\text{fr}})$ is uniformly exponentially localized in space, contributions from this term can be seen to decay much faster in time.

Similarly, using that $\mathbf{e}_-(\xi, \sigma^2) = \mathbf{q}(\xi, \sigma^2)e^{\nu_{\text{wt}}(\sigma^2)\xi}$, and that $\mathbf{q}(\xi, \sigma^2)$ is analytic in σ^2 in $L^\infty(\mathbb{R})$, with $\mathbf{q}(\cdot, 0) = \mathbf{u}'_{\text{wt}}$, we have

$$\begin{aligned} \alpha_-(\sigma)\chi_-(\xi)\mathbf{e}_-(\xi; \sigma^2) &= \alpha_-(\sigma)\chi_-(\xi)\mathbf{u}'_{\text{fr}}(\xi)e^{\nu_{\text{wt}}(\sigma^2)\xi} + \alpha_-(\sigma)\chi_-(\xi)e^{\nu_{\text{wt}}(\sigma^2)\xi}[\mathbf{u}'_{\text{fr}}(\xi) - \mathbf{u}'_{\text{wt}}(\xi)] \\ &\quad + \alpha_-(\sigma)\chi_-(\xi)e^{\nu_{\text{wt}}(\sigma^2)\xi}[\mathbf{u}'_{\text{wt}}(\xi) - \mathbf{q}(\xi; \sigma^2)]. \end{aligned}$$

The first term contributes to the leading-order time dynamics, while the second term is faster decaying since $\chi_-[\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}}]$ is uniformly exponentially localized. Since $\mathbf{q}(\cdot; \sigma^2) = \mathbf{u}'_{\text{wt}} + O(|\sigma|^2) \in L^\infty(\mathbb{R})$ the last term is also faster decaying in time.

In summary, we have obtained the resolvent decomposition

$$\mathbf{u}(\cdot; \sigma) = (\mathcal{L}_{\text{fr}} - \sigma^2)^{-1}\mathbf{g} = \mathbf{u}'_{\text{fr}}[\bar{s}_p^{\text{fr}}(\sigma)\mathbf{g}] + \bar{s}_c(\sigma)\mathbf{g} + \bar{s}_e(\sigma)\mathbf{g}, \quad (6.26)$$

where

$$\bar{s}_p^{\text{fr}}(\sigma)\mathbf{g} = \chi_- \bar{s}_p^{\text{wt}}(\sigma^2)\mathbf{g}_- + \chi_- \alpha_-(\sigma)e^{\nu_{\text{wt}}(\sigma^2)\cdot}, \quad (6.27)$$

$$\bar{s}_c(\sigma)\mathbf{g} = R_{\text{fr}}(\cdot; \sigma) + \alpha_-(\sigma)\chi_- e^{\nu_{\text{wt}}(\sigma^2)\cdot} \left([\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}}] + [\mathbf{u}'_{\text{wt}} - \mathbf{q}(\cdot; \sigma^2)] \right) + \chi_- \bar{s}_c^{\text{wt}}(\sigma^2)\mathbf{g}_- \quad (6.28)$$

$$+ \chi_- (\mathbf{u}'_{\text{wt}} - \mathbf{u}'_{\text{fr}}) \bar{s}_p^{\text{wt}}(\sigma^2)\mathbf{g}_-,$$

$$\bar{s}_e(\sigma)\mathbf{g} = \chi_- \bar{s}_e^{\text{wt}}(\sigma^2)\mathbf{g}.$$

In the next section we will extract temporal decay by combining the abstract linear estimates in Section 5 with bounds on $\bar{s}_p^{\text{fr}}(\sigma)$, $\bar{s}_c(\sigma)$ and $\bar{s}_e(\sigma)$. We recall that control on $\bar{s}_e(\sigma)$ follows from Lemma 6.2. Moreover, control on $\bar{s}_c(\sigma)$ is provided by the estimates on $R_{\text{fr}}(\cdot; \sigma)$ in Lemmas 6.14, 6.15, and 6.16 in combination with the control on $\nu_{\text{wt}}(\sigma^2)$, $Q(\cdot; \sigma^2)$, $P_{\text{wt}}^{\text{cu}}(\sigma^2)$ and $\alpha_-(\sigma)$, obtained in Propositions 3.1 and 6.13 and Lemmas 6.1 and 6.9, using the explicit expressions (6.7) and (6.8) of $\bar{s}_c^{\text{wt}}(\sigma)$ and $\bar{s}_p^{\text{wt}}(\sigma)$. Finally, using again the explicit expressions (6.8) of $\bar{s}_p^{\text{wt}}(\sigma)$, control on $\bar{s}_p^{\text{fr}}(\sigma)$ follows by control on $\nu_{\text{wt}}(\sigma^2)$ in combination with the following estimate on the interaction term $\chi_- \alpha_-(\sigma)e^{\nu_{\text{wt}}(\sigma^2)\cdot}$.

Lemma 6.17 (Estimate on interaction term). *Fix $1 \leq p \leq \infty$. There exist constants $C, \delta > 0$ such that we have the estimates*

$$|\alpha_-(\sigma) - \alpha_-(0)| \leq C|\sigma|\|\mathbf{g}\|_{L_{0,1}^1}, \quad |\alpha_-(0)| \leq C\|\mathbf{g}\|_{L_{0,1}^1}, \quad (6.29)$$

for all $\sigma \in \Delta_\delta^{\text{fr},1}$ and $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$, and

$$\|[\alpha_-(\sigma) - \alpha_-(0)]\chi_{-e^{\nu_{\text{wt}}(\sigma^2)}}\|_{L^p} \leq C|\sigma|^{1-\frac{4}{p}}\|\mathbf{g}\|_{L_{0,1}^1}, \quad (6.30)$$

for all $\sigma^2 \in \Omega_\delta^{\text{wt}}$ and $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$.

Proof. The first estimate (6.29) follows readily from Lemma 6.9 and Proposition 6.13. Noting that $\Omega_\delta^{\text{wt}} \subset \Omega_\delta^{\text{fr},1}$ (see Figure 4), we observe that estimate (6.29) holds for all $\sigma^2 \in \Omega_\delta^{\text{wt}}$ and $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$.

For proving the second estimate, we fix $d > 0$ and define the curve

$$\Gamma_{\text{par}}^\varepsilon = B(0, \delta) \cap \{\lambda_\varepsilon(b) : b \in \mathbb{R}\}, \quad \lambda_\varepsilon(b) := \varepsilon + ib - db^2,$$

for each $\varepsilon > 0$. By estimate (6.29) and a basic scaling argument, we obtain

$$\|[\alpha_-(\sigma) - \alpha_-(0)]\chi_{-e^{\nu_{\text{wt}}(\sigma^2)}}\|_{L^p} \lesssim \frac{|\sigma|}{|\text{Re } \nu_{\text{wt}}(\sigma^2)|^{1/p}}\|\mathbf{g}\|_{L_{0,1}^1}, \quad (6.31)$$

for all $g \in L_{0,1}^1(\mathbb{R}, \mathbb{C}^2)$ and $\sigma^2 \in \Omega_\delta^{\text{wt}}$. Using the expansion (6.4), we see that along $\Gamma_{\text{par}}^\varepsilon$, we have

$$\text{Re } \nu_{\text{wt}}(\lambda_\varepsilon(b)) = \nu_1\varepsilon - \nu_2\varepsilon^2 + b^2(-\nu_1d + \nu_2) + O((\varepsilon + b)^3).$$

From this expansion, we observe that, upon taking $d > 0$ smaller if necessary, there are constants $k_1, k_2 > 0$, uniform in $\varepsilon > 0$ sufficiently small, such that

$$\text{Re } \nu_{\text{wt}}(\lambda_\varepsilon(b)) \geq k_1\varepsilon + k_2b^2,$$

holds along $\Gamma_{\text{par}}^\varepsilon$. Note that along $\Gamma_{\text{par}}^\varepsilon$, we have $|\sigma|^2 \sim \sqrt{\varepsilon^2 + b^2}$, and so we have

$$\text{Re } \nu_{\text{wt}}(\sigma^2) \geq k|\sigma|^4,$$

for $\sigma^2 \in \Gamma_{\text{par}}^\varepsilon$, where $k > 0$ is some constant independent of ε and σ . Combining this estimate and (6.31), we obtain (6.30) for $\sigma^2 \in \Gamma_{\text{par}}^\varepsilon$ with constant $C > 0$ independent of ε and σ . If $\delta > 0$ is small, we can write $\Omega_\delta^{\text{wt}}$ as the union of all such $\Gamma_{\text{par}}^\varepsilon$, parameterized over $\varepsilon > 0$ sufficiently small, and so the result follows. $\square$

7 Linear estimates

In this section, we will apply the general linear estimates of Section 5 to the resolvent description of Section 6 to establish a decomposition of the semigroup $e^{\mathcal{L}_{\text{fr}}t}$ and corresponding estimates, which are suitable for proving sharp nonlinear stability results. The outcome of our linear analysis may be summarized as follows.

Theorem 7.1 (Linear estimates). *The semigroup $e^{\mathcal{L}_{\text{fr}}t}$, generated by $\mathcal{L}_{\text{fr}}$, may be extended to an operator on $L^p(\mathbb{R}, \mathbb{R}^2)$ for $1 \leq p < \infty$ or on $C_0(\mathbb{R}, \mathbb{R}^2)$. Moreover, $e^{\mathcal{L}_{\text{fr}}t}$ admits a decomposition*

$$[e^{\mathcal{L}_{\text{fr}}t}\mathbf{g}](\xi) = \mathbf{u}'_{\text{fr}}(\xi)[s_p(t)\mathbf{g}](\xi) + [S_c(t)\mathbf{g}](\xi) + [S_e(t)\mathbf{g}](\xi), \quad \xi \in \mathbb{R}, t > 0, \quad (7.1)$$

with $s_p(0) = 0$, and there exist constants $C, \mu > 0$ such that the following estimates hold.

1. **Estimates on exponentially damped part.** For all $t > 0$ we have

$$\|S_e(t)\|_{L^2 \rightarrow L^2} \leq Ce^{-\mu t}, \quad \|S_e(t)\|_{C_0 \rightarrow C_0} \leq Ce^{-\mu t}.$$

2. **Estimates on smoothing principal part.** For each pair of non-negative integers m and ℓ , there exists a constant $C_{m,\ell} > 0$ such that

$$\begin{aligned}\|\partial_\xi^\ell \partial_t^m s_p(t) \mathbf{g}\|_{L^2} &\leq \frac{C_{m,\ell}}{(1+t)^{(\frac{1}{4} + \frac{m+\ell}{2}) \wedge \frac{3}{2}}} \|\mathbf{g}\|_{L_{0,1}^1}, \\ \|\partial_\xi^\ell \partial_t^m s_p(t) \mathbf{g}\|_{L^\infty} &\leq \frac{C_{m,\ell}}{(1+t)^{(\frac{1}{2} + \frac{m+\ell}{2}) \wedge \frac{3}{2}}} \|\mathbf{g}\|_{L_{0,1}^1},\end{aligned}\tag{7.2}$$

for all $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$, where $a \wedge b = \min\{a, b\}$.

3. **Estimates on residual part.** We have

$$\|S_c(t) \mathbf{g}\|_{L^2} \leq \frac{C}{(1+t)^{\frac{3}{4}}} \left(\|\mathbf{g}\|_{L_{0,1}^1} + \|\mathbf{g}\|_{L^\infty} \right), \quad \|S_c(t) \mathbf{g}\|_{L^\infty} \leq \frac{C}{1+t} \left(\|\mathbf{g}\|_{L_{0,1}^1} + \|\mathbf{g}\|_{L^\infty} \right),$$

for all $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap C_0(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$.

Moreover, we have the improved decay estimate in the leading edge,

$$\|\chi_+ e^{\mathcal{L}_{\text{fr}} t} \mathbf{g}\|_{L_{0,-1}^\infty} \leq \frac{C}{(1+t)^{\frac{3}{2}}} \left(\|\mathbf{g}\|_{L_{0,1}^1} + \|\mathbf{g}\|_{L^\infty} \right),\tag{7.3}$$

for all $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap C_0(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$.

Fix $\delta > 0$ small enough so that the decomposition (6.26) of the resolvent holds in the ball $B(0, \delta)$, which is depicted in gray in the right panel of Figure 3. By Corollary 4.4, we may write the action of the semigroup $e^{\mathcal{L}_{\text{fr}} t}$ on a test function $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ via the inverse Laplace representation as

$$e^{\mathcal{L}_{\text{fr}} t} \mathbf{g} = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^2} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda,\tag{7.4}$$

where the contour Γ_R^2 is depicted in the right panel of Figure 3. To prove Theorem 7.1, we will deform the portion $\Gamma_0^2 = \Gamma_R^2 \cap B(0, \delta)$ of the contour Γ_R^2 near the origin in several different ways depending on the behavior of the different terms arising in the decomposition (6.26) of the resolvent so that we can subsequently apply the abstract linear estimates obtained in Section 5.

First, we identify the various terms in the decomposition (7.1). Recall by Corollary 4.4 that we can write the semigroup $e^{\mathcal{L}_{\text{fr}} t}$ via the inverse Laplace transform as

$$e^{\mathcal{L}_{\text{fr}} t} \mathbf{g} = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,-} \cup \Gamma_{\text{int}}^{2,-} \cup \Gamma_0^{2,+} \cup \Gamma_{\text{int}}^{1,+} \cup \Gamma_R^{1,+}} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda,$$

for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$. The segments $\Gamma_R^{1,\pm}$ and $\Gamma_{\text{int}}^{2,\pm}$ of the contour are contained strictly in the left half-plane, and so contribute to the exponentially decaying part of the semigroup, see Corollary 4.4. By Lemma 6.2 the term $\bar{s}_e^{\text{wt}}(\lambda)$ in the decomposition (6.26) of the resolvent is analytic in λ in $L^p(\mathbb{R})$ on the full ball $B(0, \delta)$. As a consequence, the integral over Γ_0^2 of this term is also exponentially decaying in time as the contour Γ_0^2 can be deformed within $B(0, \delta)$ so that it lies in the open left half-plane. Thus, collecting these exponentially decaying terms, we define

$$\begin{aligned}S_e(t) \mathbf{g} &= -\frac{\varsigma(t)}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,-} \cup \Gamma_{\text{int}}^{2,-} \cup \Gamma_{\text{int}}^{2,+} \cup \Gamma_R^{1,+}} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda - \frac{\varsigma(t)}{2\pi i} \int_{\Gamma_0^2} e^{\lambda t} \chi_- \bar{s}_e^{\text{wt}}(\lambda) \mathbf{g} d\lambda \\ &\quad + (1 - \varsigma(t)) e^{\mathcal{L}_{\text{fr}} t} \mathbf{g},\end{aligned}\tag{7.5}$$

where $\varsigma: [0, \infty) \rightarrow \mathbb{R}$ satisfies $\varsigma(t) = 0$ for $t \in [0, 1]$ and $\varsigma(t) = 1$ for $t \in [2, \infty)$.³ On the other hand, we collected all terms, contributing to the leading-order temporal dynamics, in the term $\bar{s}_p^{\text{fr}}(\sigma)$ in the decomposition (6.26) of the resolvent. Thus, we define

$$s_p(t)\mathbf{g} = -\frac{\varsigma(t)}{2\pi i} \int_{\Gamma_0^2} e^{\sigma^2 t} \bar{s}_p^{\text{fr}}(\sigma) \mathbf{g} d(\sigma^2), \quad (7.6)$$

for the principal part, which leaves only the algebraically decaying residual part

$$S_c(t)\mathbf{g} = -\frac{\varsigma(t)}{2\pi i} \int_{\Gamma_0^2} e^{\sigma^2 t} \bar{s}_c(\sigma) \mathbf{g} d(\sigma^2). \quad (7.7)$$

Recall that expressions for $\bar{s}_e^{\text{wt}}(\lambda)$, $\bar{s}_p^{\text{fr}}(\sigma)$, and $\bar{s}_c(\sigma)$ are given in (6.6), (6.8), and (6.7), respectively.

Having identified the terms in the decomposition (7.1), we now estimate them one by one by choosing appropriate integration contours based on the analysis in Section 5, thereby proving Theorem 7.1.

Lemma 7.2 (Estimates on exponentially damped part). *The operator $S_e(t)$, defined by (7.5), extends to a bounded linear operator on $L^2(\mathbb{R}, \mathbb{R}^2)$ and on $C_0(\mathbb{R}, \mathbb{R}^2)$ for each fixed $t > 0$. Moreover, there exist constants $C, \mu > 0$ such that for all $t > 0$ we have*

$$\|S_e(t)\|_{L^2 \rightarrow L^2} \leq C e^{-\mu t}, \quad \|S_e(t)\|_{C_0 \rightarrow C_0} \leq C e^{-\mu t}.$$

Proof. The estimates on the first two terms in (7.5) follow for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ from Corollary 4.4 together with analyticity in λ of $\bar{s}_e^{\text{wt}}(\lambda)$ on the full ball $B(0, \delta)$ obtained in Lemma 6.2. From the proof of Proposition 4.1, it is clear that $e^{\mathcal{L}_{\text{fr}} t}$ also generates a strongly continuous semigroup on $L^2(\mathbb{R}, \mathbb{R}^2)$ or on $C_0(\mathbb{R}, \mathbb{R}^2)$. An exponential decay estimate on the last term in (7.5) follows from these facts together with compact support in time of $1 - \varsigma(t)$. With these estimates in hand, we can then naturally extend $S_e(t)$ to the larger spaces $L^2(\mathbb{R}, \mathbb{R}^2)$ and $C_0(\mathbb{R}, \mathbb{R}^2)$ by approximating with test functions. $\square$

Lemma 7.3 (Estimates on smoothing principal part). *Let $s_p(t)$ be defined by (7.6). For each pair of non-negative integers ℓ and j , the operator $\partial_\xi^\ell \partial_t^j s_p(t)$ extends to a bounded linear operator from $L_{0,1}^1(\mathbb{R}, \mathbb{R}^2)$ into $L^2(\mathbb{R}, \mathbb{R}^2)$ or from $L_{0,1}^1(\mathbb{R}, \mathbb{R}^2)$ into $C_0(\mathbb{R}, \mathbb{R}^2)$. Moreover, there exists a constant $C_{j,\ell} > 0$ such that*

$$\|\partial_\xi^\ell \partial_t^j s_p(t)\mathbf{g}\|_{L^2} \leq \frac{C_{j,\ell}}{(1+t)^{\frac{1}{4} + \frac{j+\ell}{2} \wedge \frac{3}{2}}} \|\mathbf{g}\|_{L_{0,1}^1}, \quad \|\partial_\xi^\ell \partial_t^j s_p(t)\mathbf{g}\|_{L^\infty} \leq \frac{C_{j,\ell}}{(1+t)^{\frac{1}{2} + \frac{j+\ell}{2} \wedge \frac{3}{2}}} \|\mathbf{g}\|_{L_{0,1}^1}, \quad (7.8)$$

for all $t > 0$ and $\mathbf{g} \in L_{0,1}^1(\mathbb{R})$.

Proof. By (6.27), the action of $s_p(t)$ on a test function $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ decomposes as

$$s_p(t)\mathbf{g} = -\frac{\varsigma(t)}{2\pi i} \int_{\Gamma_0^2} e^{\lambda t} \chi_{-\bar{s}_p^{\text{wt}}(\lambda)} \mathbf{g}_- d\lambda - \frac{\varsigma(t)}{2\pi i} \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_{-\alpha_-}(\sigma) e^{\nu_{\text{wt}}(\sigma^2)} d(\sigma^2). \quad (7.9)$$

We start estimating the first integral. With the aid of (6.8) we rewrite this integral as

$$\int_{\Gamma_0^2} e^{\lambda t} [\chi_{-\bar{s}_p^{\text{wt}}(\lambda)} \mathbf{g}_-](\xi) d\lambda = \varphi \left(\int_{\Gamma_0^2} e^{\lambda t} \chi_{-}(\xi) \int_{\mathbb{R}} e^{\nu_{\text{wt}}(\lambda)(\xi-\zeta)} \chi_{-}(\xi-\zeta) P_{\text{wt}}^{\text{cu}}(0) Q(\zeta, 0) \Lambda_1 \mathbf{g}_-(\zeta) d\zeta d\lambda \right).$$

³The reason for introducing the temporal cut-off function is to assure that the critical diffusively decaying part of the semigroup vanishes at $t = 0$, that is, we have $s_p(0) = 0$. This has the advantage that temporal derivatives of the phase modulation function simplify in the nonlinear stability argument.

Using that $\chi_-(\xi - \zeta)e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)}$ is uniformly exponentially localized for λ in the compact contour Γ_0^2 , we swap the order of integration to obtain

$$\int_{\Gamma_0^2} e^{\lambda t} [\chi_- \bar{s}_p^{\text{wt}}(\lambda) \mathbf{g}_-](\xi) d\lambda = \varphi \left(\chi_-(\xi) \int_{\mathbb{R}} P_{\text{wt}}^{\text{cu}}(0) Q(\zeta, 0) \Lambda_1 \mathbf{g}_-(\zeta) \int_{\Gamma_0^2} e^{\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta)} \chi_-(\xi - \zeta) d\lambda d\zeta \right).$$

Since the map $\lambda \mapsto e^{\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta)}$ is analytic in λ on the full ball $B(0, \delta)$ for each fixed $\xi, \zeta \in \mathbb{R}$ and $t \geq 0$ by Proposition 3.1, we can deform the integration contour Γ_0^2 to the pointwise contour $\Gamma_{\xi, \zeta, t}$ obtained in Proposition 5.4. Then, using Proposition 5.4 to estimate the resulting integral and applying Young's convolution inequality, we obtain

$$\left\| \int_{\Gamma_0^2} e^{\lambda t} \chi_- \bar{s}_p^{\text{wt}}(\lambda) \mathbf{g}_- d\lambda \right\|_{L^2} \lesssim \|G^{0,0,0}(t, \cdot)\|_{L^2} \|Q(\cdot, 0) \mathbf{g}_-\|_{L^1} \lesssim \frac{\|\mathbf{g}\|_{L^1}}{(1+t)^{\frac{1}{4}}} \leq \frac{\|\mathbf{g}\|_{L_{0,1}^1}}{(1+t)^{\frac{1}{4}}},$$

and

$$\left\| \int_{\Gamma_0^2} e^{\lambda t} \chi_- \bar{s}_p^{\text{wt}}(\lambda) \mathbf{g}_- d\lambda \right\|_{L^\infty} \lesssim \|G^{0,0,0}(t, \cdot)\|_{L^\infty} \|Q(\cdot, 0) \mathbf{g}_-\|_{L^1} \lesssim \frac{\|\mathbf{g}\|_{L^1}}{(1+t)^{\frac{1}{2}}} \leq \frac{\|\mathbf{g}\|_{L_{0,1}^1}}{(1+t)^{\frac{1}{2}}},$$

for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$, which correspond to the estimates (7.8) for $j = \ell = 0$. The estimate for $j > 0$ and $\ell = 0$ follows in the same way: differentiating j times the factor $e^{\lambda t}$ produces a factor of λ^j , and we then can apply Proposition 5.4 with $j > 0$ to obtain the improved decay. Considering $\ell > 0$, the ξ -derivatives may fall on either $\chi_-(\xi)$, $e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)}$ or $\chi_-(\xi - \zeta)$. If all ℓ derivatives fall on $e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)}$, then the desired estimate follows directly by applying Proposition 5.4 with $\ell > 0$. If any derivatives fall on $\chi_-(\xi - \zeta)$, then we use Proposition 3.1 and the fact that χ'_- is compactly supported to conclude that $\lambda \mapsto e^{\nu_{\text{wt}}(\lambda)} \chi'_-$ is analytic in λ on $B(0, \delta)$ in $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$, so that the contour integral can be shifted into the left half-plane, implying exponential decay in time. Similarly, if any derivatives fall on $\chi_-(\xi)$, then note that the map $(\xi, \zeta) \mapsto \chi'_-(\xi) \mathbf{g}_-(\zeta) \chi_-(\xi - \zeta) = \chi'_-(\xi) \chi_-(\zeta) \chi_-(\xi - \zeta) \mathbf{g}(\zeta)$ is supported on the compact set $\{1 \leq \xi \leq \zeta \leq 0\}$, so that we do not have to worry about loss of spatial localization of $e^{\nu_{\text{wt}}(\lambda)(\xi - \zeta)}$. Hence, the resulting integral is again analytic in λ on $B(0, \delta)$ in $L^p(\mathbb{R})$ in this case and we gain exponential decay in time. This completes the estimates for the first term in (7.9) for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

Next, we analyze the second term in (7.9), which encodes the interaction between branched and outgoing modes. We start by further decomposing this integral as

$$\begin{aligned} \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_- \alpha_-(\sigma) e^{\nu_{\text{wt}}(\sigma^2)} d(\sigma^2) &= \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_- \alpha_-(0) e^{\nu_{\text{wt}}(\sigma^2)} d(\sigma^2) \\ &\quad + \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_- [\alpha_-(\sigma) - \alpha_-(0)] e^{\nu_{\text{wt}}(\sigma^2)} d(\sigma^2). \end{aligned} \quad (7.10)$$

The estimates (7.8) for the first integral in (7.10) with $j = \ell = 0$ follow by applying the pointwise estimates in Proposition 5.4 with $\zeta = 0$ fixed. Using Lemma 6.17 to estimate $|\alpha_-(0)|$, this leads to

$$\left\| \int_{\Gamma_0^2} e^{\lambda t} \chi_- \alpha_-(0) e^{\nu_{\text{wt}}(\lambda)} d\lambda \right\|_{L^p} \lesssim \|G^{0,0,0}(t, \cdot)\|_{L^p} |\alpha_-(0)| \lesssim \frac{\|\mathbf{g}\|_{L_{0,1}^1}}{(1+t)^{\frac{1}{2} - \frac{1}{2p}}},$$

for $p \in \{2, \infty\}$, $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$. This can be extended to positive j and ℓ by handling ξ - and t -derivatives exactly as for the first integral in (7.9).

We cannot use Proposition 5.4 for the second term in (7.10), since $\alpha_-(\sigma) - \alpha_-(0)$ is not analytic in σ^2 in a full neighborhood of the origin. Instead, by Lemma 6.17, we have

$$\|\chi_- [\alpha_-(\sigma) - \alpha_-(0)] e^{\nu_{\text{wt}}(\sigma^2)}\|_{L^p} \lesssim |\sigma|^{1 - \frac{4}{p}} \|\mathbf{g}\|_{L_{0,1}^1} = |\sigma|^{2(\frac{1}{2} - \frac{2}{p})} \|\mathbf{g}\|_{L_{0,1}^1}, \quad (7.11)$$

for $p = 2, \infty$, $\sigma^2 \in \Omega_\delta^{\text{wt}}$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$. Since $\frac{1}{2} - \frac{2}{p} > -1$ for $p > \frac{4}{3}$, we can apply Proposition 5.3 to obtain

$$\left\| \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_- [\alpha_-(\sigma) - \alpha_-(0)] e^{\nu_{\text{wt}}(\sigma^2) \cdot} d(\sigma^2) \right\|_{L^p} \leq \frac{\|\mathbf{g}\|_{L_{0,1}^1}}{(1+t)^{\frac{3}{4} - \frac{1}{p}}}, \quad (7.12)$$

for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$, $t > 0$ and $p = 2, \infty$, which leads to the estimates (7.8) for $\ell = j = 0$. The same argument applies to obtain faster decay rates for $j > 0$, since differentiating $e^{\sigma^2 t}$ with respect to t produces additional factors of $|\sigma|^2$, which contribute to faster decay by Proposition 5.3. When $\ell > 0$, if all derivatives fall on $e^{\nu_{\text{wt}}(\sigma^2) \cdot}$, then the same argument applies by Proposition 3.1. However, if some derivatives fall on χ_- , then, since χ'_- is compactly supported, the function $\partial_x^k \chi_- \partial_x^{\ell-k} e^{\nu_{\text{wt}}(\sigma^2) \cdot}$ is analytic in σ^2 on $B(0, \delta)$ in $L^p(\mathbb{R})$ provided $k \geq 1$. Hence, if we define

$$u(\sigma) = [\alpha_-(\sigma) - \alpha_-(0)] \partial_\xi^k \chi_- \partial_\xi^{\ell-k} e^{\nu_{\text{wt}}(\sigma^2) \cdot},$$

then this analyticity together with the regularity of $\alpha_-(\sigma)$ from Lemma 6.9 and Proposition 6.13 implies that $u(\sigma)$ is analytic in $L^p(\mathbb{R})$ in σ^2 on $\Omega_\delta^{\text{fr},1}$. It enjoys the estimate $\|u(\sigma)\|_{L^p} \lesssim |\sigma| \|g\|_{L_{0,1}^1}$ for $\sigma \in \Delta_\delta^{\text{fr},1}$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ by Lemma 6.17. Applying Proposition 5.1, we see that this term decays with rate $(1+t)^{-3/2}$.

Altogether, after using approximation with test functions, we have established (7.8) for any pair of non-negative integers (ℓ, j) , as desired. $\square$

Lemma 7.4 (Estimates on smoothing residual part). *Let $S_c(t)$ be defined by (7.7). The operator $S_c(t)$ extends to a bounded linear operator from $L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap C_0(\mathbb{R}, \mathbb{R}^2)$ into $L^2(\mathbb{R}, \mathbb{R}^2)$ or from $L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap C_0(\mathbb{R}, \mathbb{R}^2)$ into $C_0(\mathbb{R}, \mathbb{R}^2)$. Moreover, there exists a constant $C > 0$ such that*

$$\|S_c(t)\mathbf{g}\|_{L^2} \leq \frac{C}{(1+t)^{\frac{3}{4}}} \|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty}, \quad \|S_c(t)\mathbf{g}\|_{L^\infty} \leq \frac{C}{1+t} \|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty},$$

for all $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap C_0(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$.

Proof. Expanding $\bar{s}_c(\sigma)\mathbf{g}$ for a test function $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ via the formula (6.28), we find

$$\begin{aligned} S_c(t)\mathbf{g} = & -\frac{\varsigma(t)}{2\pi i} \left(\int_{\Gamma_0^2} e^{\sigma^2 t} R^{\text{fr}}(\sigma) d(\sigma^2) + \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_- \bar{s}_c^{\text{wt}}(\sigma^2) \mathbf{g}_- d(\sigma^2) \right. \\ & + \int_{\Gamma_0^2} e^{\sigma^2 t} (\alpha_-(\sigma) - \alpha_-(0)) e^{\nu_{\text{wt}}(\sigma^2) \cdot} \chi_- (\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}}) d(\sigma^2) \\ & + \int_{\Gamma_0^2} e^{\sigma^2 t} \chi_- (\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}}) (\alpha_-(0) + \bar{s}_p^{\text{wt}}(\sigma^2) \mathbf{g}_-) d(\sigma^2) \\ & \left. + \int_{\Gamma_0^2} e^{\sigma^2 t} \alpha_-(\sigma) \chi_- e^{\nu_{\text{wt}}(\sigma^2) \cdot} (\mathbf{u}'_{\text{wt}} - \mathbf{q}(\cdot; \sigma^2)) d(\sigma^2) \right). \end{aligned} \quad (7.13)$$

For the first integral, applying Lemma 6.15 and Proposition 5.2, we obtain the estimate

$$\left\| \int_{\Gamma_0^2} e^{\sigma^2 t} R^{\text{fr}}(\sigma) d(\sigma^2) \right\|_{L^2} \lesssim \frac{\|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty}}{(1+t)^{\frac{3}{4}}},$$

for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$. Similarly, using Lemma 6.16 instead of 6.15, we obtain

$$\left\| \int_{\Gamma_0^2} e^{\sigma^2 t} R^{\text{fr}}(\sigma) d(\sigma^2) \right\|_{L^\infty} \leq \frac{\|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty}}{1+t},$$

for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$.

The desired estimates on the second integral in (7.13) follow exactly as those on the first integral in (7.9) in the proof of Lemma 7.3, noting that each term in the formula (6.7) for $\bar{s}_c^{\text{wt}}(\sigma)$ carries by Lemma 6.1 an extra factor of λ compared to $\bar{s}_p^{\text{wt}}(\lambda)$, which improves the temporal decay rate by a factor $(1+t)^{-1/2}$.

For the third integral in (7.13), note that $\chi_- e^{\nu_{\text{wt}}(\sigma^2) \cdot} (\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}})$ is analytic in σ^2 in a full neighborhood of the origin in $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$ using Proposition 3.1 and the fact that $\chi_- (\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}})$ is exponentially localized. Hence, the worst behavior in this term is in $\alpha_-(\sigma) - \alpha_-(0)$, which can be bounded with the aid of Lemma 6.17. Altogether, we can use Proposition 5.1 to obtain

$$\left\| \int_{\Gamma_0^2} e^{\sigma^2 t} (\alpha_-(\sigma) - \alpha_-(0)) \chi_- e^{\nu_{\text{wt}}(\sigma^2) \cdot} (\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}}) d(\sigma^2) \right\|_X \lesssim \frac{\|\mathbf{g}\|_{L_{0,1}^1}}{(1+t)^{\frac{3}{2}}},$$

for $g \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$, where $X = L^2(\mathbb{R})$ or $X = L^\infty(\mathbb{R})$.

For the fourth integral in (7.13), we observe that the integrand is analytic in σ^2 in a full neighborhood of the origin in $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$, where we use the expression (6.8) and the facts that $\nu_{\text{wt}}(\lambda)$ is analytic in λ by Proposition 3.1 and $\chi_- (\mathbf{u}'_{\text{fr}} - \mathbf{u}'_{\text{wt}})$ is exponentially localized. Thus, we gain exponential decay in time in $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$ of the fourth integral.

The last integral in (7.13) can be estimated exactly as the second term in (7.9) in the proof of Lemma 7.3. That is, we further decompose

$$\begin{aligned} \int_{\Gamma_0^2} e^{\sigma^2 t} \alpha_-(\sigma) \chi_- e^{\nu_{\text{wt}}(\sigma^2) \cdot} (\mathbf{u}'_{\text{wt}} - \mathbf{q}(\cdot; \sigma^2)) d(\sigma^2) &= \int_{\Gamma_0^2} e^{\sigma^2 t} \alpha_-(0) \chi_- e^{\nu_{\text{wt}}(\sigma^2) \cdot} (\mathbf{u}'_{\text{wt}} - \mathbf{q}(\cdot; \sigma^2)) d(\sigma^2) \\ &\quad + \int_{\Gamma_0^2} e^{\sigma^2 t} [\alpha_-(\sigma) - \alpha_-(0)] \chi_- e^{\nu_{\text{wt}}(\sigma^2) \cdot} (\mathbf{u}'_{\text{wt}} - \mathbf{q}(\cdot; \sigma^2)) d(\sigma^2). \end{aligned}$$

Lemma 6.10 then yields

$$\|\mathbf{u}'_{\text{wt}} - \mathbf{q}(\cdot; \lambda)\|_{L^\infty} \lesssim |\lambda|, \quad (7.14)$$

for $|\lambda|$ sufficiently small. The desired estimates on the first term then follow by applying the pointwise estimates of Proposition 5.4, using Lemma 6.17 to bound $\alpha_-(0)$ and noting that we carry an extra factor of $|\lambda|$ by estimate (7.14). Similarly, the second term carries an extra factor of $|\sigma|^2$ compared to (7.12). Hence, we obtain the desired estimates by combining Proposition 5.3 with the estimates (7.11) and (7.14).

Altogether, the result now follows by approximation with test functions. $\square$

To complete the proof of Theorem 7.1, it only remains to establish the improved decay (7.3) in the leading edge.

Lemma 7.5 (Improved decay in the leading edge). *There exists a constant $C > 0$ such that for all $\mathbf{g} \in L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap C_0(\mathbb{R}, \mathbb{R}^2)$ and all $t > 0$, we have the estimate*

$$\|\chi_+ e^{\mathcal{L}_{\text{fr}} t} \mathbf{g}\|_{L_{0,-1}^\infty} \leq \frac{C}{(1+t)^{\frac{3}{2}}} \|\mathbf{g}\|_{L_{0,1}^1 \cap L^\infty}.$$

Proof. Using the inverse Laplace representation (7.4), we write

$$\chi_+ e^{\mathcal{L}_{\text{fr}} t} \mathbf{g} = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,-} \cup \Gamma_{\text{int}}^{2,-} \cup \Gamma_{\text{int}}^{2,+} \cup \Gamma_R^{1,+}} e^{\lambda t} \chi_+ (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda - \frac{1}{2\pi i} \int_{\Gamma_0^2} e^{\lambda t} \chi_+ (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda.$$

For the first term, we apply Corollary 4.4 to find a constant $r > 0$ such that

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,-} \cup \Gamma_{\text{int}}^{2,-} \cup \Gamma_{\text{int}}^{2,+} \cup \Gamma_R^{1,+}} e^{\lambda t} \chi_+(\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda \right\|_{L_{0,-1}^\infty} \lesssim e^{-rt} \|\mathbf{g}\|_{L^\infty}, \quad (7.15)$$

for all $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $t > 0$, where we use the continuous embedding $L_{0,-1}^\infty(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

For the integral over Γ_0^2 , we find from the resolvent decomposition (6.26) that, for $\lambda = \sigma^2$ in a neighborhood of the origin, we have

$$\chi_+(\mathcal{L}_{\text{fr}} - \sigma^2)^{-1} \mathbf{g} = \chi_+ R_{\text{fr}}(\cdot; \sigma),$$

which satisfies the estimate

$$\|\chi_+[R_{\text{fr}}(\cdot; \sigma) - R_{\text{fr}}(\cdot; 0)]\|_{L_{0,-1}^\infty} \lesssim |\sigma| \|\mathbf{g}\|_{L_{0,1}^1},$$

for all $\sigma \in \Delta_\delta^{\text{fr},1}$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ by Lemma 6.14. Using Proposition 5.1, we therefore obtain

$$\left\| \int_{\Gamma_0^2} e^{\lambda t} \chi_+(\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda \right\|_{L_{0,-1}^\infty} \lesssim \frac{\|\mathbf{g}\|_{L_{0,1}^1}}{(1+t)^{\frac{3}{2}}},$$

for $t > 0$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$. Combining this with (7.15) completes the proof of the lemma by approximation with test functions. $\square$

8 Nonlinear iteration scheme and nonlinear estimates

In this section we formulate a nonlinear iteration scheme and state associated nonlinear estimates. In the next section we then prove our nonlinear stability result by closing an iterative argument based on this scheme with the nonlinear estimates.

We are interested in the long-time dynamics of the perturbed solution $\mathbf{u}(t)$ of (1.5) with initial condition $\mathbf{u}(0) = \mathbf{u}_{\text{fr}} + \mathbf{w}_0$, where $\mathbf{v}_0 := \omega \mathbf{w}_0 \in L_{0,1}^1(\mathbb{R}, \mathbb{R}^2) \cap (Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R}))$ is sufficiently small. The initial perturbation $\mathbf{w}_0$ is L^1 -localized on $(-\infty, 0]$ and exponentially localized on $[0, \infty)$. Recall that the latter is necessary to marginally stabilize the essential spectrum associated with the leading edge of the front.

The perturbed solution can be written as $\mathbf{u}(t) = \mathbf{u}_{\text{fr}} + \mathbf{w}(t)$, where the perturbation $\mathbf{w}(t)$ has initial condition $\mathbf{w}(0) = \mathbf{w}_0$. Consequently, the weighted perturbation $\tilde{\mathbf{v}}(t) = \omega \mathbf{w}(t)$ with $\tilde{\mathbf{v}}(0) = \mathbf{v}_0$ satisfies

$$\tilde{\mathbf{v}}_t = \mathcal{L}_{\text{fr}} \tilde{\mathbf{v}} + \tilde{\mathcal{N}}(\tilde{\mathbf{v}}), \quad (8.1)$$

with continuously Fréchet-differentiable nonlinearity $\tilde{\mathcal{N}}: Z_1(\mathbb{R}, \mathbb{R}^2) \rightarrow Z_1(\mathbb{R}, \mathbb{R}^2)$ given by

$$\tilde{\mathcal{N}}(\tilde{\mathbf{v}}) = \omega \tilde{N}\left(\frac{\tilde{\mathbf{v}}}{\omega}\right), \quad \tilde{N}(\mathbf{w}) = F(\mathbf{u}_{\text{fr}} + \mathbf{w}) - F(\mathbf{u}_{\text{fr}}) - F'(\mathbf{u}_{\text{fr}}) \mathbf{w}.$$

As stated in Proposition 4.1, the linearization $\mathcal{L}_{\text{fr}}$ generates a C^0 -semigroup on $Z_1(\mathbb{R}, \mathbb{R}^2)$ with domain $Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R})$. Hence, the following local well-posedness result follows from classical semigroup theory, see e.g. [55, Theorem 6.1.5].

Proposition 8.1 (Local well-posedness of the unmodulated perturbation). *There exists a maximal time $T_{\text{max}} \in (0, \infty]$ such that (8.1) possesses a unique solution*

$$\tilde{\mathbf{v}} \in C([0, T_{\text{max}}], Z_3(\mathbb{R}, \mathbb{R}) \times Z_2(\mathbb{R}, \mathbb{R})) \cap C^1([0, T_{\text{max}}], Z_1(\mathbb{R}, \mathbb{R}^2)),$$

with initial condition $\tilde{\mathbf{v}}(0) = \mathbf{v}_0$. Moreover, if $T_{\text{max}} < \infty$, then we have

$$\lim_{t \nearrow T_{\text{max}}} \|\tilde{\mathbf{v}}(t)\|_{Z_1} = \infty.$$

First, we recall the discussion from Section 2.2 about the difficulties and overall strategy of the nonlinear iteration argument. The principal difficulty is that the overall linear decay rate $(1+t)^{-1/2}$ of Theorem 7.1 is too slow to directly close a nonlinear iteration scheme. To overcome this, we draw on ideas from the nonlinear stability of wave trains [26, 44, 45, 46, 66] and introduce the *inverse-modulated perturbation*

$$\mathbf{v}(\xi, t) = \omega(\xi) (\mathbf{u}(\xi - \psi(\xi, t), t) - \mathbf{u}_{\text{fr}}(\xi)). \quad (8.2)$$

We will choose the spatio-temporal phase modulation ψ to capture the leading-order behavior of the solution, crucially relying on the detailed semigroup decomposition of Theorem 7.1. Using this phase modulation, we will obtain improved decay for $\mathbf{v}$ which will ultimately allow us to close a nonlinear iteration.

We then, however, encounter the additional difficulty that the coupled system for $\mathbf{v}$ and ψ is quasilinear, and we need to overcome an apparent loss of regularity in our nonlinear iteration scheme. To control regularity, we will introduce the forward-modulated perturbation,

$$\hat{\mathbf{v}}(\xi, t) = \omega(\xi) (\mathbf{u}(\xi, t) - \mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t))). \quad (8.3)$$

By putting the phase modulation in $\mathbf{u}_{\text{fr}}$ rather than the solution $\mathbf{u}$ itself, we ensure that the equation for $\hat{\mathbf{v}}$ remains semilinear, so that it is easier to control regularity for $\hat{\mathbf{v}}$. We rely on ideas recently developed in [76] to show that decay rates for $\mathbf{v}$ and $\hat{\mathbf{v}}$ are equivalent, so that we may enjoy improved temporal decay of $\mathbf{v}$ while using nonlinear damping estimates on $\hat{\mathbf{v}}$ to control regularity, ultimately closing a nonlinear iteration argument to prove Theorem 2.1.

The rest of this section is structured as follows. First, we derive an equation for the inverse-modulated perturbation $\mathbf{v}(t)$ and establish estimates on the corresponding nonlinearity. We then choose the phase modulation function $\psi(t)$ in such a way that it accounts for supercritical terms in the Duhamel formula for $\mathbf{v}(t)$. Subsequently, we consider the equation for the forward-modulated perturbation and derive an associated nonlinear damping estimate, which yields regularity control for the inverse-modulated perturbation by bounding the relevant norms of $\mathbf{v}(t)$ in terms of those of $\hat{\mathbf{v}}(t)$ plus controllable error terms in $\psi_\xi(t)$.

8.1 The inverse-modulated perturbation

Using that both the perturbed solution $\mathbf{u}(t)$ and the pattern-forming front $\mathbf{u}_{\text{fr}}$ solve the FitzHugh-Nagumo system (1.5), we derive an equation for the inverse-modulated perturbation (8.2) in Appendix E, which reads

$$(\partial_t - \mathcal{L}_{\text{fr}}) [\mathbf{v} + \omega \mathbf{u}'_{\text{fr}} \psi] = \mathcal{N}(\mathbf{v}, \psi, \partial_t \psi) + (\partial_t - \mathcal{L}_{\text{fr}}) [\psi_\xi \mathbf{v}], \quad (8.4)$$

with nonlinearity $\mathcal{N}$ given by

$$\mathcal{N}(\mathbf{v}, \psi, \psi_t) = \omega \left(\mathcal{Q} \left(\frac{\mathbf{v}}{\omega}, \psi \right) + \partial_\xi \mathcal{R} \left(\frac{\mathbf{v}}{\omega}, \psi, \psi_t \right) \right),$$

where

$$\mathcal{Q}(\mathbf{z}, \psi) = (F(\mathbf{u}_{\text{fr}} + \mathbf{z}) - F(\mathbf{u}_{\text{fr}}) - F'(\mathbf{u}_{\text{fr}})\mathbf{z}) (1 - \psi_\xi), \quad (8.5)$$

is quadratic in $\mathbf{z}$ and

$$\mathcal{R}(\mathbf{z}, \psi) := \mathbf{z} (c\psi_\xi - \psi_t) + D \left(\frac{(\mathbf{z}_\xi + \mathbf{u}'_{\text{fr}} \psi_\xi) \psi_\xi}{1 - \psi_\xi} + (\mathbf{z} \psi_\xi)_\xi \right), \quad (8.6)$$

contains all terms which are linear in $\mathbf{z}$. We observe that equation (8.4) is quasilinear in $\mathbf{v}(t)$. Using the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the fact that $\omega(\xi)^{-1}$ is bounded and decays exponentially as $\xi \rightarrow \infty$, we establish the relevant nonlinear estimate.

Lemma 8.2 (Nonlinear estimates). *There exists a constant $C > 0$ such that the inequality*

$$\|\mathcal{N}(\mathbf{v}, \psi, \psi_t)\|_{L^1_{0,1} \cap Z_0} \leq C \left(\|v_1\|_{Z_0}^2 + \|\omega(\psi_\xi, \psi_t)\|_{Z_2 \times Z_1} \left(\|\mathbf{v}\|_{Z_2 \times Z_1} + \|\omega\psi_\xi\|_{Z_0} \right) \right),$$

holds for each $\mathbf{v} = (v_1, v_2) \in Z_2(\mathbb{R}) \times Z_1(\mathbb{R})$ and $(\psi, \psi_t) \in Z_3(\mathbb{R}) \times Z_1(\mathbb{R})$ satisfying $\|v_1\|_{L^\infty}, \|\psi_\xi\|_{W^{1,\infty}} \leq \frac{1}{2}$ and $\|\omega(\psi_\xi, \psi_t)\|_{Z_2 \times Z_1} < \infty$.

For the moment we assume that the phase modulation function $\psi(t)$ vanishes identically at $t = 0$. Then, the Duhamel formulation of (8.4) reads

$$\mathbf{v}(t) + \omega \mathbf{u}'_{\text{fr}} \psi(t) = e^{\mathcal{L}_{\text{fr}} t} \mathbf{v}_0 + \int_0^t e^{\mathcal{L}_{\text{fr}}(t-s)} \mathcal{N}(\mathbf{v}(s), \psi(s), \partial_t \psi(s)) ds + \psi_\xi(t) \mathbf{v}(t). \quad (8.7)$$

We make a choice for $\psi(t)$ such that the linear term $\mathbf{u}'_{\text{fr}} \psi(t)$ compensates for the critical nonlinear contributions on the right-hand side of equation (8.7). More precisely, motivated by the semigroup decomposition (7.1), we make the implicit choice

$$\psi(t) = s_p(t) \mathbf{v}_0 + \int_0^t s_p(t-s) \mathcal{N}(\mathbf{v}(s), \psi(s), \partial_t \psi(s)) ds, \quad (8.8)$$

which, upon substitution in (8.7), leads to the integral equation

$$\mathbf{v}(t) = (S_c(t) + S_e(t)) \mathbf{v}_0 + \int_0^t (S_c(t-s) + S_e(t-s)) \mathcal{N}(\mathbf{v}(s), \psi(s), \partial_t \psi(s)) ds + \psi_\xi(t) \mathbf{v}(t), \quad (8.9)$$

for the inverse-modulated perturbation, where we use that the identities (6.27) and (7.6) imply that $\psi(t)$ vanishes on $[-1, \infty)$. We note that, due to the introduction of the temporal cut-off function $\varsigma(t)$, we have $s_p(0) = 0$ and thus we have $\psi(0) = 0$. Moreover, we emphasize that, with this choice of $\psi(t)$, the inverse-modulated perturbation exhibits, at least on the linear level, higher-order algebraic pointwise decay at rate t^{-1} , see Theorem 7.1, whereas the phase-modulation $\psi(t)$ decays diffusively at rate $t^{-\frac{1}{2}}$. Nevertheless, the nonlinearity $\mathcal{N}$ in (8.8) and (8.9) only contains spatial and temporal derivatives of ψ , which satisfy

$$\partial_\xi^\ell \partial_t^j \psi(t) = \partial_\xi^\ell \partial_t^j s_p(t) \mathbf{v}_0 + \int_0^t \partial_\xi^\ell \partial_t^j s_p(t-s) \mathcal{N}(\mathbf{v}(s), \psi(s), \partial_t \psi(s)) ds, \quad (8.10)$$

for $\ell, j \in \mathbb{N}_0$, where we use that $s_p(0) = 0$. The estimates in Theorem 7.1 show that the linear terms in (8.10) also decay pointwise at rate t^{-1} for $\ell + j \geq 1$. These higher-order decay rates suggest that, as in the stability analyses [45, 46] of periodic wave trains, an iteration scheme consisting of $\mathbf{v}(t)$ and derivatives of $\psi(t)$ might close, which we indeed confirm in the next section. Note that we also need to address the loss of derivatives in the quasilinear integral scheme (8.9)-(8.10), an issue we tackle in the next section through nonlinear damping estimates on the forward-modulated perturbation (8.3).

We end this section by establishing local well-posedness of the phase modulation and the inverse-modulated perturbation. To that end, we note that we can express the inverse-modulated perturbation as

$$\mathbf{v}(\xi, t) = \tilde{\mathbf{v}}(\xi - \psi(\xi, t), t) + \omega(\xi) (\mathbf{u}_{\text{fr}}(\xi - \psi(\xi, t)) - \mathbf{u}_{\text{fr}}(\xi)), \quad (8.11)$$

and recall that local well-posedness of the unmodulated perturbation $\tilde{\mathbf{v}}(t)$ has been established in Proposition 8.1. Thus, the integral equation (8.8) forms a closed system in ψ for which a standard contraction mapping argument yields local well-posedness.

Proposition 8.3 (Local well-posedness of the phase modulation). *Let T_{max} and $\tilde{\mathbf{v}}(t)$ be as in Proposition 8.1. Then, there exists a maximal time $\tau_{\text{max}} \in (0, T_{\text{max}}]$ such that (8.8), with $\mathbf{v}$ given by (8.11), has a unique solution*

$$\psi \in C([0, \tau_{\text{max}}], H^4(\mathbb{R})) \cap C^1([0, \tau_{\text{max}}], H^2(\mathbb{R})).$$

In addition, if $\tau_{\max} < T_{\max}$, then we have

$$\lim_{t \nearrow \tau_{\max}} \|(\psi(t), \partial_t \psi(t))\|_{H^4 \times H^2} = \infty.$$

Finally, we have $\psi(0) = 0$ and $\psi(t)$ vanishes on $[-1, \infty)$ for each $t \in [0, \tau_{\max})$.

For the sake of completeness we provide a proof of Proposition 8.3 in Appendix F.

Local well-posedness of the inverse-modulated perturbation (8.11) now follows by Propositions 8.1 and 8.3 using the mean value theorem and the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Corollary 8.4 (Local well-posedness of the inverse-modulated perturbation). *Let $\tilde{\mathbf{v}}(t)$ be the unmodulated perturbation from Proposition 8.1 and let $\psi(t)$ and $\tau_{\max}$ be as in Proposition 8.3. Then, the inverse-modulated perturbation, defined by (8.11), satisfies $\mathbf{v} = (v_1, v_2) \in C([0, \tau_{\max}), Z_2(\mathbb{R}) \times Z_1(\mathbb{R}))$. Moreover, the Duhamel formulation (8.7) holds for all $t \in [0, \tau_{\max})$.*

8.2 The forward-modulated perturbation

In this section we derive a nonlinear damping estimate for the forward-modulated perturbation $\hat{\mathbf{v}}(t)$ in order to control regularity in the quasilinear iteration scheme (8.9)-(8.10). That is, we derive an energy estimate controlling the $(H^3 \times H^2)$ -norm of $\hat{\mathbf{v}}(t)$ in terms of its L^2 -norm and the $(H^3 \times H^2)$ -norm of the initial condition $\mathbf{v}_0$.

We start by establishing local well-posedness of the forward-modulated perturbation, which readily follows by combining Propositions 8.1 and 8.3 and applying the mean value theorem.

Corollary 8.5 (Local well-posedness of the forward-modulated perturbation). *Let $\tilde{\mathbf{v}}(t)$ be the unmodulated perturbation from Proposition 8.1 and let $\psi(t)$ and $\tau_{\max}$ be as in Proposition 8.3. Then, the forward-modulated perturbation, defined by (8.3), satisfies $\hat{\mathbf{v}} \in C([0, \tau_{\max}), H^3(\mathbb{R}) \times H^2(\mathbb{R})) \cap C^1([0, \tau_{\max}), H^1(\mathbb{R}))$.*

Using that both the perturbed solution $\mathbf{u}(t)$ and the pattern-forming front $\mathbf{u}_{\text{fr}}$ solve the FitzHugh-Nagumo system (1.5), we find that the forward-modulated perturbation $\hat{\mathbf{v}}(t)$ satisfies the equation

$$\begin{aligned} \hat{\mathbf{v}}_t = & D\hat{\mathbf{v}}_{\xi\xi} + c\hat{\mathbf{v}}_{\xi} + \omega \left(F \left(\frac{\hat{\mathbf{v}}}{\omega} + \hat{\mathbf{u}}_{\text{fr},0} \right) - F(\hat{\mathbf{u}}_{\text{fr},0}) \right) \\ & + \omega(c\psi_{\xi} + \partial_t \psi) \hat{\mathbf{u}}_{\text{fr},1} + \omega D(\psi_{\xi\xi} \hat{\mathbf{u}}_{\text{fr},1} + \psi_{\xi}(\psi_{\xi} + 2) \hat{\mathbf{u}}_{\text{fr},2}), \end{aligned} \quad (8.12)$$

where we denote $\hat{\mathbf{u}}_{\text{fr},j}(\xi, t) = (\partial_{\xi}^j \mathbf{u}_{\text{fr}})(\xi + \psi(\xi, t))$.

We emphasize that, in contrast to the equation (8.4) for the inverse-modulated perturbation $\mathbf{v}(t)$, the equation (8.12) is semilinear in $\hat{\mathbf{v}}$, which implies that all nonlinear terms in (8.12) can be controlled by linear damping terms. As mentioned before, the linear damping terms in (8.12) are $\partial_{\xi\xi} \hat{v}_1$ in the first component and $-\epsilon \gamma \hat{v}_2$ in the second component. Using these observations we establish the following nonlinear damping estimate for the forward-modulated perturbation.

Proposition 8.6 (Nonlinear damping estimate for the forward-modulated perturbation). *Let $\hat{\mathbf{v}}(t)$ be as in Proposition 8.5 and $\psi(t)$ and $\tau_{\max}$ as in Proposition 8.3. Fix $R > 0$. There exist constants $C, \mu > 0$ such that the forward-modulated perturbation $\hat{\mathbf{v}}(t)$ satisfies the nonlinear damping estimate*

$$\begin{aligned} \|\hat{\mathbf{v}}(t)\|_{H^3 \times H^2}^2 \leq & C \left(e^{-\mu t} \|\mathbf{v}_0\|_{H^3 \times H^2}^2 + \|\hat{\mathbf{v}}(t)\|_{L^2}^2 \right. \\ & \left. + \int_0^t e^{-\mu(t-s)} \left(\|\hat{\mathbf{v}}(s)\|_{L^2}^2 + \|\psi_{\xi}(s)\|_{H^3}^2 + \|\partial_s \psi(s)\|_{H^2}^2 \right) ds \right), \end{aligned} \quad (8.13)$$

for each $t \in [0, \tau_{\max})$ with

$$\sup_{0 \leq s \leq t} (\|\hat{v}_1(s)\|_{W^{1,\infty}} + \|\psi_{\xi}(s)\|_{W^{1,\infty}}) \leq R. \quad (8.14)$$

Proof. Our aim is to infer an estimate for the energy

$$E(t) = \frac{1}{2} \left\| \partial_\xi^3 \hat{v}_1(t) \right\|_{L^2}^2 + \frac{1}{2\epsilon\gamma} \left\| \partial_\xi^2 \hat{v}_2(t) \right\|_{L^2}^2.$$

In order to be able to differentiate $E(t)$ with respect to t , we restrict ourselves for the moment to initial conditions $\mathbf{v}_0 \in Z_5(\mathbb{R}) \times Z_4(\mathbb{R})$. One then obtains, using the same reasoning as in the proofs of Propositions 8.1 and 8.3, analogous local well-posedness statements with two additional degrees of regularity. That is, we have $\tilde{\mathbf{v}} \in C([0, T_{\max}), Z_5(\mathbb{R}) \times Z_4(\mathbb{R})) \cap C^1([0, T_{\max}), Z_3(\mathbb{R}))$, $\psi \in C([0, \tau_{\max}), H^5(\mathbb{R})) \cap C^1([0, \tau_{\max}), H^3(\mathbb{R}))$ and, thus, $\hat{\mathbf{v}} \in C([0, \tau_{\max}), H^5(\mathbb{R}) \times H^4(\mathbb{R})) \cap C^1([0, \tau_{\max}), H^3(\mathbb{R}))$.

Fix a constant $R > 0$. Let $t \in [0, \tau_{\max})$ be such that (8.14) holds. Differentiating $E(s)$ and integrating by parts we establish a t -independent constant $C > 0$ such that

$$\begin{aligned} E'(s) \leq & - \left\| \partial_\xi^4 \hat{v}_1(s) \right\|_{L^2}^2 - \left\| \partial_\xi^2 \hat{v}_2(s) \right\|_{L^2}^2 + \left\| \partial_\xi^4 \hat{v}_1(s) \right\|_{L^2} \left\| \partial_\xi^2 \hat{v}_2(s) \right\|_{L^2} + C \left(\left\| \partial_\xi^2 \hat{v}_1(s) \right\|_{L^2} \left\| \partial_\xi^2 \hat{v}_2(s) \right\|_{L^2} \right. \\ & \left. + \left\| \hat{v}_1(s) \right\|_{H^3}^2 + \left(\left\| \hat{\mathbf{v}}(s) \right\|_{H^3 \times H^1} + \left\| \partial_x^4 \hat{v}_1(s) \right\|_{L^2} + \left\| \partial_\xi^2 \hat{v}_2(s) \right\|_{L^2} \right) (\left\| \psi_\xi(s) \right\|_{H^3} + \left\| \partial_s \psi(s) \right\|_{H^2}) \right), \end{aligned}$$

for $s \in [0, t]$, where we used the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the fact that $\psi(s)$ vanishes on $[-1, \infty)$ by Proposition 8.3. Next, we apply Young's inequality to the above estimate to yield a t -independent constant $C > 0$ such that

$$E'(s) \leq -\frac{1}{4} \left\| \partial_\xi^4 \hat{v}_1(s) \right\|_{L^2}^2 - \frac{1}{4} \left\| \partial_\xi^2 \hat{v}_2(s) \right\|_{L^2}^2 + C \left(\left\| \hat{\mathbf{v}}(s) \right\|_{H^3 \times H^1}^2 + \left\| \psi_\xi(s) \right\|_{H^3}^2 + \left\| \partial_s \psi(s) \right\|_{H^2}^2 \right).$$

Subsequently, we use Sobolev interpolation to obtain t -independent constants $C, \mu > 0$ such that

$$E'(s) \leq -\mu E(s) + C \left(\left\| \hat{\mathbf{v}}(s) \right\|_{L^2}^2 + \left\| \psi_\xi(s) \right\|_{H^3}^2 + \left\| \partial_s \psi(s) \right\|_{H^2}^2 \right),$$

for $s \in [0, t]$. Integrating the above inequality we arrive at

$$E(t) \leq e^{-\mu t} E(0) + C \int_0^t e^{-\mu(t-s)} \left(\left\| \hat{\mathbf{v}}(s) \right\|_{L^2}^2 + \left\| \psi_\xi(s) \right\|_{H^3}^2 + \left\| \partial_s \psi(s) \right\|_{H^2}^2 \right) ds,$$

Finally, using Sobolev interpolation again, we establish a t -independent constant $C > 0$ such that (8.13) holds.

For the case $\mathbf{v}_0 \in Z_3(\mathbb{R}) \times Z_2(\mathbb{R})$, we approximate $\mathbf{v}_0$ in $(Z_3 \times Z_2)$ -norm by a sequence $(\mathbf{v}_{0,n})_{n \in \mathbb{N}}$ in $Z_5(\mathbb{R}) \times Z_4(\mathbb{R})$. By continuity with respect to initial data, see [19, Proposition 4.3.7], we obtain sequences of solutions $\tilde{\mathbf{v}}_n(t)$ of (8.1) with $\tilde{\mathbf{v}}_n(0) = \mathbf{v}_{0,n}$ and of solutions $\psi_n(t)$ of (8.8) (with $\mathbf{v}_0$ replaced by $\mathbf{v}_{0,n}$) such that $\tilde{\mathbf{v}}_n(t)$ converges to $\tilde{\mathbf{v}}(t)$ in $Z_3(\mathbb{R}) \times Z_2(\mathbb{R})$, $\psi_n(t)$ converges to $\psi(t)$ in $H^4(\mathbb{R})$ and $\partial_t \psi_n(t)$ converges to $\partial_t \psi(t)$ in $H^2(\mathbb{R})$. Since (8.13) only depends on the $(H^3 \times H^2)$ -norm of $\hat{\mathbf{v}}(t) = \tilde{\mathbf{v}}(t) + \hat{\mathbf{u}}_{\text{fr}} - \mathbf{u}_{\text{fr}}$, on the H^4 -norm of $\psi(t)$ and on the H^2 -norm of $\partial_t \psi(t)$, the desired result follows by approximation. $\square$

As long as the phase modulation $\psi(t)$ and its spatial derivative stay sufficiently small, one can express the forward- and inverse-modulated perturbation in terms of each other by inverting the function $\xi \mapsto \xi - \psi(\xi, t)$. With the aid of the mean value theorem, one then establishes that the $W^{k,p}$ -norms of the forward- and inverse-modulated perturbations are equivalent modulo controllable norms of ψ_ξ . This has been established in [76, Corollary 5.3] for the case without exponential weight. For the sake of completeness, we obtain this equivalence for the relevant norms in the current setting in the following lemma.

Lemma 8.7 (Equivalence of the forward- and inverse-modulated Perturbations). *Let $\mathbf{v}(t)$ be as in Corollary 8.4, $\hat{\mathbf{v}}(t)$ as in Corollary 8.5 and $\psi(t)$ and $\tau_{\max}$ as in Proposition 8.3. Then, there exists a constant $C > 0$ such that*

$$\left\| \mathbf{v}(t) \right\|_{Z_2 \times Z_1} \leq C \left(\left\| \hat{\mathbf{v}}(t) \right\|_{Z_2 \times Z_1} + \left\| \psi_\xi(t) \right\|_{Z_1} \right), \quad (8.15)$$

and

$$\begin{aligned}\|\dot{\mathbf{v}}(t)\|_{L^\infty} &\leq C(\|\mathbf{v}(t)\|_{L^\infty} + \|\psi_\xi(t)\|_{L^\infty}), & \|\dot{\mathbf{v}}(t)\|_{W^{1,\infty}} &\leq C(\|\mathbf{v}(t)\|_{W^{1,\infty}} + \|\psi_\xi(t)\|_{L^\infty}), \\ \|\dot{\mathbf{v}}(t)\|_{L^2} &\leq C(\|\mathbf{v}(t)\|_{L^2} + \|\psi_\xi(t)\|_{L^\infty}), & \|\dot{\mathbf{v}}(t)\|_{H^1} &\leq C(\|\mathbf{v}(t)\|_{H^1} + \|\psi_\xi(t)\|_{Z_0}),\end{aligned}\quad (8.16)$$

for any $t \in [0, \tau_{\max})$ with $\|\psi(t)\|_{Z_2} \leq \frac{1}{2}$.

Proof. Let $t \in [0, \tau_{\max})$ with $\|\psi(t)\|_{Z_2} \leq \frac{1}{2}$. Since we have $\|\psi_\xi(t)\|_{L^\infty} \leq \frac{1}{2}$, the function $h_t: \mathbb{R} \rightarrow \mathbb{R}$ given by $h_t(\xi) = \xi - \psi(\xi, t)$ is strictly increasing and invertible. Moreover, using $\xi = h_t(h_t^{-1}(\xi)) = h_t^{-1}(\xi) - \psi(h_t^{-1}(\xi), t)$ and $\|\psi(t)\|_{L^\infty} \leq \frac{1}{2}$, we find $|\xi - h_t^{-1}(\xi)| \leq \frac{1}{2}$ for all $\xi \in \mathbb{R}$.

Substituting (8.3) into (8.2) we obtain

$$\begin{aligned}\mathbf{v}(\xi, t) &= \omega(\xi) \left(\frac{\dot{\mathbf{v}}(\xi - \psi(\xi, t), t)}{\omega(\xi - \psi(\xi, t))} + \mathbf{u}_{\text{fr}}(\xi - \psi(\xi, t) + \psi(\xi - \psi(\xi, t), t)) - \mathbf{u}_{\text{fr}}(\xi) \right), \\ &= \dot{\mathbf{v}}(\xi - \psi(\xi, t), t) + \mathbf{u}_{\text{fr}}(\xi - \psi(\xi, t) + \psi(\xi - \psi(\xi, t), t)) - \mathbf{u}_{\text{fr}}(\xi),\end{aligned}$$

where the second equality follows from the facts that for $\xi \leq -1$ we have $\xi - \psi(\xi, t) \leq -1 + \|\psi(t)\|_{L^\infty} \leq -\frac{1}{2}$, for $\xi \leq 0$ we have $\omega(\xi) = 1$ and for $\xi \geq -1$ we have $\psi(\xi, t) = 0$ by Proposition 8.3. Taking spatial derivatives yields

$$\begin{aligned}\mathbf{v}_\xi(\xi, t) &= \dot{\mathbf{v}}_\xi(\xi - \psi(\xi, t), t) (1 - \psi_\xi(\xi, t)) - \mathbf{u}'_{\text{fr}}(\xi) \\ &\quad + \mathbf{u}'_{\text{fr}}(\xi - \psi(\xi, t) + \psi(\xi - \psi(\xi, t), t)) (1 - \psi_\xi(\xi, t) + \psi_\xi(\xi - \psi(\xi, t), t) (1 - \psi_\xi(\xi, t))), \\ \mathbf{v}_{\xi\xi}(\xi, t) &= \dot{\mathbf{v}}_{\xi\xi}(\xi - \psi(\xi, t), t) (1 - \psi_\xi(\xi, t))^2 - \mathbf{u}''_{\text{fr}}(\xi) - \dot{\mathbf{v}}_\xi(\xi - \psi(\xi, t), t) \psi_{\xi\xi}(\xi, t) \\ &\quad + \mathbf{u}''_{\text{fr}}(\xi - \psi(\xi, t) + \psi(\xi - \psi(\xi, t), t)) (1 - \psi_\xi(\xi, t) + \psi_\xi(\xi - \psi(\xi, t), t) (1 - \psi_\xi(\xi, t)))^2 \\ &\quad + \mathbf{u}'_{\text{fr}}(\xi - \psi(\xi, t) + \psi(\xi - \psi(\xi, t), t)) \left(\psi_\xi(\xi - \psi(\xi, t), t) (1 - \psi_\xi(\xi, t))^2 \right. \\ &\quad \left. - \psi_{\xi\xi}(\xi, t) (1 + \psi_\xi(\xi - \psi(\xi, t), t)) \right).\end{aligned}$$

Using the mean value theorem twice we estimate

$$\left| \left(\partial_\xi^j \mathbf{u}_{\text{fr}} \right) (\xi - \psi(\xi, t) + \psi(\xi - \psi(\xi, t), t)) - \partial_\xi^j \mathbf{u}_{\text{fr}}(\xi) \right| \leq \left\| \partial_\xi^{j+1} \mathbf{u}_{\text{fr}} \right\|_{L^\infty} \|\psi_\xi(t)\|_{L^\infty} |\psi(\xi, t)|.$$

for $j = 0, 1, 2$ and $\xi \in \mathbb{R}$. In addition, using $\|\psi_\xi(t)\|_{L^\infty} \leq \frac{1}{2}$ and applying the substitution $y = h_t(\xi)$, we bound

$$\|f(\cdot - \psi(\cdot, t))\|_{L^2}^2 = \int_{\mathbb{R}} |f(\xi - \psi(\xi, t))|^2 d\xi = \int_{\mathbb{R}} \frac{|f(y)|^2}{1 - \psi_\xi(h_t^{-1}(y), t)} dy \leq \|f\|_{L^2}^2.$$

for $f \in L^2(\mathbb{R})$. With the aid of the latter two estimates and using the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the fact that $\|\psi(t)\|_{Z_2} \leq \frac{1}{2}$, we thus establish (8.15).

Conversely, substituting (8.2) into (8.3), we obtain

$$\begin{aligned}\dot{\mathbf{v}}(\xi, t) &= \omega(\xi) \left(\frac{\mathbf{v}(h_t^{-1}(\xi), t)}{\omega(h_t^{-1}(\xi))} + \mathbf{u}_{\text{fr}}(h_t^{-1}(\xi)) - \mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t)) \right) \\ &= \mathbf{v}(h_t^{-1}(\xi), t) + \mathbf{u}_{\text{fr}}(h_t^{-1}(\xi)) - \mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t)),\end{aligned}$$

where we use that for $\xi \leq -\frac{1}{2}$ we have $h_t^{-1}(\xi) = \xi + \psi(h_t^{-1}(\xi), t) \leq -\frac{1}{2} + \|\psi(t)\|_\infty \leq 0$, for $\xi \leq 0$ we have $\omega(\xi) = 1$, for $\xi \geq -\frac{1}{2}$ we have $h_t^{-1}(\xi) = \xi + \psi(h_t^{-1}(\xi), t) \geq -\frac{1}{2} - \|\psi(t)\|_{L^\infty} \geq -1$, and for $\xi \geq -1$ we have $\psi(\xi, t) = 0$ by Proposition 8.3. Taking spatial derivatives yields

$$\dot{\mathbf{v}}_\xi(\xi, t) = \left(\mathbf{v}_\xi(h_t^{-1}(\xi), t) + \mathbf{u}'_{\text{fr}}(h_t^{-1}(\xi)) \right) \partial_\xi(h_t^{-1}(\xi)) - \mathbf{u}'_{\text{fr}}(\xi + \psi(\xi, t))(1 + \psi_\xi(\xi, t)).$$

First, the inverse function theorem implies

$$\partial_\xi \left(h_t^{-1}(\xi) \right) = 1 + \frac{\psi_\xi \left(h_t^{-1}(\xi), t \right)}{1 - \psi_\xi \left(h_t^{-1}(\xi), t \right)},$$

for $\xi \in \mathbb{R}$. Next, recalling $h_t^{-1}(\xi) = \xi + \psi(h_t^{-1}(\xi), t)$ and using the mean value theorem twice, we arrive at

$$\left| \left(\partial_\xi^j \mathbf{u}_{\text{fr}} \right) \left(h_t^{-1}(\xi) \right) - \left(\partial_\xi^j \right) \mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t)) \right| \leq \left\| \partial_\xi^{j+1} \mathbf{u}_{\text{fr}} \right\|_{L^\infty} \|\psi_\xi(t)\|_{L^\infty} \left| \psi \left(h_t^{-1}(\xi), t \right) \right|.$$

for $j = 0, 1$ and $\xi \in \mathbb{R}$. Lastly, using $\|\psi_\xi(t)\|_{L^\infty} \leq \frac{1}{2}$ and applying the substitution $\xi = h_t(y)$, we estimate

$$\left\| f \circ h_t^{-1} \right\|_{L^2}^2 = \int_{\mathbb{R}} \left| f \left(h_t^{-1}(\xi) \right) \right|^2 d\xi = \int_{\mathbb{R}} |f(y)|^2 (1 + \psi_y(y, t)) dy \leq \|f\|_{L^2}^2,$$

for $f \in L^2(\mathbb{R})$. With these three observations, recalling the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the fact that $\|\psi(t)\|_{Z_1} \leq \frac{1}{2}$, we have established (8.16). $\square$

9 Nonlinear stability argument

In this section we prove our nonlinear stability result, Theorem 2.1 by applying the linear estimates, obtained in Theorem 7.1, and the nonlinear estimates, obtained in Lemma 8.2 and Proposition 8.6, to the nonlinear interaction scheme consisting of the integral equations (8.9) and (8.10) for the inverse-modulated perturbation $\mathbf{v}(t)$ and the phase modulation $\psi(t)$, respectively.

Proof of Theorem 2.1. Set $\mathbf{v}_0 = \omega \mathbf{w}_0$ and let $\tilde{\mathbf{v}}(t)$, $\mathbf{v}(t)$ and $\dot{\mathbf{v}}(t)$ be the associated unmodulated, inverse- and forward-modulated perturbations, established in Proposition 8.1 and Corollaries 8.4 and 8.5. In addition, let $\psi(t)$ be the corresponding phase modulation, established in Proposition 8.3. It follows by Propositions 8.1 and 8.3 and Corollary 8.4 that the functions $\eta_1, \eta_2: [0, \tau_{\max}) \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \eta_1(t) &= \sup_{0 \leq s \leq t} \left[(1+s) \|\mathbf{v}(s)\|_{L^\infty} + (1+s)^{\frac{3}{4}} (\|\mathbf{v}(s)\|_{L^2} + \|\psi_\xi(s)\|_{H^3} + \|\partial_s \psi(s)\|_{H^2}) + (1+s)^{\frac{1}{4}} \|\psi(s)\|_{L^2} \right], \\ \eta_2(t) &= \sup_{0 \leq s \leq t} \left[(1+s)^{\frac{3}{4}} \|\partial_\xi \mathbf{v}(s)\|_{H^1} + (1+s)^{\frac{1}{4}} \|\tilde{\mathbf{v}}(s)\|_{Z_1} \right], \end{aligned}$$

are well-defined, continuous, positive and monotonically increasing. We will control terms that appear in $\eta_1(t)$ by iterative estimates on their Duhamel formulas, whereas terms in $\eta_2(t)$ are controlled by the nonlinear damping estimate stated in Proposition 8.6. This leads to different type of inequalities for $\eta_1(t)$ and $\eta_2(t)$, which we eventually combine to an inequality for $\eta(t) = \eta_1(t) + \eta_2(t)$. Next, recall that if $\tau_{\max} < T_{\max}$ or $T_{\max} < \infty$, then we have

$$\lim_{t \nearrow \tau_{\max}} \eta(t) = \infty. \quad (9.1)$$

As common in nonlinear iteration arguments, our aim is to establish a nonlinear inequality for the template function $\eta(t)$, which, by continuity, implies that $\eta(t)$ must stay small and precludes (9.1), yielding global existence. More specifically, our goal is to obtain a t -independent constant $C \geq 1$ such that for any $t \in [0, \tau_{\max})$ with $\eta(t) \leq \frac{1}{2}$ the inequality

$$\eta(t) \leq C \left(E_0 + \eta(t)^2 \right), \quad (9.2)$$

holds. Then, upon taking $\delta = \frac{1}{4C^2}$, it follows from continuity, monotonicity and non-negativity of η that, provided $E_0 \in (0, \delta)$, we have $\eta(t) \leq 2CE_0 \leq \frac{1}{2}$ for all $t \in [0, \tau_{\max})$. Indeed, for any given $t \in [0, \tau_{\max})$ with $\eta(s) \leq 2CE_0$ for each $s \in [0, t]$, we conclude

$$\eta(t) \leq C(E_0 + 4C^2E_0^2) < 2CE_0,$$

using (9.2). Therefore, once we have established (9.2) and, thus, $\eta(t) \leq 2CE_0$ for all $t \in [0, \tau_{\max})$, it follows by (9.1) that necessarily $\tau_{\max} = T_{\max} = \infty$. Consequently, one obtains that $\eta(t) \leq 2CE_0$ for all $t \geq 0$, which then yields the desired estimates.

Thus, we start by showing the nonlinear inequality (9.2). In fact, we first obtain such an inequality for $\eta_1(t)$, while using that both $\eta_1(t)$ and $\eta_2(t)$ are bounded. We stress that boundedness of $\eta_2(t)$ is required here to apply the nonlinear damping estimate stated in Proposition 8.6 in order to control derivatives in the nonlinearity $\mathcal{N}$ in (8.9) and (8.10). Subsequently, we bound $\eta_2(t)$ in terms of $\eta_1(t)$ using the same nonlinear damping estimate. Finally, adding the inequalities for $\eta_1(t)$ and $\eta_2(t)$ yields an estimate of the form (9.2). Thus, let $t \in [0, \tau_{\max})$ be such that $\eta(t) \leq \frac{1}{2}$. Lemma 8.7 and the continuous embedding

$$H^{k+1}(\mathbb{R}) \hookrightarrow Z_k(\mathbb{R}), \quad k \in \mathbb{N}_0, \quad (9.3)$$

yield the following bounds on the forward-modulated perturbation

$$\|\dot{\mathbf{v}}(\tau)\|_{L^2} \lesssim \frac{\eta_1(t)}{(1+\tau)^{\frac{3}{4}}}, \quad \|\dot{v}_1(\tau)\|_{W^{1,\infty}} \lesssim \frac{\eta(t)}{(1+\tau)^{\frac{3}{4}}}, \quad (9.4)$$

for all $\tau \in [0, t]$. Thus, applying the nonlinear damping estimate in Proposition 8.6 and using (9.3) and (9.4), while noting that $\eta(t) \leq \frac{1}{2}$, we establish

$$\|\dot{\mathbf{v}}(s)\|_{H^3 \times H^2}^2 \lesssim e^{-\mu s} E_0^2 + \frac{\eta_1(t)^2}{(1+s)^{\frac{3}{2}}} + \int_0^s e^{-\mu(s-r)} \frac{\eta_1(t)^2}{(1+r)^{\frac{3}{2}}} dr \lesssim \frac{E_0^2 + \eta_1(t)^2}{(1+s)^{\frac{3}{2}}}, \quad (9.5)$$

for all $s \in [0, t]$. Hence, combining Lemma 8.7 with (9.5), recalling (9.3), and using Young's inequality, we arrive at

$$\|\mathbf{v}(s)\|_{Z_2 \times Z_1} \lesssim \frac{E_0 + \eta_1(t)}{(1+s)^{\frac{3}{4}}}, \quad (9.6)$$

for each $s \in [0, t]$. Finally, combining the latter estimate with Lemma 8.2, using $\eta(t) \leq \frac{1}{2}$, recalling (9.3) and noting that $\psi(s)$ vanishes on $[-1, \infty)$ by Proposition 8.3, we obtain the nonlinear bound

$$\|\mathcal{N}(\mathbf{v}(s), \psi(s), \partial_s \psi(s))\|_{L_{0,1}^1 \cap Z_0} \lesssim \frac{\eta_1(t)(E_0 + \eta_1(t))}{(1+s)^{\frac{3}{2}}}, \quad (9.7)$$

for each $s \in [0, t]$.

We are now in the position to establish an inequality for $\eta_1(t)$ by iterative estimates on the Duhamel formulas (8.9) and (8.10) for the inverse-modulated perturbation $\mathbf{v}(t)$ and the phase modulation $\psi(t)$, respectively. Thus, we apply the linear estimates in Theorem 7.1 and the nonlinear bound in (9.7) to (8.9) and establish

$$\|\mathbf{v}(t)\|_{L^2} \lesssim \frac{E_0 + \eta_1(t)^2}{(1+t)^{\frac{3}{4}}} + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{e^{\mu(t-s)}(1+s)^{\frac{3}{2}}} ds + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{(1+t-s)^{\frac{3}{4}}(1+s)^{\frac{3}{2}}} ds \lesssim \frac{E_0 + \eta_1(t)^2}{(1+t)^{\frac{3}{4}}}, \quad (9.8)$$

and

$$\|\mathbf{v}(t)\|_{L^\infty} \lesssim \frac{E_0 + \eta_1(t)^2}{1+t} + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{e^{\mu(t-s)}(1+s)^{\frac{3}{2}}} ds + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{(1+t-s)(1+s)^{\frac{3}{2}}} ds \lesssim \frac{E_0 + \eta_1(t)^2}{1+t}, \quad (9.9)$$

where we used $\eta_1(t) \leq \frac{1}{2}$. Similarly, we apply Theorem 7.1 and estimate (9.7) to (8.10) and obtain

$$\|\psi(t)\|_{L^p} \lesssim \frac{E_0}{(1+t)^{\frac{1}{2}-\frac{1}{2p}}} + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{(1+t-s)^{\frac{1}{2}-\frac{1}{2p}}(1+s)^{\frac{3}{2}}} ds \lesssim \frac{E_0 + \eta_1(t)^2}{(1+t)^{\frac{1}{2}-\frac{1}{2p}}}, \quad (9.10)$$

and

$$\|\partial_\xi^\ell \partial_t^j \psi(t)\|_{L^p} \lesssim \frac{E_0}{(1+t)^{1-\frac{1}{2p}}} + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{(1+t-s)^{1-\frac{1}{2p}}(1+s)^{\frac{3}{2}}} ds \lesssim \frac{E_0 + \eta_1(t)^2}{(1+t)^{1-\frac{1}{2p}}}, \quad (9.11)$$

for $p = 2, \infty$ and $\ell, j \in \mathbb{N}_0$ with $1 \leq \ell + 2j \leq 4$, where we used $\eta(t) \leq \frac{1}{2}$. Combining the estimates (9.8), (9.9), (9.10), and (9.11), we establish a t -independent constant $C_1 \geq 1$ such that

$$\eta_1(t) \leq C_1 (E_0 + \eta_1(t)^2). \quad (9.12)$$

Next, we obtain an inequality for $\eta_2(t)$. First, we express the unmodulated perturbation as

$$\tilde{\mathbf{v}}(\xi, t) = \dot{\mathbf{v}}(\xi, t) + \omega(\xi) (\mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t)) - \mathbf{u}_{\text{fr}}(\xi)) = \dot{\mathbf{v}}(\xi, t) + \mathbf{u}_{\text{fr}}(\xi + \psi(\xi, t)) - \mathbf{u}_{\text{fr}}(\xi) \quad (9.13)$$

where we use that $\omega(\xi) = 1$ for $\xi \leq 0$ and $\psi(t)$ vanishes on $[-1, \infty)$ by Proposition 8.3. Thus, Lemma 8.7 and the mean value theorem yield

$$\|\tilde{\mathbf{v}}(t)\|_{Z_1} \lesssim \|\dot{\mathbf{v}}(t)\|_{Z_1} + \|\psi(t)\|_{Z_1} \lesssim \|\mathbf{v}(t)\|_{Z_1} + \|\psi(t)\|_{Z_1},$$

where we use $\eta(t) \leq \frac{1}{2}$. Combining the latter with (9.6), while recalling (9.3), we establish the bounds

$$\|\tilde{\mathbf{v}}(t)\|_{Z_1} \leq \frac{E_0 + \eta_1(t)}{(1+t)^{\frac{1}{4}}}, \quad \|\mathbf{v}_\xi(s)\|_{L^\infty} \leq \frac{E_0 + \eta_1(t)}{(1+t)^{\frac{3}{4}}}.$$

Thus, we establish a t -independent constant $C_2 \geq 1$ such that

$$\eta_2(t) \leq C_2 (E_0 + \eta_1(t)). \quad (9.14)$$

Finally, (9.12) and (9.14) afford the estimate

$$\eta(t) \leq C_2 E_0 + (1 + C_2) \eta_1(t) \leq (C_2 + (1 + C_2) C_1) E_0 + (1 + C_2) C_1 \eta_1(t)^2,$$

which implies (9.2) for some t -independent constant $C \geq 1$. As mentioned before, this yields $\tau_{\max} = T_{\max} = \infty$ and $\eta(t) \leq 2CE_0$ for all $t \geq 0$. Hence, recalling Propositions 8.1 and 8.3, we readily establish (2.16). Moreover, the mean value theorem, identity (9.13) and Lemma 8.7 yield the estimates

$$\begin{aligned} \|\tilde{\mathbf{v}}(t)\|_{H^3 \times H^2} &\lesssim \|\dot{\mathbf{v}}(t)\|_{H^3 \times H^2} + \|\psi(t)\|_{H^3}, & \|\tilde{\mathbf{v}}(t)\|_{L^\infty} &\lesssim \|\dot{\mathbf{v}}(t)\|_{L^\infty} + \|\psi(t)\|_{L^\infty}, \\ \|\dot{\mathbf{v}}(t)\|_{L^\infty} &\lesssim \|\mathbf{v}(t)\|_{L^\infty} + \|\psi_\xi(t)\|_{L^\infty}, \end{aligned} \quad (9.15)$$

for $t \geq 0$, where we use that $\eta(t) \leq 2CE_0$. All in all, combining the estimates (9.5), (9.10), (9.11) and (9.15) with the fact that $\eta(t) \leq 2CE_0$ holds for all $t \geq 0$, yields a t -independent constant $M > 0$ such that (2.17) and (2.18) are satisfied.

It only remains to prove (2.19), that is, to transfer the improved linear decay estimate (7.3) in the leading edge to the nonlinear level. Thus, we multiply the Duhamel formula (8.7) with χ_+ and recall that $\psi(t)$ vanishes on $[-1, \infty)$ to arrive at

$$\chi_+ \tilde{\mathbf{v}}(t) = \chi_+ \mathbf{v}(t) = \chi_+ e^{\mathcal{L}_{\text{fr}} t} \mathbf{v}_0 + \int_0^t \chi_+ e^{\mathcal{L}_{\text{fr}}(t-s)} \mathcal{N}(\mathbf{v}(s), \psi(s), \partial_t \psi(s)) ds. \quad (9.16)$$

Applying the linear bound (7.3) in Theorem 7.1 and the nonlinear bound (9.7) to (9.16), while using $\eta(t) \leq 2CE_0$, we obtain

$$\|\rho_{0,-1} \chi_+ \tilde{\mathbf{v}}(t)\|_{L^\infty} \lesssim \frac{E_0}{(1+t)^{\frac{3}{2}}} + \int_0^t \frac{\eta_1(t)(E_0 + \eta_1(t))}{(1+t-s)^{\frac{3}{2}}(1+s)^{\frac{3}{2}}} ds \lesssim \frac{E_0}{(1+t)^{\frac{3}{2}}},$$

for $t \geq 0$. We conclude that there exists a t -independent constant $M > 0$ such that estimate (2.19) holds for all $t \geq 0$, which completes the proof. $\square$

10 Discussion

We believe that the methods developed here can be useful for analyzing diffusive stability problems in many contexts, and we conclude by discussing potential applications of our methods to several related problems.

Pushed pattern-forming fronts. The analysis in [18] also gives a rigorous construction of pushed pattern-forming invasion fronts in the FitzHugh-Nagumo system (1.1) for $0 < a < \frac{1}{3}$. For pushed fronts, the propagation is no longer driven by the dynamics in the leading edge, but instead by a localized mode near the front interface. Hence, the essential spectrum in the leading edge may be fully stabilized by introducing an exponential weight, but there is a resonance pole of the Evans function at $\lambda = 0$. As a result, the linearized dynamics exhibit a non-decaying mode associated to spatial translation. Key to the dynamics then is the interaction of this non-decaying mode with the outgoing diffusive mode. We expect that our approach to resolvent decompositions can still be used in this setting to give a precise description of the linearized dynamics which could then be used to close a nonlinear stability argument. The stability problem here is conceptually similar to the stability of source defects, which also focuses on the interaction of non-decaying, translational modes with outgoing diffusive modes, and has been studied in the complex Ginzburg-Landau equation in [13]. We emphasize that, for pushed pattern-forming fronts, proving nonlinear stability against the natural class of perturbations readily yields selection from steep initial data, since the exponential weights involved automatically allow one to consider perturbations which cut off the front tail. This strongly contrasts to the case of pulled pattern-forming fronts, where substantial additional steps are necessary to extend sharp nonlinear stability results, as presented in this work, to selection results (see further discussion below).

Pulled pattern-forming fronts beyond the FitzHugh-Nagumo system. The essential feature of the stability of pulled pattern-forming fronts is the interaction between the branched diffusive mode associated with the linear spreading speed and the outgoing diffusive mode associated with the pattern in the wake. The methods developed here should generally be successful in proving nonlinear stability of pulled pattern-forming fronts also in other systems. A particular case of interest is front invasion in the wake of a Turing instability, for instance in the Swift-Hohenberg equation or other reaction-diffusion systems. The new challenge, compared to the FitzHugh-Nagumo fronts studied here, is that the invasion dynamics in the leading edge are oscillatory in time, so that the pattern-forming fronts are *modulated* traveling waves, which are time-periodic in the comoving frame, rather than stationary. We expect that this can be overcome by combining the analysis here with ideas from [14], which develops a framework for studying time-periodic diffusive stability problems via an inverse Laplace transform. We expect that this approach can be used to establish a general result on nonlinear stability of pulled pattern-forming invasion fronts in reaction-diffusion systems near a supercritical Turing instability.

Selection of pulled pattern-forming fronts from steep initial data. The challenge in establishing selection of pulled pattern-forming fronts from steep initial data is that perturbations which cut off the front tail induce a logarithmic delay $-\frac{3}{2\eta_{\text{lin}}} \log t$ in the position of the front. In this frame, perturbations in the leading edge no longer decay at all, which has been overcome for fronts selecting constant states in [2, 9], but presents substantial additional difficulties for pattern-forming fronts, since these non-decaying modes interact with the outgoing diffusive dynamics in the wake. We are hopeful that combining the present analysis with recent work on selection of pulled fronts [2, 9] and on stability of periodic wave trains against nonlocalized perturbations [22], which also do not exhibit temporal decay, may make progress towards establishing front selection in this context. In particular, the $(1+t)^{-3/2}$ decay for localized perturbations is a key ingredient in closing the front selection argument in [2, 9], so the fact that we recover this decay rate in the leading edge here is a promising sign for establishing selection of pulled pattern-forming fronts via this program.

A Generation of a C^0 -semigroup on $Z_k(\mathbb{R})$

In this section we prove Proposition 4.1, showing that $e^{\mathcal{L}_{\text{fr}} t}$ generates a strongly continuous semigroup on $Z_k(\mathbb{R}, \mathbb{C}^2)$ for any fixed non-negative integer k . We rely on the following formulation of the Hille-Yosida theorem.

Theorem A.1 ([55, Section 1.5, Theorem 5.3]). *Let X be a Banach space. A linear operator $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X satisfying $\|T(t)\|_{X \rightarrow X} \leq M e^{\mu_* t}$ for some constants $M > 0$, $\mu_* \in \mathbb{R}$ if and only if:*

1. *A is closed and densely defined.*
2. *The resolvent set $\rho(A)$ of A contains the interval (μ_*, ∞) and for all $\lambda \in \mathbb{R}$ with $\lambda > \mu_*$, we have the estimate*

$$\|(A - \lambda)^{-n}\|_{X \rightarrow X} \leq \frac{M}{(\lambda - \mu_*)^n}, \quad n \in \mathbb{N}. \quad (\text{A.1})$$

We will also use the following result on robustness of generators of C^0 -semigroups.

Proposition A.2 (Robustness of generation of C^0 -semigroups under bounded perturbations, [55, Section 3.1, Theorem 1.1]). *Let X be a Banach space and let A_0 be the infinitesimal generator of a strongly continuous semigroup $T(t)$ on X , satisfying $\|T(t)\|_{X \rightarrow X} \leq M e^{\mu_* t}$ for some constants $M > 0$ and $\mu_* \in \mathbb{R}$. If A_1 is a bounded linear operator on X , then $A_0 + A_1$ is the infinitesimal generator of a C^0 -semigroup $S(t)$ on X , satisfying $\|S(t)\|_{X \rightarrow X} \leq M e^{(\mu_* + M\|A_1\|_{X \rightarrow X})t}$.*

Fix a non-negative integer k , and consider $\mathcal{L}_{\text{fr}}$ as an operator

$$\mathcal{L}_{\text{fr}} : Z_{k+2}(\mathbb{R}, \mathbb{C}) \times Z_{k+1}(\mathbb{R}, \mathbb{C}) \subseteq Z_k(\mathbb{R}, \mathbb{C}^2) \rightarrow Z_k(\mathbb{R}, \mathbb{C}^2).$$

Standard computations verify that with this domain, $\mathcal{L}_{\text{fr}}$ is a closed and densely defined operator. The main work is then in identifying the principal part A_0 and establishing the resolvent estimates (A.1).

Let u_* denote the first component of $\mathbf{u}_{\text{fr}}$ and let $c = c_{\text{lin}}$. Consider the resolvent equation $(\mathcal{L}_{\text{fr}} - \lambda)\mathbf{u} = \mathbf{g}$, which takes the form

$$\begin{aligned} u_{\xi\xi} + (c + a_1)u_{\xi} + a_0 u + F'_1(u_*)u - v - \lambda u &= g_1 \\ cv_{\xi} + \varepsilon u - \varepsilon \gamma v - \lambda v + b_0 v &= g_2, \end{aligned}$$

where a_j, b_j are generated by conjugation with the exponential weight ω , with the expressions

$$a_1 = 2\omega(\omega^{-1})', \quad a_0 = \omega(\omega^{-1})'' + c\omega(\omega^{-1})', \quad b_0 = c\omega(\omega^{-1})'. \quad (\text{A.2})$$

Note that a_j, b_j are all smooth and bounded functions on $\mathbb{R}$. We define operators

$$A_0 = \begin{pmatrix} \partial_{\xi\xi} + (c + a_1)\partial_{\xi} & 0 \\ 0 & c\partial_{\xi} \end{pmatrix}, \quad A_1 = \begin{pmatrix} F'_1(u_*) + a_0 & -1 \\ \varepsilon & -\varepsilon\gamma + b_0 \end{pmatrix},$$

so that $\mathcal{L}_{\text{fr}} = A_0 + A_1$. Notice that $A_0 : Z_{k+2}(\mathbb{R}, \mathbb{C}) \times Z_{k+1}(\mathbb{R}, \mathbb{C}) \rightarrow Z_k(\mathbb{R}, \mathbb{C}^2)$ is closed and densely defined, and $A_1 : Z_k(\mathbb{R}, \mathbb{C}^2) \rightarrow Z_k(\mathbb{R}, \mathbb{C}^2)$ is bounded.

Lemma A.3. *Fix a non-negative integer k . The operator*

$$A_0 : Z_{k+2}(\mathbb{R}, \mathbb{C}) \times Z_{k+1}(\mathbb{R}, \mathbb{C}) \subset Z_k(\mathbb{R}, \mathbb{C}^2) \rightarrow Z_k(\mathbb{R}, \mathbb{C}^2)$$

generates a strongly continuous semigroup on $Z_k(\mathbb{R}, \mathbb{C}^2)$.

Proof. Since $A_0 - \lambda$ is block diagonal, we may invert each component separately. By standard theory of analytic semigroups (see e.g. [38, 50]), the operator $\partial_{\xi\xi} + (c + a_1)\partial_{\xi} : Z_{k+2}(\mathbb{R}, \mathbb{C}) \subset Z_k(\mathbb{R}, \mathbb{C}) \rightarrow Z_k(\mathbb{R}, \mathbb{C})$ is sectorial, and hence generates an analytic semigroup. Analytic semigroups are in particular strongly continuous [55], so by Theorem A.1, there exist $M, \mu_* > 0$ so that

$$\|(\partial_{\xi\xi} + (c + a_1)\partial_{\xi} - \lambda)^{-n}\|_{Z_k \rightarrow Z_k} \leq \frac{M}{(\lambda - \mu_*)^n}, \quad n \in \mathbb{N}, \quad (\text{A.3})$$

for all $\lambda \in \mathbb{R}$ with $\lambda > \mu_*$.

By explicitly computing the Green's function for the operator $c\partial_{\xi} - \lambda$ and computing its L^1 norm, we readily obtain

$$\|(c\partial_{\xi} - \lambda)^{-1}\|_{Z_k \rightarrow Z_k} \leq \frac{1}{\lambda}, \quad (\text{A.4})$$

for all $\lambda > 0$. Combining (A.3) and (A.4), we find that there are $M, \mu_* > 0$ such that

$$\|(A_0 - \lambda)^{-n}\|_{Z_k \rightarrow Z_k} \leq \frac{M}{(\lambda - \mu_*)^n}, \quad n \in \mathbb{N},$$

for all $\lambda \in \mathbb{R}$ with $\lambda > \mu_*$. Hence, by Theorem A.1, A_0 generates a strongly continuous semigroup on $Z_k(\mathbb{R}, \mathbb{C}^2)$, as desired. $\square$

Proof of Proposition 4.1. Since $A_1 : Z_k(\mathbb{R}, \mathbb{C}^2) \rightarrow Z_k(\mathbb{R}, \mathbb{C}^2)$ is bounded, this follows immediately from Corollary A.3 and Proposition A.2. $\square$

B Shifting the integration contour

We prove Proposition 4.3 in this section by carefully analyzing the high frequency behavior of the resolvent $(\mathcal{L}_{\text{fr}} - \lambda)^{-1}$. The analysis of this section adapts some ideas from [52], although we use a different method to obtain a description of the high frequency behavior of the resolvent. We again let u_* denote the first component of $\mathbf{u}_{\text{fr}}$, and abbreviate $c = c_{\text{lin}} > 0$. We have

$$\mathcal{L}_{\text{fr}} = \omega \mathcal{A}_{\text{fr}} \omega^{-1} = \begin{pmatrix} \partial_{\xi\xi} + (c + a_1)\partial_{\xi} + F_1'(u_*) + a_0 & -1 \\ \varepsilon & c\partial_{\xi} - \varepsilon\gamma + b_0 \end{pmatrix},$$

where a_j, b_j are given by (A.2). The key observations to retaining spectral mapping properties for the operator $\mathcal{L}_{\text{fr}}$ is damping induced by the second derivative in the first component and by the term $-\varepsilon\gamma$ in the second component. In addition, since ω is non-decreasing, b_0 is non-positive and, thus, only contributes to additional damping.

B.1 Resolvent decomposition for $\text{Im } \lambda \gg 1$

Let $\eta > 0$ be as in Corollary 4.2 and fix $\Omega_0 > 0$. We consider the resolvent equation

$$(\mathcal{L}_{\text{fr}} - \lambda) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad (\text{B.1})$$

for $\mathbf{g} = (g_1, g_2) \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $\lambda \in \rho(\mathcal{L}_{\text{fr}})$ of the form $\lambda = b + i\Omega$, with $b \in [-\frac{3}{4}\varepsilon\gamma, \eta]$ and Ω real with $|\Omega| \geq \Omega_0$. We set $\kappa = \text{sign}(\Omega)$, so that $\Omega = \kappa|\Omega|$. We rescale equation (B.1) by defining $X = \sqrt{|\Omega|}\xi$ together

with the rescaled functions

$$\mathbf{U}(X) = \begin{pmatrix} U(X) \\ V(X) \end{pmatrix} = \begin{pmatrix} u \left(\frac{X}{\sqrt{|\Omega|}} \right) \\ v \left(\frac{X}{\sqrt{|\Omega|}} \right) \end{pmatrix}, \quad \mathbf{G}(X) = \begin{pmatrix} G_1(X) \\ G_2(X) \end{pmatrix} = \begin{pmatrix} g_1 \left(\frac{X}{\sqrt{|\Omega|}} \right) \\ g_2 \left(\frac{X}{\sqrt{|\Omega|}} \right) \end{pmatrix},$$

and

$$U_*(X) = u_* \left(\frac{X}{\sqrt{|\Omega|}} \right), \quad A_1(X) = a_1 \left(\frac{X}{\sqrt{|\Omega|}} \right), \quad A_0(X) = a_0 \left(\frac{X}{\sqrt{|\Omega|}} \right), \quad B_0(X) = b_0 \left(\frac{X}{\sqrt{|\Omega|}} \right),$$

so that (B.1) becomes

$$\begin{pmatrix} |\Omega|(\partial_{XX} - i\kappa) + \sqrt{|\Omega|}(c + A_1)\partial_X + F'_1(U_*) - b + A_0 & -1 \\ \varepsilon & c\sqrt{|\Omega|}\partial_X - \varepsilon\gamma - b - i\kappa|\Omega| + B_0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}.$$

Setting $\mathbf{U} = (U, V)^\top$, multiplying both sides by $\text{diag}(|\Omega|^{-1}, |\Omega|^{-1/2})$ and setting $\mu = |\Omega|^{-1/2}$, we obtain the system

$$[L_0(\mu) + L_1(\mu)]\mathbf{U} = \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix}, \quad (\text{B.2})$$

where

$$L_0(\mu) = \begin{pmatrix} \partial_{XX} - i\kappa & 0 \\ 0 & c\partial_X - \tilde{b}\mu - \frac{i\kappa}{\mu} + \mu B_0 \end{pmatrix} =: \begin{pmatrix} L_0^{11} & 0 \\ 0 & L_0^{22}(\mu) \end{pmatrix},$$

$\tilde{b} = b + \varepsilon\gamma$, $\kappa \in \{-1, 1\}$, and

$$L_1(\mu) = \begin{pmatrix} \mu(c + A_1)\partial_x + \mu^2(F'_1(U_*) - b + A_0) & -\mu^2 \\ \varepsilon\mu & 0 \end{pmatrix} =: \begin{pmatrix} L_1^{11}(\mu) & L_1^{12}(\mu) \\ L_1^{21}(\mu) & 0 \end{pmatrix}.$$

The operator $L_0(\mu)$ represents a principal part which is responsible for the damping, whereas $L_1(\mu)$ contains higher-order corrections which do not interfere with the damping from $L_0(\mu)$. In particular, we will see that the precise form of U_* does not matter, and the coefficients from the exponential weight are irrelevant as long as B_0 is non-positive.

Our strategy is to invert $L(\mu) := L_0(\mu) + L_1(\mu)$ by using the factorization

$$L(\mu) = [1 + L_1(\mu)L_0(\mu)^{-1}]L_0(\mu),$$

inverting $L_0(\mu)$ explicitly, and showing that $[1 + L_1(\mu)L_0(\mu)^{-1}]$ may be inverted via a Neumann series. Since $L_0(\mu)$ is diagonal, we can invert it by inverting each block separately. The first block may be inverted by standard spectral theory, since $\pm i$ is not in the spectrum of ∂_{XX} on $L^p(\mathbb{R})$. For the second block, we can use an integrating factor to write the inverse explicitly as

$$\left(c\partial_X - \tilde{b}\mu - \frac{i\kappa}{\mu} + \mu B_0 \right)^{-1} G_2(X) = \int_{\mathbb{R}} e^{\frac{i\kappa}{\mu c}(X-Y)} G_\mu^{\text{tr}}(X, Y) G_2(Y) dY,$$

where

$$G_\mu^{\text{tr}}(X, Y) = \begin{cases} \frac{1}{c} e^{\frac{\tilde{b}\mu(X-Y)}{c}} e^{\frac{\mu}{c} \int_X^Y B_0(Z) dZ}, & X \leq Y, \\ 0, & X > Y. \end{cases} \quad (\text{B.3})$$

Exploiting the fact that $B_0(Z)$ is non-positive, we readily obtain the following estimates.

Lemma B.1. *Let $1 \leq p \leq \infty$. There exists a constant $C > 0$ such that, for $\mu > 0$ sufficiently small, we have the estimates*

$$\|(L_0^{11})^{-1}\|_{L^p \rightarrow W^{2,p}} \leq C, \quad \|L_0^{22}(\mu)^{-1}\|_{L^p \rightarrow L^p} \leq \frac{C}{|\mu|}. \quad (\text{B.4})$$

We also readily obtain the following basic estimates on $L_1(\mu)$.

Lemma B.2. *Let $1 \leq p \leq \infty$. There exists a constant $C > 0$ such that, for $\mu > 0$ sufficiently small, we have the estimates*

$$\|L_1^{11}(\mu)\|_{W^{1,p} \rightarrow L^p} \leq C|\mu|, \quad \|L_1^{12}(\mu)\|_{L^p \rightarrow L^p} \leq C|\mu|^2, \quad \|L_1^{21}(\mu)\|_{L^p \rightarrow L^p} \leq C|\mu|.$$

We may then invert $[1 + L_1(\mu)L_0(\mu)^{-1}]$ via the Neumann series

$$[1 + L_1(\mu)L_0(\mu)^{-1}]^{-1} = \sum_{n=0}^{\infty} \left(-L_1(\mu)L_0(\mu)^{-1}\right)^n,$$

provided $\mu > 0$ is sufficiently small.

Lemma B.3. *Let $1 \leq p \leq \infty$. The series*

$$\sum_{n=0}^{\infty} (-L_1(\mu)L_0(\mu)^{-1})^n$$

converges in the space of bounded operators on $L^p(\mathbb{R})$, provided $\mu > 0$ is sufficiently small. Hence, for $\mu > 0$ sufficiently small, the operator $[1 + L_1(\mu)L_0(\mu)^{-1}]$ is invertible on $L^p(\mathbb{R})$, with its inverse given by this Neumann series. Moreover, there exists a constant $C > 0$ such that

$$\left\| \sum_{n=3}^{\infty} (-L_1(\mu)L_0(\mu)^{-1})^n \right\|_{L^p \rightarrow L^p} \leq C|\mu|^3. \quad (\text{B.5})$$

Proof. Combining Lemmas B.1 and B.2, we find that $L_1(\mu)L_0(\mu)^{-1}$ may be written in the form

$$L_1(\mu)L_0(\mu)^{-1} = \begin{pmatrix} \mu B_{11}(\mu) & \mu B_{12}(\mu) \\ \mu B_{21}(\mu) & 0 \end{pmatrix},$$

where $B_{ij}(\mu)$ are operators on L^p , which are μ -uniformly bounded. Hence, there exists a μ -independent constant $C > 0$ such that

$$\|[L_1(\mu)L_0(\mu)^{-1}]^n\|_{L^p \rightarrow L^p} \leq (C|\mu|)^n,$$

which implies that the series converges for $|\mu| < \frac{1}{C}$, and that the estimate (B.5) holds. $\square$

Collecting relevant terms. In order to shift the integration contour in (4.1), we want to express the solution to the resolvent equation (B.1) as an explicit leading-order part, plus a remainder which is integrable in Ω . From Lemma B.3, we can get an expansion of the solution to (B.1) in powers of $\mu = |\Omega|^{-1/2}$. To get an integrable remainder, we therefore need to explicitly capture all terms up to order μ^2 , since $\mu^3 = |\Omega|^{-3/2}$ is integrable near $|\Omega| = \infty$. Thus, we employ Lemma B.3 to express the solution to the rescaled resolvent equation (B.2) as

$$L(\mu)^{-1} \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix} = L_0(\mu)^{-1} \sum_{n=0}^{\infty} (-L_1(\mu)L_0(\mu)^{-1})^n \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix}. \quad (\text{B.6})$$

By Lemmas B.1 and B.2, we see that

$$L_0(\mu)^{-1} \sim \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \quad L_1(\mu) \sim \begin{pmatrix} \mu & \mu^2 \\ \mu & 0 \end{pmatrix},$$

which we can use to check the order in μ of the terms in (B.6). The first term, corresponding to $n = 0$, in (B.6), is

$$\begin{pmatrix} U_0(\mu) \\ V_0(\mu) \end{pmatrix} = L_0(\mu)^{-1} \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \mu^2 \\ \mu \end{pmatrix} \sim \begin{pmatrix} \mu^2 \\ 1 \end{pmatrix}.$$

The next term, corresponding to $n = 1$ in (B.6), is

$$\begin{pmatrix} \tilde{U}_1(\mu) \\ \tilde{V}_1(\mu) \end{pmatrix} = -L_0(\mu)^{-1} L_1(\mu) \begin{pmatrix} U_0(\mu) \\ V_0(\mu) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \mu & \mu^2 \\ \mu & 0 \end{pmatrix} \begin{pmatrix} \mu^2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} \mu^2 \\ \mu^2 \end{pmatrix}.$$

We denote this term by $(\tilde{U}_1, \tilde{V}_1)^\top$ since it contains some higher-order terms which we will not compute explicitly, and so we will later extract the principal contributions U_1, V_1 . The term corresponding to $n = 2$ in (B.6) is

$$\begin{pmatrix} U_2(\mu) \\ V_2(\mu) \end{pmatrix} = -L_0(\mu) L_1(\mu) \begin{pmatrix} \tilde{U}_1(\mu) \\ \tilde{V}_1(\mu) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} \mu & \mu^2 \\ \mu & 0 \end{pmatrix} \begin{pmatrix} \mu^2 \\ \mu^2 \end{pmatrix} \sim \begin{pmatrix} \mu^3 \\ \mu^2 \end{pmatrix}. \quad (\text{B.7})$$

The residual error from truncating the series at $n = 2$ will then be of order $\mu^3 = \Omega^{-3/2}$.

Explicitly computing (U_0, V_0) , we find

$$\begin{pmatrix} U^0(X, \mu) \\ V^0(X, \mu) \end{pmatrix} = L_0(\mu)^{-1} \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}} G^p(X - Y) \mu^2 G_1(Y) dY \\ \mu \int_{\mathbb{R}} e^{\frac{i\kappa}{\mu c}(X - Y)} G_\mu^{\text{tr}}(X, Y) G_2(Y) dY \end{pmatrix}, \quad (\text{B.8})$$

where G_μ^{tr} is given by (B.3), and

$$G^p(X) = \frac{1}{2\sqrt{i\kappa}} e^{-\sqrt{i\kappa}|X - Y|} \quad (\text{B.9})$$

is the spatial Green's function corresponding to the resolvent operator $(\partial_{XX} - i\kappa)^{-1}$.

Translating back to the original coordinates $u(\xi)$ and $v(\xi)$, these leading-order terms are then given by

$$u_0(\xi) = \frac{1}{\sqrt{|\Omega|}} \int_{\mathbb{R}} G^p \left(\sqrt{|\Omega|}(\xi - \zeta) \right) g_1(\zeta) d\zeta, \quad (\text{B.10})$$

$$v_0(\xi) = \int_{\mathbb{R}} \tilde{G}^{\text{tr}}(\xi, \zeta) e^{i\frac{\Omega}{c}(\xi - \zeta)} g_2(\zeta) d\zeta, \quad (\text{B.11})$$

where

$$\tilde{G}^{\text{tr}}(\xi, \zeta) = \begin{cases} \frac{1}{c} e^{\frac{\tilde{b}(\xi - \zeta)}{c}} e^{\frac{1}{c} \int_{\xi}^{\zeta} b_0(z) dz}, & \xi \leq \zeta, \\ 0, & \xi > \zeta. \end{cases} \quad (\text{B.12})$$

The next term is

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{V}_1 \end{pmatrix} = -L_0(\mu)^{-1} L_1(\mu) \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = -L_0(\mu)^{-1} \begin{pmatrix} L_1^{11}(\mu) U_0 - \mu^2 V_0 \\ \varepsilon \mu U_0 \end{pmatrix}.$$

Notice from (B.8) and (B.4) that $\|U_0\|_{W^{2,p}} \lesssim |\mu|^2 \|G_1\|_{L^p}$ for $G_1 \in C_c^\infty(\mathbb{R})$ and $\mu > 0$ sufficiently small. Then, also applying Lemma B.2, we find

$$\|L_1^{11}(\mu)U_0\|_{L^p} \lesssim |\mu|^3 \|\mathbf{G}\|_{L^p}, \quad (\text{B.13})$$

for $\mathbf{G} = (G_1, G_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $\mu > 0$ sufficiently small. Hence we identify the leading-order contributions as

$$\begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} \mu^2(\partial_{XX} - i\kappa)^{-1}V_0 \\ -\varepsilon\mu L_0^{22}(\mu)^{-1}U_0 \end{pmatrix},$$

while the term $-(L_0^{11})^{-1}L_1^{11}(\mu)U_0$ is higher order. Returning to original coordinates $u(\xi)$ and $v(\xi)$, we obtain the corresponding expressions

$$u_1(\xi) = \frac{1}{\sqrt{|\Omega|}} \int_{\mathbb{R}} G^p \left(\sqrt{|\Omega|}(\xi - \zeta) \right) \int_{\mathbb{R}} e^{i\frac{\Omega}{c}(\zeta - z)} \tilde{G}^{\text{tr}}(\zeta, z) g_2(z) dz d\zeta, \quad (\text{B.14})$$

$$v_1(\xi) = -\frac{\varepsilon}{\sqrt{|\Omega|}} \int_{\mathbb{R}} \tilde{G}^{\text{tr}}(\xi, \zeta) e^{i\frac{\Omega}{c}(\xi - \zeta)} \int_{\mathbb{R}} G^p \left(\sqrt{|\Omega|}(\zeta - z) \right) g_1(z) dz d\zeta. \quad (\text{B.15})$$

For the final explicit term, corresponding to $n = 2$ in (B.6), we only need to compute V_2 explicitly, since $U_2 = O(\mu^3)$ as argued before. In X coordinates, following the formal calculation (B.7) while keeping track of which entries are relevant, we have

$$V_2 = -\varepsilon\mu^4 L_0^{22}(\mu)^{-1} \left[(L_0^{11})^{-1} \left(L_0^{22}(\mu)^{-1} G_2 \right) \right].$$

Note from Lemma B.1 that each factor of $L_0^{22}(\mu)^{-1}$ contributes a factor of μ^{-1} in norm, so V_2 is in fact of order μ^2 rather than μ^4 . Reverting to ξ coordinates, we obtain

$$v_2(\xi) = -\varepsilon \int_{\mathbb{R}^3} \tilde{G}^{\text{tr}}(\xi, \zeta) e^{i\frac{\Omega}{c}(\xi - \zeta)} \frac{G^p(\sqrt{|\Omega|}(\zeta - z))}{\sqrt{|\Omega|}} \tilde{G}^{\text{tr}}(z, w) e^{i\frac{\Omega}{c}(z - w)} g_2(w) dw dz d\zeta. \quad (\text{B.16})$$

It follows from the proof of Lemma B.3 that all remaining terms in the Neumann series expansion are $O(\mu^3)$, and so do not need to be computed explicitly. We now translate this remainder estimate into the original coordinates.

Lemma B.4. *There exist constants $C, \Omega_0 > 0$ such that for each $\lambda = b + i\Omega \in \mathbb{C}$ with $-\frac{3}{4}\varepsilon\gamma \leq b \leq \eta$ and $|\Omega| \geq \Omega_0$, the solution $(u(\xi; \lambda), v(\xi; \lambda))$ to the resolvent equation (B.1) with $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ obeys the estimate*

$$\left\| \begin{pmatrix} u(\cdot; \lambda) \\ v(\cdot; \lambda) \end{pmatrix} - \begin{pmatrix} u_0(\cdot; \lambda) \\ v_0(\cdot; \lambda) \end{pmatrix} - \begin{pmatrix} u_1(\cdot; \lambda) \\ v_1(\cdot; \lambda) \end{pmatrix} - \begin{pmatrix} 0 \\ v_2(x; \lambda) \end{pmatrix} \right\|_{L^p} \leq C\Omega^{-\frac{3}{2}} \|\mathbf{g}\|_{L^p},$$

where (u_0, v_0) are given by (B.10)-(B.11), (u_1, v_1) are given by (B.14)-(B.15), and v_2 is given by (B.16).

Proof. The remainder terms, in X coordinates introduced above, are given by

$$\begin{aligned} \mathbf{U}_{\text{res}}(\mu) &= \begin{pmatrix} U_{\text{res}}(\mu) \\ V_{\text{res}}(\mu) \end{pmatrix} = -L_0(\mu)^{-1} \begin{pmatrix} L_1^{11}(\mu)U_0 \\ 0 \end{pmatrix} \\ &\quad + P_1 L_0(\mu)^{-1} (-L_1(\mu) L_0(\mu)^{-1})^2 \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix} + L_0(\mu)^{-1} \sum_{n=3}^{\infty} (-L_1(\mu) L_0(\mu)^{-1})^n \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix}, \end{aligned}$$

where $P_1(U, V)^\top = (U, 0)^\top$ is the projection onto the first coordinate, with this term accounting for the fact that we computed V_2 but not U_2 explicitly.

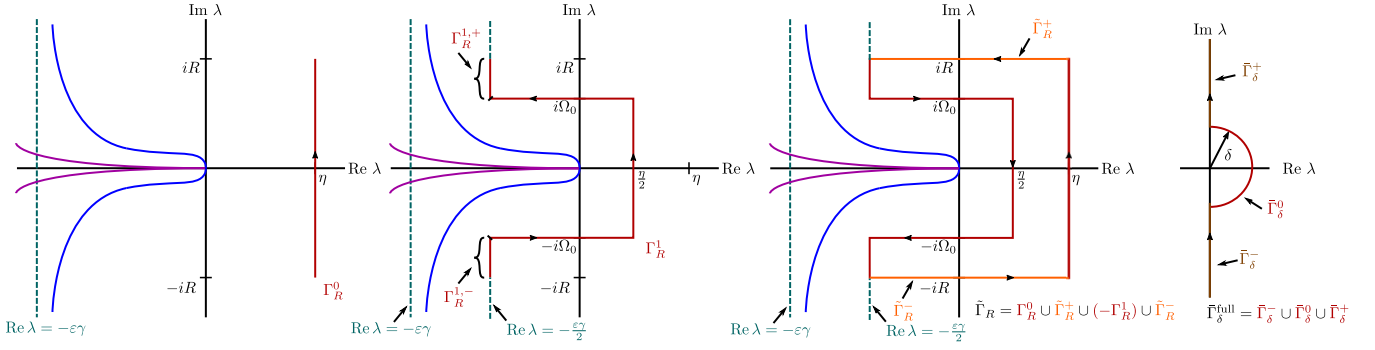


Figure 6: Far left: Fredholm borders of $\mathcal{L}$ (blue, purple) and starting contour Γ_R^0 (dark red) from Corollary 4.2. Left center: the new contour Γ_R^1 (dark red). Right center: the closed contour $\tilde{\Gamma}_R$. Far right: the contour $\tilde{\Gamma}_\delta^{\text{full}}$.

Using the estimates of Lemmas B.1 and B.2, we have

$$\left\| P_1 L_0(\mu)^{-1} (-L_1(\mu) L_0(\mu)^{-1})^2 \begin{pmatrix} \mu^2 G_1 \\ \mu G_2 \end{pmatrix} \right\|_{L^p} \lesssim |\mu|^3 \|\mathbf{G}\|_{L^p}.$$

for $\mu > 0$ sufficiently small and $\mathbf{G} = (G_1, G_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$. Together with (B.13) and (B.5), we obtain

$$\|\mathbf{U}_{\text{res}}(\cdot; \mu)\|_{L^p} \lesssim |\mu|^3 \|\mathbf{G}\|_{L^p},$$

for $\mu > 0$ sufficiently small and $\mathbf{G} = (G_1, G_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$. Hence, reverting to ξ coordinates with $\mathbf{u}_{\text{res}}(\xi; \lambda) = \mathbf{U}_{\text{res}}(X; \mu)$, we arrive at

$$\|\mathbf{u}_{\text{res}}(\cdot; \lambda)\|_{L^p} \lesssim \Omega^{-\frac{3}{2}} \|\mathbf{g}\|_{L^p},$$

for $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$ and $\lambda = b + i\Omega \in \mathbb{C}$ with $-\frac{3}{4}\varepsilon\gamma \leq b \leq \eta$ and $|\Omega|$ sufficiently large. Note that transforming back to ξ coordinates yields a factor of $|\mu|^{-1/p}$ in the L^p norm, but this factor is present on both sides of the above estimate, and hence can be canceled. $\square$

B.2 Shifting the integration contour

Let $\eta > 0$ be as in Corollary 4.2. Letting $g_1, g_2 \in C_c^\infty(\mathbb{R})$ and

$$\begin{pmatrix} u \\ v \end{pmatrix} = (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

for $\lambda \in \rho(\mathcal{L}_{\text{fr}})$, we may write

$$e^{\mathcal{L}_{\text{fr}} t} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\eta - iR}^{\eta + iR} e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda,$$

for $t > 0$. Our goal in this section is to use Lemma B.4 to shift this contour into the left half-plane, except for a region bounded near the origin. To do this, we first fix Ω_0 sufficiently large so that Lemma B.4 applies for $|\Omega| \geq \Omega_0$. We then let Γ_R^0 , Γ_R^1 , and $\tilde{\Gamma}_R$ be the contours depicted in Figure 6.

Lemmas B.1 and B.4 show that the only term in the expansion of $(u(\cdot; \lambda), v(\cdot; \lambda))^\top$ that does not carry a factor $|\Omega|^{-1/2}$ is the term $(0, v_0(\cdot; \lambda))^\top$. However, this term also vanishes as $|\Omega| \rightarrow \infty$ by the Riemann-Lebesgue lemma. We conclude that the contributions from the horizontal segments on the top and bottom of $\tilde{\Gamma}_R$ vanish in the limit $R \rightarrow \infty$, so that we have

$$\lim_{R \rightarrow \infty} \int_{\tilde{\Gamma}_R} e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda = \lim_{R \rightarrow \infty} \left(\int_{\Gamma_R^0} - \int_{\Gamma_R^1} \right) e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda.$$

Since

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R^0} e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda$$

exists for $g_1, g_2 \in C_c^\infty(\mathbb{R})$ by Corollary 4.2, we will have

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R^1} e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda = \lim_{R \rightarrow \infty} \int_{\Gamma_R^0} e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda,$$

with the limit on the left-hand side existing, provided

$$\lim_{R \rightarrow \infty} \int_{\tilde{\Gamma}_R} e^{\lambda t} \begin{pmatrix} u(\lambda) \\ v(\lambda) \end{pmatrix} d\lambda = 0.$$

But this integral is zero for each $R > 0$ by analyticity of $(u(\lambda), v(\lambda))$, where we use that the contour $\tilde{\Gamma}_R$ lies fully to the right of the spectrum of $\mathcal{L}_{\text{fr}}$; see Figure 6. Hence we may write

$$e^{\mathcal{L}_{\text{fr}} t} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma_R^1} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} d\lambda,$$

for $g_1, g_2 \in C_c^\infty(\mathbb{R})$ and $t > 0$. The goal of the remainder of this section is to obtain the following estimate on the integral over $\Gamma_R^{1,+} \cup \Gamma_R^{1,-}$, which will (together with the subsequent analysis of the resolvent near the origin) allow us to extend the contour definition of the semigroup to $\mathbf{g} \in L^p(\mathbb{R}, \mathbb{R}^2)$, $1 \leq p < \infty$ or $\mathbf{g} \in C_0(\mathbb{R}, \mathbb{R}^2)$ and prove exponential decay estimates for the high frequency contributions.

Proposition B.5. *Fix $1 \leq p \leq \infty$. Provided Ω_0 is sufficiently large, there exists a constant $C > 0$ such that*

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} (\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g} d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon\gamma}{4}t} \|\mathbf{g}\|_{L^p}$$

for all $t > 0$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

To prove this proposition, we split up $(\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g}$ into several terms according to the decomposition in Lemma B.4 and estimate the integral for each term separately.

Lemma B.6 (u_0 estimates). *Fix $1 \leq p \leq \infty$. Let u_0 be as in Lemma B.4. There exists a constant $C > 0$ such that*

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} u_0(\lambda) d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon\gamma}{4}t} \|\mathbf{g}\|_{L^p}$$

for all $t > 0$ and for all $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

Proof. Recalling that the contour $\Gamma_R^{1,-}$ is the line segment connecting the point $-\varepsilon\gamma/2 - iR$ with $-\varepsilon\gamma/2 - i\Omega_0$ and $\Gamma_R^{1,+}$ joins $-\varepsilon\gamma/2 + i\Omega_0$ with $-\varepsilon\gamma/2 + iR$, we have by (B.10)

$$\left[\lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} u_0(\lambda) d\lambda \right] (\xi) = i e^{-\frac{\varepsilon\gamma}{2}t} \int_{|\Omega| > \Omega_0} e^{i\Omega t} \frac{1}{\sqrt{|\Omega|}} \int_{\mathbb{R}} G^p \left(\sqrt{|\Omega|}(\xi - \zeta) \right) g_1(\zeta) d\zeta d\Omega,$$

where we use the notation

$$\int_{|\Omega| > \Omega_0} f(\Omega) d\Omega = \int_{-\Omega_0}^{-\infty} f(\Omega) d\Omega + \int_{\Omega_0}^{\infty} f(\Omega) d\Omega,$$

that is, the orientation is consistent with the second panel of Figure 6.

Using the formula (B.9) for G^P , we obtain

$$\left[\lim_{R \rightarrow \infty} \int_{\Gamma_R^{\text{new},+} \cup \Gamma_R^{\text{new},-}} e^{\lambda t} u_0(\lambda) d\lambda \right] (\xi) = i e^{-\frac{\varepsilon\gamma}{2}t} \int_{|\Omega| > \Omega_0} e^{i\Omega t} \int_{\mathbb{R}} \frac{e^{-\sqrt{i\Omega}|\xi-\zeta|}}{2\sqrt{i\Omega}} g_1(\zeta) d\zeta d\Omega.$$

We swap the order of integration and rewrite the inner integral as

$$\begin{aligned} i e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_1(\zeta) \int_{|\Omega| > \Omega_0} e^{i\Omega t} \frac{e^{-\sqrt{i\Omega}|\xi-\zeta|}}{2\sqrt{i\Omega}} d\Omega d\zeta &= e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_1(\zeta) \int_{\bar{\Gamma}_{\Omega_0}^{\text{full}}} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda d\zeta \\ &\quad - e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_1(\zeta) \int_{\bar{\Gamma}_{\Omega_0}^0} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda d\zeta, \end{aligned} \quad (\text{B.17})$$

where for any $\delta > 0$ we denote by $\bar{\Gamma}_{\delta}^0$ the right semi-circle centered at the origin with radius δ , and by $\bar{\Gamma}_{\delta}^{\text{full}}$ the contour that joins this semi-circle to the vertical rays extending to $\pm i\infty$; see Figure 6. Since the singularity $\lambda^{-1/2}$ is integrable, we find that

$$\lim_{\delta \rightarrow 0^+} \int_{\bar{\Gamma}_{\delta}^{\text{full}}} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda = \text{i p.v.} \int_{\mathbb{R}} e^{i\Omega t} \frac{e^{-\sqrt{i\Omega}|\xi-\zeta|}}{2\sqrt{i\Omega}} d\Omega = i \sqrt{\frac{\pi}{t}} e^{-|\xi-\zeta|^2/4t} =: iG^{\text{heat}}(t, \xi - \zeta),$$

recognizing the integral as the inverse Fourier transform (in time) of the Fourier representation of the heat kernel. On the other hand, the value of the integral is independent of $\delta > 0$ since the integrand is analytic away from the origin in the closed right half-plane. In particular, we obtain

$$\int_{\bar{\Gamma}_{\Omega_0}^{\text{full}}} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda = iG^{\text{heat}}(t, \xi - \zeta).$$

Now to bound the integral over $\bar{\Gamma}_{\Omega_0}^0$ in (B.17) we use that the integrand is analytic away from the negative real axis to deform $\bar{\Gamma}_{\Omega_0}^0$ to a new contour

$$\bar{\Gamma}^2 = \bar{\Gamma}_-^1 \cup \bar{\Gamma}_{\delta_0}^0 \cup \bar{\Gamma}_+^1$$

where $\delta_0 = \frac{\varepsilon\gamma}{4}$, $\bar{\Gamma}_-^1$ is the line segment joining $-i\Omega_0$ with $-i\delta_0$, and $\bar{\Gamma}_+^1$ is the line segment joining $i\delta_0$ with $i\Omega_0$. There exist (ε -, Ω_0 - and γ -dependent) constants $C, \alpha > 0$ such that for $\lambda \in \bar{\Gamma}^2$ we have

$$|\lambda|^{-1/2} \left| e^{-\sqrt{\lambda}|\xi-\zeta|} \right| \leq C e^{-\alpha|\xi-\zeta|}, \quad (\text{B.18})$$

while also $|e^{\lambda t}| \leq e^{\frac{\varepsilon\gamma}{4}t}$. Applying Young's convolution inequality we obtain

$$\left\| \xi \mapsto \int_{\mathbb{R}} g_1(\zeta) \int_{\bar{\Gamma}^2} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda d\zeta \right\|_{L^p} \lesssim e^{\frac{\varepsilon\gamma}{4}t} \|g_1\|_{L^p},$$

for $t > 0$ and $g_1 \in C_c^\infty(\mathbb{R})$. Absorbing the slowly exponentially growing factor into the factor of $e^{-\frac{\varepsilon\gamma}{2}t}$ from (B.17) gives the desired estimate for the second term

$$i e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_1(\zeta) \int_{\bar{\Gamma}_{\Omega_0}^0} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda d\zeta = i e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_1(\zeta) \int_{\bar{\Gamma}^2} e^{\lambda t} \frac{e^{-\sqrt{\lambda}|\xi-\zeta|}}{2\sqrt{\lambda}} d\lambda d\zeta,$$

in (B.17). Similarly, the estimate

$$\left\| \xi \mapsto \int_{\mathbb{R}} g_1(\zeta) iG^{\text{heat}}(t, \xi - \zeta) d\zeta \right\|_{L^p} \lesssim \|G^{\text{heat}}(t, \cdot)\|_{L^1} \|g_1\|_{L^p} \lesssim \|g_1\|_{L^p} \lesssim \|\mathbf{g}\|_{L^p}$$

for $t > 0$ and $\mathbf{g} = (g_1, g_2)^\top \in \mathbb{C}_c^\infty(\mathbb{R}, \mathbb{R}^2)$, leads to the desired estimate for the remaining term in (B.17), completing the proof. $\square$

Before estimating the other contributions identified in Lemma B.4, we first record some basic estimates on $\tilde{G}^{\text{tr}}$ which can be readily inferred from (B.12).

Lemma B.7 ($\tilde{G}^{\text{tr}}$ estimates). *Let $\tilde{G}^{\text{tr}}(\xi, \zeta)$ be defined by (B.12). There exists a constant $C > 0$ such that*

$$\|\tilde{G}^{\text{tr}}(\cdot, \cdot)\|_{L^\infty(\mathbb{R}^2, \mathbb{R})} \leq C, \quad \sup_{\zeta \in \mathbb{R}} \|\tilde{G}^{\text{tr}}(\cdot, \zeta)\|_{L^1} \leq C.$$

As a consequence, it follows by Minkowski's integral inequality that for any $1 \leq p \leq \infty$ and $g_2 \in C_c^\infty(\mathbb{R})$, we have

$$\left\| \int_{\mathbb{R}} \tilde{G}^{\text{tr}}(\cdot, \zeta) g_2(\zeta) d\zeta \right\|_{L^p} \leq \left(\sup_{\zeta \in \mathbb{R}} \|\tilde{G}^{\text{tr}}(\cdot, \zeta)\|_{L^1} \right) \|g_2\|_{L^p} \leq C \|g_2\|_{L^p}.$$

Lemma B.8 (v_0 estimates). *Fix $1 \leq p \leq \infty$. Let v_0 be as in Lemma B.4. There exists a constant $C > 0$ such that*

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} v_0(\lambda) d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon\gamma}{2}t} \|\mathbf{g}\|_{L^p}$$

for all $t > 0$ and all $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

Proof. Here we must take advantage of the symmetry in the principal value integral. The indefinite integrals here are all to be understood in the principal value sense. We have by (B.11)

$$\left[\lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} v_0(\lambda) d\lambda \right] (\xi) = i e^{-\frac{\varepsilon\gamma}{2}t} \int_{|\Omega| > \Omega_0} e^{i\Omega t} \int_{\mathbb{R}} \tilde{G}^{\text{tr}}(\xi, \zeta) e^{i\frac{\Omega}{c}(\xi - \zeta)} g_2(\zeta) d\zeta d\Omega$$

where $\tilde{G}^{\text{tr}}(\xi, \zeta)$ is given in (B.12). Swapping the order of integration, we have

$$\left[\int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} v_0(\lambda) d\lambda \right] (\xi) = i e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_2(\zeta) \tilde{G}^{\text{tr}}(\xi, \zeta) \int_{|\Omega| > \Omega_0} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega d\zeta.$$

Note that

$$\begin{aligned} \text{p.v.} \int_{|\Omega| > \Omega_0} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega &= \text{p.v.} \int_{\mathbb{R}} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega - \int_{|\Omega| \leq \Omega_0} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega \\ &= c 2\pi \delta(\xi - \zeta + ct) - \int_{|\Omega| \leq \Omega_0} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega \end{aligned}$$

Hence

$$\begin{aligned} \left[\int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} v_0(\lambda) d\lambda \right] (\xi) &= i e^{-\frac{\varepsilon\gamma}{2}t} \left(2\pi c \int_{\mathbb{R}} g_2(\zeta) \tilde{G}^{\text{tr}}(\xi, \zeta) \delta(\xi - \zeta + ct) d\zeta \right. \\ &\quad \left. - \int_{\mathbb{R}} g_2(\zeta) \tilde{G}^{\text{tr}}(\xi, \zeta) \int_{|\Omega| \leq \Omega_0} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega d\zeta \right). \end{aligned}$$

For the first term, we have

$$2\pi c e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_2(\zeta) \tilde{G}^{\text{tr}}(\xi, \zeta) \delta(\xi - \zeta + ct) d\zeta = 2\pi c e^{-\frac{\varepsilon\gamma}{2}t} g_2(\xi + ct) \tilde{G}^{\text{tr}}(\xi, \xi + ct).$$

Measuring in $L^p(\mathbb{R})$, we obtain

$$\left\| 2\pi c e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_2(\zeta) \tilde{G}^{\text{tr}}(\cdot, \zeta) \delta(\cdot - \zeta + ct) d\zeta \right\|_{L^p} \lesssim e^{-\frac{\varepsilon\gamma}{2}t} \|g_2\|_{L^p} \|\tilde{G}^{\text{tr}}\|_{L^\infty} \lesssim e^{-\frac{\varepsilon\gamma}{2}t} \|\mathbf{g}\|_{L^p},$$

for $t > 0$ and $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$. For the second term, we use Lemma B.7 to obtain

$$\begin{aligned} \left\| e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} g_2(\zeta) \tilde{G}^{\text{tr}}(\xi, \zeta) \int_{|\Omega| \leq \Omega_0} e^{i\frac{\Omega}{c}(\xi - \zeta + ct)} d\Omega d\zeta \right\|_{L^p} &\leq 2\Omega_0 e^{-\frac{\varepsilon\gamma}{2}t} \|g_2\|_{L^p} \left(\sup_{\zeta \in \mathbb{R}} \|\tilde{G}^{\text{tr}}(\cdot, \zeta)\|_{L^1} \right) \\ &\lesssim e^{-\frac{\varepsilon\gamma}{2}t} \|\mathbf{g}\|_{L^p}, \end{aligned}$$

for $t > 0$ and $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$, which concludes the proof of the desired estimate. $\square$

Lemma B.9 (Estimates on u_1 and v_1). *Fix $1 \leq p \leq \infty$. Let u_1 and v_1 be as in Lemma B.4. There exists a constant $C > 0$ such that*

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} u_1(\lambda) d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon\gamma}{4}t} \|\mathbf{g}\|_{L^p},$$

and

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} v_1(\lambda) d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon\gamma}{4}t} \|\mathbf{g}\|_{L^p},$$

for all $t > 0$ and all $\mathbf{g} = (g_1, g_2)^\top \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

Proof. We will prove the cases $p = 1$ and $p = \infty$ directly, the estimates for $1 < p < \infty$ then following by interpolation. Using (B.14) and swapping the order of integration, we have

$$\left[\lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} u_1(\lambda) d\lambda \right] (\xi) = i e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{G}^{\text{tr}}(\zeta, z) g_2(z) \int_{|\Omega| > \Omega_0} e^{i\Omega(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{i\Omega}|\xi - \zeta|}}{2\sqrt{i\Omega}} d\Omega dz d\zeta.$$

As in the proof of Lemma B.6, we would like to close the integration contour so that we may recognize the contour integral as an inverse Fourier transform. However, in this case we can only evaluate this inverse Fourier transform provided $t + \frac{\zeta - z}{c} \geq 0$, in which case we will obtain the heat kernel evaluated at time $t + \frac{\zeta - z}{c} \geq 0$. Since $\tilde{G}^{\text{tr}}(\zeta, z)$ is supported on $\{\zeta \leq z\}$, the quantity $t + \frac{\zeta - z}{c}$ will be negative for some portions of the overall integral, and so we estimate the integral over these portions separately. Defining the set

$$E_t = \left\{ (\zeta, z) \in \mathbb{R}^2 : t + \frac{\zeta - z}{c} \geq 0 \right\},$$

and denoting by E_t^c its complement in $\mathbb{R}^2$, we write

$$\left[\lim_{R \rightarrow \infty} \int_{\Gamma_R^{1,+} \cup \Gamma_R^{1,-}} e^{\lambda t} u_1(\lambda) d\lambda \right] (\xi) = \frac{i}{e^{\frac{\varepsilon\gamma}{2}t}} \left(\int_{E_t} + \int_{E_t^c} \right) \left[\tilde{G}^{\text{tr}}(\zeta, z) g_2(z) \int_{|\Omega| > \Omega_0} e^{i\Omega(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{i\Omega}|\xi - \zeta|}}{2\sqrt{i\Omega}} d\Omega dz d\zeta \right].$$

For $(\zeta, z) \in E_t$, arguing as in the proof of Lemma B.6, we find

$$i \int_{|\Omega| > \Omega_0} e^{i\Omega(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{i\Omega}|\xi - \zeta|}}{2\sqrt{i\Omega}} d\Omega = i G^{\text{heat}} \left(t + \frac{\zeta - z}{c}, \xi - \zeta \right) - \int_{\bar{\Gamma}^2} e^{\lambda(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{\lambda}|\xi - \zeta|}}{2\sqrt{\lambda}} d\lambda.$$

Using Minkowski's inequality and Lemma B.7 together with the estimates (B.18) and $|e^{\lambda(t + (\zeta - z)/c)}| \leq e^{\frac{\varepsilon\gamma}{4}t}$ for $\lambda \in \bar{\Gamma}^2$, $t \geq 0$ and $(\zeta, z) \in E_t$ with $\zeta \leq z$, we obtain

$$\left\| \xi \mapsto \int_{E_t} \tilde{G}^{\text{tr}}(\zeta, z) g_2(z) \int_{|\Omega| > \Omega_0} e^{i\Omega(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{i\Omega}|\xi - \zeta|}}{2\sqrt{i\Omega}} d\Omega dz d\zeta \right\|_{L^1} \lesssim e^{\frac{\varepsilon\gamma}{4}t} \|g_2\|_{L^1},$$

for $t > 0$ and $g_2 \in C_c^\infty(\mathbb{R})$. The exponentially growing factor may again be absorbed by the original factor of $e^{-\frac{\varepsilon\gamma}{2}t}$. By a similar argument, we obtain the desired control in $L^\infty(\mathbb{R})$ as well.

It remains to control the integral over E_t^c . For $(\zeta, z) \in E_t^c$, we again add an additional piece to close the integration contour, writing

$$i \int_{|\Omega| > \Omega_0} e^{i\Omega(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{i\Omega}|\xi - \zeta|}}{2\sqrt{i\Omega}} d\Omega = \left(\int_{\bar{\Gamma}_{\Omega_0}^{\text{full}}} - \int_{\bar{\Gamma}_{\Omega_0}^0} \right) \left[e^{\lambda(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{\lambda}|\xi - \zeta|}}{2\sqrt{\lambda}} d\lambda \right].$$

Since $\text{Re } \lambda \geq 0$ for $\lambda \in \bar{\Gamma}_{\Omega_0}^0$ and $t + \frac{\zeta - z}{c} < 0$ if $(\zeta, z) \in E_t^c$, there exist t -, ζ -, z -, and ξ -independent constants $C, \alpha > 0$ such that

$$\left| \int_{\bar{\Gamma}_{\Omega_0}^0} e^{\lambda(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{\lambda}|\xi - \zeta|}}{2\sqrt{\lambda}} d\lambda \right| \leq C e^{-\alpha|\xi - \zeta|},$$

for $t \geq 0$ and $(\zeta, z) \in E_t^c$, from which the desired estimates follow readily for this term via Minkowski's inequality. For fixed $\eta > 0$, the integral over $\bar{\Gamma}_{\Omega_0}^{\text{full}}$ can be deformed to the contour $\text{Re } \lambda = \eta$, since the integrand is analytic away from the negative real axis and vanishes as $\text{Im } \lambda \rightarrow \infty$, from which we obtain

$$\int_{\bar{\Gamma}_{\Omega_0}^{\text{full}}} e^{\lambda(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{\lambda}|\xi - \zeta|}}{2\sqrt{\lambda}} d\lambda = e^{\eta(t + \frac{\zeta - z}{c})} \text{p.v.} \int_{-\infty}^{\infty} e^{i\Omega(t + \frac{\zeta - z}{c})} \frac{e^{-\sqrt{\eta + i\Omega}|\xi - \zeta|}}{2\sqrt{\eta + i\Omega}} d\Omega.$$

The remaining integral is uniformly bounded in $\eta \geq 1, t > 0, \xi \in \mathbb{R}$ and $(\zeta, z) \in E_t^c$ as a principal value integral. Hence, sending $\eta \rightarrow \infty$, we conclude that the contribution from $\bar{\Gamma}_{\Omega_0}^{\text{full}}$ vanishes. The proof of the estimate involving v_1 proceeds analogously. $\square$

Lemma B.10 (Estimate on v_2). *Fix $1 \leq p \leq \infty$. Let v_2 be as in Lemma B.4. There exists a constant $C > 0$ such that*

$$\left\| \lim_{R \rightarrow \infty} \int_{\Gamma_R^{\text{new},+} \cup \Gamma_R^{\text{new},-}} e^{\lambda t} v_2(\lambda) d\lambda \right\|_{L^p} \leq C e^{-\frac{\varepsilon\gamma}{4}t} \|\mathbf{g}\|_{L^p}$$

for all $t > 0$ and $\mathbf{g} \in C_c^\infty(\mathbb{R}, \mathbb{R}^2)$.

Proof. The proof is analogous to that of Lemma B.9, except that it involves an additional convolution with kernel $e^{i\frac{\Omega}{c}} G^{\text{tr}}$. As a result we recognize the principal contribution, in the inverse Fourier transform, as the heat kernel convolved twice with two separate Dirac delta distributions, leading to an integral of the form

$$e^{-\frac{\varepsilon\gamma}{2}t} \int_{\mathbb{R}^3} \tilde{G}^{\text{tr}}(\xi, \zeta) G^{\text{heat}} \left(t - \frac{\xi - \zeta + z - w}{c}, \zeta - z \right) \tilde{G}^{\text{tr}}(z, w) g_2(w) dw d\zeta dz.$$

Applying Lemma B.7 and estimating the innermost integral with the heat kernel scaling in a similar manner to the proof of Lemma B.9 then gives the desired estimate. $\square$

Having estimated the principal terms arising in the expansion of the resolvent we are now able to complete the proof of Proposition B.5.

Proof of Proposition B.5. We decompose $(\mathcal{L}_{\text{fr}} - \lambda)^{-1} \mathbf{g}$ as in Lemma B.4. The estimates on terms involving u_0, v_0, u_1, v_1 , and v_2 have already been established in Lemmas B.6-B.10. The remainder terms are controlled in $L^p(\mathbb{R})$ by $|\Omega|^{-3/2}$ by Lemma B.4, which is integrable on $|\Omega| > \omega_0$, completing the proof. $\square$

C Estimates on the leading edge resolvent

From the analysis in Section 6.2, we know that for σ^2 to the right of $\Sigma_{\text{ess}}(\mathcal{L}_+)$, we can write the leading edge resolvent $(\mathcal{L}_+ - \sigma^2)^{-1}$ as

$$(\mathcal{L}_+ - \sigma^2)^{-1} \mathbf{g}(\xi) = \int_{\mathbb{R}} \Pi_{13} T_{\sigma}^{\text{fr}}(\xi - \zeta) \Lambda_1 \mathbf{g}(\zeta) d\zeta,$$

with $\mathbf{g} = (g_1, g_2)^{\top}$ and

$$T_{\sigma}^{\text{fr}}(\zeta) = G_{\sigma}^{\text{heat}}(\zeta) P_{\text{pole}} + \tilde{G}_{\sigma}^c(\zeta) + [G_{\sigma}^c(\zeta) - G_{\sigma}^{\text{heat}}(\zeta) P_{\text{pole}}] + G_{\sigma}^h(\zeta),$$

where

$$\begin{aligned} G_{\sigma}^{\text{heat}}(\zeta) &= \frac{1}{\sigma} e^{-\nu_{\text{fr}}^1 \sigma |\zeta|}, \\ G_{\sigma}^c(\zeta) &= \frac{1}{\sigma} P_{\text{pole}} \left(e^{\nu_{\text{fr}}^+(\sigma) \zeta} 1_{\{\zeta < 0\}} + e^{\nu_{\text{fr}}^-(\sigma) \zeta} 1_{\{\zeta \geq 0\}} \right), \\ \tilde{G}_{\sigma}^c(\zeta) &= \left[e^{\nu_{\text{fr}}^-(\sigma) \zeta} \left(P_{\text{fr}}^{\text{cs}}(\sigma) - \frac{P_{\text{pole}}}{\sigma} \right) \right] 1_{\{\zeta \geq 0\}} - \left[e^{\nu_{\text{fr}}^+(\sigma) \zeta} \left(P_{\text{fr}}^{\text{cu}}(\sigma) + \frac{P_{\text{pole}}}{\sigma} \right) \right] 1_{\{\zeta < 0\}}, \\ G_{\sigma}^h(\zeta) &= -e^{\nu_3(\sigma^2) \zeta} P_{\text{fr}}^{\text{uu}}(\sigma^2) 1_{\{\zeta \leq 0\}}. \end{aligned}$$

If $\mathbf{g}$ is odd, then we can rewrite

$$G_{\sigma}^{\text{heat}} * \mathbf{g}(\xi) = \int_0^{\infty} G_{\sigma}^{\text{odd}}(\xi, \zeta) \mathbf{g}(\zeta) d\zeta, \quad (\text{C.1})$$

for $\xi \in \mathbb{R}$, where

$$G_{\sigma}^{\text{odd}}(\xi, \zeta) = \frac{1}{\sigma} \left[e^{-\nu_{\text{fr}}^1 \sigma |\xi - \zeta|} - e^{-\nu_{\text{fr}}^1 \sigma |\xi + \zeta|} \right].$$

Note that $G_{\sigma}^{\text{odd}}(-\xi, \zeta) = -G_{\sigma}^{\text{odd}}(\xi, \zeta)$, so that $G_{\sigma}^{\text{heat}} * \mathbf{g}$ is odd provided $\mathbf{g}$ is.

In the following four lemmas we obtain bounds on G_{σ}^{odd} , G_{σ}^c , $\tilde{G}_{\sigma}^c$ and G_{σ}^h , which eventually lead to the proofs of Lemmas 6.7 and 6.8.

Lemma C.1. *There exist positive constants C and δ such that*

$$|G_{\sigma}^{\text{odd}}(\xi, \zeta)| + |\partial_{\xi} G_{\sigma}^{\text{odd}}(\xi, \zeta)| \leq C \langle \zeta \rangle e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\xi - \zeta|},$$

for all $\xi, \zeta \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$.

Proof. We start with the estimate with no derivative. First, assume $\xi \geq \zeta \geq 0$, so that $|\xi - \zeta| = \xi - \zeta$. Then, we have

$$G_{\sigma}^{\text{odd}}(\xi, \zeta) = \frac{1}{\sigma} e^{-\nu_{\text{fr}}^1 \sigma |\xi - \zeta|} \left[1 - e^{-2\nu_{\text{fr}}^1 \sigma \zeta} \right].$$

Using that there exists a constant $C > 0$ such that $|1 - e^z| \leq C|z|$ for all $z \in \mathbb{C}$ with $\text{Re } z \leq 0$, we have $|1 - e^{-2\nu_{\text{fr}}^1 \sigma \zeta}| \lesssim |\sigma| |\zeta|$, and so

$$|G_{\sigma}^{\text{odd}}(\xi, \zeta)| \lesssim |\zeta| e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\xi - \zeta|} \leq \langle \zeta \rangle e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\xi - \zeta|},$$

for $\xi \geq \zeta \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$.

Now assume $\zeta \geq \xi \geq 0$, so that $|\xi - \zeta| = \zeta - \xi$. Then we have

$$G_\sigma^{\text{odd}}(\xi, \zeta) = \frac{1}{\sigma} e^{-\nu_{\text{fr}}^1 \sigma |\xi - \zeta|} \left[1 - e^{-2\nu_{\text{fr}}^1 \sigma \xi} \right].$$

Since $\text{Re}(-2\nu_{\text{fr}}^1 \sigma \xi) \leq 0$, we have $|1 - e^{-2\nu_{\text{fr}}^1 \sigma \xi}| \lesssim |\sigma| |\xi|$ for $\xi \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$. Since in the present regime, $|\xi| \leq |\zeta| \leq \langle \zeta \rangle$, we conclude

$$|G_\sigma^{\text{odd}}(\xi, \zeta)| \lesssim |\xi| e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\xi - \zeta|} \lesssim \langle \zeta \rangle e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\xi - \zeta|},$$

for $\zeta \geq \xi \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$, which completes the proof of the desired estimate for $|G_\sigma^{\text{odd}}(\xi, \zeta)|$.

For the derivative estimates, note that

$$\partial_\xi G_\sigma^{\text{odd}}(\xi, \zeta) = -\nu_{\text{fr}}^1 e^{-\nu_{\text{fr}}^1 \sigma |\xi - \zeta|} \text{sign}(\xi - \zeta) + e^{-\nu_{\text{fr}}^1 \sigma |\xi + \zeta|} \text{sign}(\xi + \zeta),$$

Since we are assuming $\xi, \zeta \geq 0$, we have $|\xi + \zeta| \geq |\xi - \zeta|$, so we readily obtain the stronger estimate $|\partial_\xi G_\sigma^{\text{odd}}(\xi, \zeta)| \lesssim e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\xi - \zeta|}$ for $\xi, \zeta \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$. $\square$

Lemma C.2. *There exist positive constants C and δ such that*

$$|G_\sigma^c(\zeta) - G_\sigma^{\text{heat}}(\zeta) P_{\text{pole}}| + |\partial_\zeta (G_\sigma^c(\zeta) - G_\sigma^{\text{heat}}(\zeta) P_{\text{pole}})| \leq C |\sigma| |\zeta| e^{-\frac{1}{2} \nu_{\text{fr}}^1 \text{Re } \sigma |\zeta|}$$

for all $\zeta \in \mathbb{R}$ provided $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$.

Proof. We assume $\zeta \geq 0$; the case $\zeta < 0$ is similar. In this case, we have

$$G_\sigma^c(\zeta) - G_\sigma^{\text{heat}}(\zeta) P_{\text{pole}} = \frac{P_{\text{pole}}}{\sigma} e^{-\nu_{\text{fr}}^1 \sigma \zeta} \left[e^{\tilde{\nu}_{\text{fr}}^-(\sigma) \zeta} - 1 \right],$$

where $\tilde{\nu}_{\text{fr}}^-(\sigma) = O(\sigma^2)$. First we assume $|\sigma^2 \zeta| \leq 1$, in which case, there exists by Taylor's theorem a σ - and ζ -independent constant $C > 0$ such that $|e^{\tilde{\nu}_{\text{fr}}^-(\sigma) \zeta} - 1| \leq C |\sigma|^2 |\zeta|$. So, we have

$$|G_\sigma^c(\zeta) - G_\sigma^{\text{heat}}(\zeta) P_{\text{pole}}| \lesssim |\sigma| |\zeta| e^{-\nu_{\text{fr}}^1 \text{Re } \sigma |\zeta|}$$

for $\zeta \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$ and $|\sigma^2 \zeta| \leq 1$.

On the other hand, the expansion (6.12) yields

$$|G_\sigma^c(\zeta) - G_\sigma^{\text{heat}}(\zeta) P_{\text{pole}}| \lesssim \frac{|\sigma|^2 |\zeta|}{|\sigma|} (|e^{\nu_{\text{fr}}^-(\sigma) \zeta}| + |e^{-\nu_{\text{fr}}^1 \sigma \zeta}|) \lesssim |\sigma| |\zeta| e^{-\frac{1}{2} \nu_{\text{fr}}^1 \text{Re } \sigma |\zeta|},$$

for $\zeta \geq 0$ and $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$ and $|\sigma^2 \zeta| \geq 1$. So we obtain the desired estimate in either case. The estimate on the derivative is similar, in fact easier since we gain a factor of σ after differentiating. $\square$

Lemma C.3. *There exist positive constants C and δ such that*

$$|\tilde{G}_\sigma^c(\zeta)| + |\partial_\zeta \tilde{G}_\sigma^c(\zeta)| \leq C e^{-\frac{1}{2} \nu_{\text{fr}}^1 \text{Re } \sigma |\zeta|},$$

for all $\zeta \in \mathbb{R}$ provided $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$.

Proof. This follows readily from the following facts: 1. $P_{\text{fr}}^{\text{cs}}(\sigma) - \frac{P_{\text{pole}}}{\sigma}$ and $P_{\text{fr}}^{\text{cu}}(\sigma) + \frac{P_{\text{pole}}}{\sigma}$ are analytic in σ in a full neighborhood of the origin by Lemma 6.4; 2. we have $\text{Re } \nu_{\text{fr}}^-(\sigma) \leq -\frac{1}{2} \nu_{\text{fr}}^1 \text{Re } \sigma$ and $\text{Re } \nu_{\text{fr}}^+(\sigma) \geq \frac{1}{2} \nu_{\text{fr}}^1 \text{Re } \sigma$ for $\sigma \in B(0, \sqrt{\delta})$ with $\text{Re } \sigma \geq 0$. $\square$

Lemma C.4. *There exist positive constants C, δ , and μ such that*

$$|G_\sigma^h(\zeta)| + |\partial_\zeta G_\sigma^h(\zeta)| \leq C e^{-\mu|\zeta|},$$

for all $\zeta \in \mathbb{R}$ provided $\sigma \in B(0, \sqrt{\delta})$.

Proof. This follows from the fact that the eigenvalue $\nu_3(\sigma^2)$ stays in the open right half-plane, uniformly bounded away from the imaginary axis, for $|\sigma|$ small, which in turn follows from Corollary 3.4. $\square$

We are now able to prove Lemmas 6.7 and 6.8, which provide control over the leading edge resolvent.

Proof of Lemma 6.7. Let $\mathbf{u}$ satisfy $(\mathcal{L}_+ - \sigma^2)\mathbf{u} = \mathbf{g}$.

First we prove (6.17). Observe that there exists a constant $\tilde{\mu} > 0$ such that

$$-\operatorname{Re} \sigma \leq -\tilde{\mu}|\sigma|, \quad (\text{C.2})$$

for all $\sigma \in \Delta_\delta^{\text{fr},2}$. Hence, combining Lemmas C.1 through C.4 and recalling (C.1) and the fact that we have $G_\sigma^{\text{odd}}(-\xi, \zeta) = -G_\sigma^{\text{odd}}(\xi, \zeta)$ for $\xi \in \mathbb{R}$ and $\zeta \geq 0$, we obtain a constant $\mu > 0$ such that the estimate

$$|\mathbf{u}(\xi; \sigma)| + |\partial_\xi \mathbf{u}(\xi; \sigma)| \lesssim \int_{\mathbb{R}} e^{-\mu|\sigma||\xi-\zeta|} \langle \zeta \rangle |\mathbf{g}(\zeta)| d\zeta \quad (\text{C.3})$$

holds for $\sigma \in \Delta_\delta^{\text{fr},2}$ and $\xi \in \mathbb{R}$. Recalling that $\mathbf{g}$ is odd, we obtain

$$\|\mathbf{u}(\cdot; \sigma)\|_{W^{1,\infty}} \lesssim \|\mathbf{g}\|_{L_{0,1}^1}$$

for $\sigma \in \Delta_\delta^{\text{fr},2}$. To upgrade to an $W^{2,\infty}$ estimate in the first component, note that if $\mathbf{u} = (u, v)^\top$, then we also have

$$u_\xi(\xi; \sigma) = \int_{\mathbb{R}} \Pi_2 T_\sigma^{\text{fr}}(\xi - \zeta) \Lambda_1 \begin{pmatrix} g_1(\zeta) \\ g_2(\zeta) \end{pmatrix} d\zeta. \quad (\text{C.4})$$

Hence we can apply the same argument to estimate u_ξ in $W^{1,\infty}$, and thereby get the desired $W^{2,\infty}$ control on u , which completes the proof of estimate (6.17).

To prove (6.18), note that $G^{\text{odd}}(\xi, \zeta) = G^{\text{odd}}(\zeta, \xi)$ and, thus, Lemma C.1 yields

$$|G_\sigma^{\text{odd}}(\xi, \zeta)| \leq C \langle \xi \rangle e^{-\nu_{\text{fr}}^{-1} \operatorname{Re} \sigma |\xi - \zeta|},$$

for $\xi, \zeta \geq 0$ and $\sigma \in \Delta_\delta^{\text{fr},2}$. Then, combining the latter with Lemmas C.2 through C.4, while recalling (C.1) and (C.2) and the fact that $G_\sigma^{\text{odd}}(-\xi, \zeta) = -G_\sigma^{\text{odd}}(\xi, \zeta)$ holds for $\xi \in \mathbb{R}$ and $\zeta \geq 0$, we establish

$$|\mathbf{u}(\xi; \sigma)| + |\partial_\xi \mathbf{u}(\xi; \sigma)| \lesssim \int_{\mathbb{R}} \langle \xi \rangle |\mathbf{g}(\zeta)| d\zeta,$$

for $\sigma \in \Delta_\delta^{\text{fr},2}$ and $\xi \in \mathbb{R}$. We can control $\chi_+(\xi) \langle \xi \rangle$ with the weight $\rho_{0,-1}(\xi)$ in the $L_{0,-1}^\infty$ norm, leading readily to (6.18), again extending to higher regularity by examining u_ξ through (C.4). $\square$

Proof of Lemma 6.8. As above, combining Lemmas C.1 through C.4 gives the pointwise estimate (C.3). Using Young's convolution inequality we readily obtain

$$\|\mathbf{u}(\cdot, \sigma)\|_{L^2} \lesssim \|\mathbf{g}\|_{L_{1,1}^1} \|e^{-\mu|\sigma|\cdot}\|_{L^2} \lesssim \frac{1}{|\sigma|^{\frac{1}{2}}} \|\mathbf{g}\|_{L_{0,1}^1},$$

for $\sigma \in \Delta_\delta^{\text{fr},2}$. $\square$

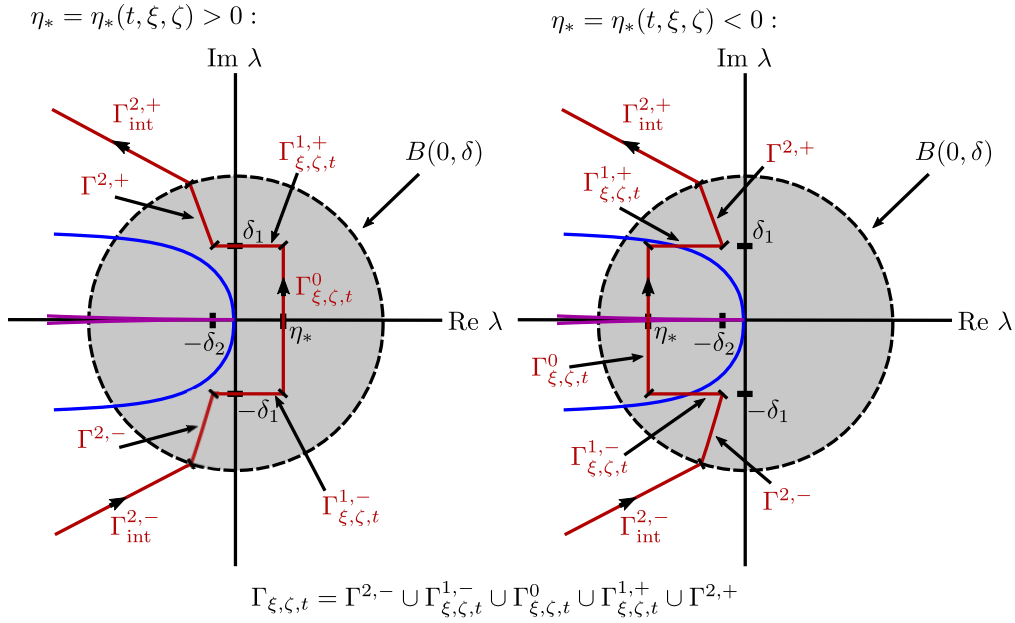


Figure 7: Fredholm borders of $\mathcal{L}_{\text{fr}}$ (blue, purple), together with pointwise contours $\Gamma_{\xi,\zeta,t}^0$ (dark red), in the cases where $\eta_*(t, \xi, \zeta) > 0$ (left) and where $\eta_*(t, \xi, \zeta) < 0$ (right). Note that the figure is not to scale, since in particular we require $|\eta_*| + \delta_2 \ll \delta_1^2$.

D Proof of pointwise estimates

In this appendix, we give a detailed exposition of a proof of Proposition 5.4. We emphasize that the proof we give to a large extent rephrases arguments from [77, Section 8]. Note that we only have to consider $\xi - \zeta \leq 0$ due to the presence of the term $\chi_-(\xi - \zeta)$ on the left-hand side of (5.13). Recall from Proposition 3.1 that the critical Floquet exponent $\nu_{\text{wt}}(\lambda) = \nu(\lambda)$ occurring in (5.13) is analytic in λ on $B(0, \delta)$, has positive real part for $\lambda \in B(0, \delta)$ lying to the right of $\Sigma(\mathcal{L}_{\text{wt}})$, and admits the expansion

$$\nu(\lambda) = \nu_1 \lambda - \nu_2 \lambda^2 + O(\lambda^3), \quad (\text{D.1})$$

with coefficients $\nu_{1,2} > 0$, where we have abbreviated $\nu(\lambda) = \nu_{\text{wt}}(\lambda)$ and $\nu_{1,2} = \nu_{\text{wt}}^{1,2}$ for simplicity of notation. We further assume that $t > 1$, since estimates for small times may be readily obtained using any fixed contour in $B(0, \delta)$, which lies to the right of $\Sigma(\mathcal{L}_{\text{wt}})$ and connects λ_0^* to λ_0 . Indeed, for $t \in [0, 1]$, λ in such a contour, and $\xi \leq \zeta$ we have that $\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta))$ is uniformly bounded, which readily leads to the estimate (5.13) for $t \in [0, 1]$.

Choice of pointwise contour. The basic idea is to choose a contour which approximately minimizes the exponent $\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta))$ in (5.13). We start by computing the minimum of the quadratic approximation $\text{Re}[\lambda t + (\nu_1 \lambda - \nu_2 \lambda^2)(\xi - \zeta)]$, restricting to $\lambda \in \mathbb{R}$. From a simple calculation, we find that the minimum is attained at $\lambda = \lambda_{\min}(\xi, \zeta, t)$ with

$$\lambda_{\min}(\xi, \zeta, t) = \frac{t + \nu_1(\xi - \zeta)}{2\nu_2(\xi - \zeta)}.$$

The objective is to integrate along a short segment of the line $\text{Re } \lambda = \lambda_{\min}$, which then approximately minimizes the exponent $\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta))$.

We first fix $\delta_1 > 0$ small, and then fix $\delta_0, \delta_2 > 0$ such that $\delta_0 + \delta_2 \ll \delta_1^2$, and such that the points $-\delta_2 + i\delta_1$

lie in $B(0, \delta)$ to the right of $\Sigma_{\text{ess}}(\mathcal{L}_{\text{wt}})$. We define

$$\eta_*(t, \xi, \zeta) = \begin{cases} \lambda_{\min}, & |\lambda_{\min}| \leq \delta_0, \\ \delta_0, & \lambda_{\min} > \delta_0, \\ -\delta_0, & \lambda_{\min} < -\delta_0. \end{cases}$$

Let $\Gamma_{\xi, \zeta, t}^0$ be the straight line segment from $\eta_* - i\delta_1$ to $\eta_* + i\delta_1$. Let $\Gamma_{\xi, \zeta, t}^{1,+}$ be the line segment connecting $\eta_* + i\delta_1$ to $-\delta_2 + i\delta_1$, and let $\Gamma^{2,+}$ be a fixed curve to the right of $\Sigma_{\text{ess}}(\mathcal{L}_{\text{wt}})$ which connects $-\delta_2 + i\delta_1$ to λ_0 . Similarly, let $\Gamma_{\xi, \zeta, t}^{1,-}$ be the line segment connecting $\eta_* - i\delta_1$ to $-\delta_2 - i\delta_1$, and let $\Gamma^{2,-}$ be a fixed curve to the right of $\Sigma_{\text{ess}}(\mathcal{L}_{\text{wt}})$ which connects $-\delta_2 - i\delta_1$ to λ_0^* . Let

$$\Gamma_{\xi, \zeta, t} = \Gamma^{2,-} \cup \Gamma_{\xi, \zeta, t}^{1,-} \cup \Gamma_{\xi, \zeta, t}^0 \cup \Gamma_{\xi, \zeta, t}^{1,+} \cup \Gamma_{\xi, \zeta, t}^{2,+}.$$

See Figure 7 for schematic of the contours. Note in particular that when $\eta_* < 0$, the contour $\Gamma_{\xi, \zeta, t}$ passes into the essential spectrum of $\mathcal{L}_{\text{fr}}$.

To prove Proposition 5.4, we estimate the contribution from each portion of the contour, breaking into the cases $|\lambda_{\min}| \leq \delta_0$, and $|\lambda_{\min}| > \delta_0$. Throughout, we will use the expansion

$$\text{Re } \nu(\lambda) = \nu_1 \text{Re } \lambda - \nu_2 \left[(\text{Re } \lambda)^2 - (\text{Im } \lambda)^2 \right] + O(|\lambda|^3), \quad (\text{D.2})$$

obtained by taking the real part of (D.1).

Case I: $|\lambda_{\min}| \leq \delta_0$.

Lemma D.1. *Fix non-negative integers ℓ and j , and let g satisfy the assumptions of Proposition 5.4 for some non-negative integer m . Suppose $t > 1$ and $\xi, \zeta \in \mathbb{R}$ are such that $\xi \leq \zeta$ and $|\lambda_{\min}(\xi, \zeta, t)| \leq \delta_0$. Then, there exist ξ -, ζ - and t -independent constants $C, M > 0$ such that*

$$\int_{\Gamma_{\xi, \zeta, t}^0} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_-(\xi - \zeta) |d\lambda| \leq C \frac{\chi_-(\xi - \zeta)}{(1+t)^{\frac{1}{2} + \frac{\ell+j+m}{2}}} e^{-\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt}}.$$

Proof. For $\lambda \in \Gamma_{\xi, \zeta, t}^0$, we have $\text{Re } \lambda = \eta_*$, so by (D.2) we have

$$\text{Re } \nu(\lambda) = \nu_1 \eta_* - \nu_2 \eta_*^2 + \nu_2 (\text{Im } \lambda)^2 + O(|\lambda|^3).$$

Noting that $\eta_* = \lambda_{\min}$ in the regime, we find after some algebra

$$\eta_* t + \nu_1 \eta_*(\xi - \zeta) - \nu_2 \eta_*^2 (\xi - \zeta) = -\frac{\nu_1^2 (\xi - \zeta + \nu_1^{-1}t)^2}{\nu_2 4|\xi - \zeta|},$$

and so we have

$$\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta)) = -\frac{\nu_1^2 (\xi - \zeta + \nu_1^{-1}t)^2}{\nu_2 4|\xi - \zeta|} + \nu_2 (\text{Im } \lambda)^2 (\xi - \zeta) + O(|\lambda|^3)(\xi - \zeta).$$

The assumption $|\lambda_{\min}| \leq \delta_0$ implies that there exist t -, ξ - and ζ -independent constants $C_1, C_2 > 0$ such that

$$C_1(\zeta - \xi) \leq t \leq C_2(\zeta - \xi), \quad (\text{D.3})$$

We obtain

$$\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta)) \leq -\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt} - \mu_1 (\text{Im } \lambda)^2 t + O(|\lambda|^3)(\xi - \zeta)$$

for some λ -, t -, ξ - and ζ -independent constants $M, \mu_1 > 0$. For $\lambda \in \Gamma_{\xi, \zeta, t}^0$ we have $|\lambda|^2 = \eta_*^2 + (\text{Im } \lambda)^2$. There then exists a λ -, t -, ξ - and ζ -independent constant $C_3 > 0$ such that $|\lambda|^3 \leq C_3[|\eta_*|^3 + |\text{Im } \lambda|^3]$ for $\lambda \in \Gamma_{\xi, \zeta, t}^0$. Hence, we arrive at

$$\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta)) \leq -\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt} - \mu_1(\text{Im } \lambda)^2 t + \mu_2[|\eta_*|^3 + |\text{Im } \lambda|^3]t$$

for some λ -, t -, ξ - and ζ -independent constant $\mu_2 > 0$. Since $\text{Im } \lambda$ is small, we can absorb $\mu_2|\text{Im } \lambda|^3 t$ into $-\mu_1(\text{Im } \lambda)^2 t$ at the cost of changing μ_1 . Similarly, note that (D.3) yields

$$|\eta_*|^2 t \leq \frac{C_2^2 \nu_1^2}{4\nu_2^2} \frac{(\xi - \zeta + \nu_1^{-1}t)^2}{t}. \quad (\text{D.4})$$

Then, since $|\eta_*| < \delta_0$, we can absorb the factor of $\mu_2|\eta_*|^3 t$ into the first term by adjusting M , provided $\delta_0 > 0$ is sufficiently small. Altogether, we obtain

$$e^{\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta))} \leq e^{-\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt}} e^{-\mu_1(\text{Im } \lambda)^2 t}$$

for some λ -, t -, ξ - and ζ -independent constants $M, \mu_1 > 0$.

For the contour integral over $\Gamma_{\xi, \zeta, t}^0$, we use this together with (D.1) and the fact that, for any non-negative integer k , there exists a λ -, t -, ξ - and ζ -independent constant $C_k > 0$ such that $|\lambda|^k \leq C_k(|\eta_*|^k + |\text{Im } \lambda|^k)$ for $\lambda \in \Gamma_{\xi, \zeta, t}^0$, to find a t -, ξ - and ζ -independent constant $\tilde{C} > 0$ such that

$$\begin{aligned} \int_{\Gamma_{\xi, \zeta, t}^0} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_-(\xi - \zeta) |d\lambda| \\ \leq \tilde{C} e^{-\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt}} \int_{\Gamma_{\xi, \zeta, t}^0} (|\eta_*|^{\ell+j+m} + |\text{Im } \lambda|^{\ell+j+m}) e^{-\mu_1(\text{Im } \lambda)^2 t} |d\lambda|. \end{aligned} \quad (\text{D.5})$$

Using (D.4) and writing $z = (\xi - \zeta + \nu_1^{-1}t)/\sqrt{t}$, we establish a t -, ξ - and ζ -independent constant $C > 0$ such that

$$\begin{aligned} e^{-\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt}} \int_{\Gamma_{\xi, \zeta, t}^0} |\eta_*|^{\ell+j+m} e^{-\mu_1(\text{Im } \lambda)^2 t} |d\lambda| &\leq C \frac{|z|^{\frac{\ell+j+m}{2}}}{t^{\frac{\ell+j+m}{2}}} e^{-z^2/M} \int_{\Gamma_{\xi, \zeta, t}^0} e^{-\mu_1(\text{Im } \lambda)^2 t} |d\lambda| \\ &\leq \frac{C}{t^{\frac{\ell+j+m}{2}}} e^{-z^2/M} \int_{\Gamma_{\xi, \zeta, t}^0} e^{-\mu_1(\text{Im } \lambda)^2 t} |d\lambda|, \end{aligned}$$

upon adjusting the values of the constants C and M to absorb the polynomially growing factor in the last inequality. Now, for any non-negative integer k , we have the scaling estimate

$$\int_{\Gamma_{\xi, \zeta, t}^0} |\text{Im } \lambda|^k e^{-\mu_1(\text{Im } \lambda)^2 t} |d\lambda| \leq K \int_{-\infty}^{\infty} |\xi|^k e^{-\mu_1 \xi^2 t} d\xi \leq \frac{\tilde{C}_k}{t^{\frac{1}{2} + \frac{k}{2}}}.$$

for some constants $K, \tilde{C}_k > 0$, which are independent of ξ, ζ and t . Using these two estimates in (D.5), we obtain the desired result upon noting that $t > 1$. $\square$

Lemma D.2. *Fix non-negative integers ℓ and j , and let g satisfy the assumptions of Proposition 5.4 for some non-negative integer m . Suppose $t > 1$ and $\xi, \zeta \in \mathbb{R}$ are such that $\xi \leq \zeta$ and $|\lambda_{\min}(\xi, \zeta, t)| \leq \delta_0$. Then, there exist ξ -, ζ - and t -independent constants $C, \mu_2 > 0$ such that*

$$\int_{\Gamma_{\xi, \zeta, t}^{1, \pm}} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_-(\xi - \zeta) |d\lambda| \leq C e^{-\mu_2 t} e^{-\mu_2 |\xi - \zeta|} \chi_-(\xi - \zeta).$$

Proof. For $\lambda \in \Gamma_{\xi, \zeta, t}^{1, \pm}$, we have $|\operatorname{Im} \lambda| = \delta_1$, and $|\operatorname{Re} \lambda| \leq \max(\delta_2, \delta_0)$. Also, the assumption $|\lambda_{\min}| < \delta_0$ implies (D.3). If we assume that δ_2 and δ_0 are sufficiently small relative to δ_1 , we can, upon recalling (D.2), control all terms in

$$\begin{aligned} \operatorname{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) &= \operatorname{Re} \lambda t + \nu_1 \operatorname{Re} \lambda (\xi - \zeta) - \nu_2 (\operatorname{Re} \lambda)^2 (\xi - \zeta) + \nu_2 (\operatorname{Im} \lambda)^2 (\xi - \zeta) + O(|\lambda|^3)(\xi - \zeta) \\ &\text{by } -\mu_2 (\operatorname{Im} \lambda)^2 |\xi - \zeta| - \mu_2 (\operatorname{Im} \lambda)^2 t, \text{ from which the desired result readily follows.} \end{aligned} \quad \square$$

Case II: $|\lambda_{\min}| > \delta_0$.

Lemma D.3. *Fix non-negative integers ℓ and j , and let g satisfy the assumptions of Proposition 5.4 for some non-negative integer m . Suppose $t > 1$ and $\xi, \zeta \in \mathbb{R}$ are such that $\xi \leq \zeta$ and $|\lambda_{\min}(\xi, \zeta, t)| > \delta_0$. Then, there exist ξ -, ζ - and t -independent constants $C, \mu > 0$ such that*

$$\int_{\Gamma_{\xi, \zeta, t}^0} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\operatorname{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_{-(\xi - \zeta)} |\mathrm{d}\lambda| \leq C e^{-\mu t} e^{-\mu |\xi - \zeta|} \chi_{-(\xi - \zeta)}. \quad (\text{D.6})$$

Proof. First, we assume $\lambda_{\min} > \delta_0$, which implies

$$t < (\zeta - \xi)(\nu_1 - 2\delta_0\nu_2). \quad (\text{D.7})$$

For $\lambda \in \Gamma_{\xi, \zeta, t}^0$ in this regime, we have $\operatorname{Re} \lambda = \delta_0$. Combining these two facts with (D.2), we find

$$\begin{aligned} \operatorname{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) &= \delta_0 t + \nu_1 \delta_0 (\xi - \zeta) - \nu_2 \delta_0^2 (\xi - \zeta) + \nu_2 (\operatorname{Im} \lambda)^2 (\xi - \zeta) + O(|\lambda|^3)(\xi - \zeta) \\ &\leq \delta_0 (\zeta - \xi) [\nu_1 - 2\delta_0\nu_2 - \nu_1 + \nu_2 \delta_0] - \nu_2 (\operatorname{Im} \lambda)^2 (\zeta - \xi) + O(|\lambda|^3 |\xi - \zeta|) \\ &= -\nu_2 \delta_0^2 |\zeta - \xi| - \nu_2 (\operatorname{Im} \lambda)^2 |\zeta - \xi| + O(|\lambda|^3 |\xi - \zeta|). \end{aligned}$$

Since there exists a λ -, ξ -, ζ - and t -independent constant $C_3 > 0$ such that $|\lambda|^3 \leq C_3(|\delta_0|^3 + |\operatorname{Im} \lambda|^3)$ for $\lambda \in \Gamma_{\xi, \zeta, t}^0$, we can absorb the $O(|\lambda|^3)$ factor into the other two, obtaining

$$\operatorname{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) \leq -\mu_1 |\zeta - \xi| - \mu_2 (\operatorname{Im} \lambda)^2 |\zeta - \xi| \leq -\mu_1 |\zeta - \xi|,$$

for some λ -, ξ -, ζ - and t -independent constants $\mu_1, \mu_2 > 0$. Note that (D.7) implies that there exists a ξ -, ζ - and t -independent constant $\tilde{C} > 0$ such that $-|\zeta - \xi| \leq -\tilde{C}t$, so that we may estimate

$$\operatorname{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) \leq -\frac{\mu_1}{2} |\zeta - \xi| - \frac{\mu_1}{2} |\zeta - \xi| \leq -\frac{\mu_1}{2} |\zeta - \xi| - \tilde{C} \frac{\mu_1}{2} t,$$

from which we conclude the desired estimate (D.6) in the case $\lambda_{\min} > \delta_0$.

The argument for the case $\lambda_{\min} < -\delta_0$ is similar. Specifically, the restriction (D.7) is replaced by

$$(\zeta - \xi) < \frac{t}{\nu_1 + 2\delta_0\nu_2}, \quad (\text{D.8})$$

which leads for $\lambda \in \Gamma_{\xi, \zeta, t}^0$ to an estimate

$$\begin{aligned} \operatorname{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) &= -\delta_0 t - \nu_1 \delta_0 (\xi - \zeta) - \nu_2 \delta_0^2 (\xi - \zeta) + \nu_2 (\operatorname{Im} \lambda)^2 (\xi - \zeta) + O(|\lambda|^3)(\xi - \zeta) \\ &\leq \delta_0 t \left(-1 + \frac{\nu_1 + \delta_0\nu_2}{\nu_1 + 2\delta_0\nu_2} \right) - \nu_2 (\operatorname{Im} \lambda)^2 |\zeta - \xi| + O(|\lambda|^3 |\xi - \zeta|) \\ &\leq -M \delta_0^2 t - \nu_2 (\operatorname{Im} \lambda)^2 |\zeta - \xi| + O(|\lambda|^3 |\xi - \zeta|), \end{aligned}$$

for some λ -, ξ -, ζ -, and t -independent constant $M > 0$. Since now in this regime there exists a ξ -, ζ - and t -independent constant $\tilde{C} > 0$ such that we have $-t \leq -\tilde{C}|\zeta - \xi|$ and we have $|\lambda|^3 \leq C_3(|\delta_0|^3 + |\operatorname{Im} \lambda|^3)$ for $\lambda \in \Gamma_{\xi, \zeta, t}^0$, we can extract a factor of $-\mu_3 |\xi - \zeta|$ from the first term, and use this term together with the $-\nu_2 (\operatorname{Im} \lambda)^2 |\zeta - \xi|$ term to control the $O(|\lambda|^3)$ term, and thereby obtain

$$\operatorname{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) \leq -\mu_2 t - \mu_3 |\xi - \zeta|,$$

for some λ -, ξ -, ζ - and t -independent constants $\mu_2, \mu_3 > 0$, from which the desired estimate (D.6) follows. $\square$

Lemma D.4. Fix non-negative integers ℓ and j , and let g satisfy the assumptions of Proposition 5.4 for some non-negative integer m . Suppose $t > 1$ and $\xi, \zeta \in \mathbb{R}$ are such that $\xi \leq \zeta$ and $|\lambda_{\min}(\xi, \zeta, t)| > \delta_0$. Then, there exist ξ -, ζ - and t -independent constants $C, \mu > 0$ such that

$$\int_{\Gamma_{\xi, \zeta, t}^{1, \pm}} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_-(\xi - \zeta) |d\lambda| \leq C e^{-\mu t} e^{-\mu |\xi - \zeta|} \chi_-(\xi - \zeta).$$

Proof. First, assume $\lambda_{\min} > \delta_0$, so that (D.7) holds. For $\lambda \in \Gamma_{\xi, \zeta, t}^{1, \pm}$, we have $|\text{Im } \lambda|^2 = \delta_1^2$. Using (D.2) we arrive at

$$\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) = \text{Re } \lambda t + \nu_1 \text{Re } \lambda (\xi - \zeta) - \nu_2 (\text{Re } \lambda)^2 (\xi - \zeta) - \nu_2 \delta_1^2 |\xi - \zeta| + O(|\lambda|^3) |\xi - \zeta|$$

for $\lambda \in \Gamma_{\xi, \zeta, t}^{1, \pm}$. If $\text{Re } \lambda > 0$, then the function $t \mapsto \text{Re } \lambda t$ is increasing, while if $\text{Re } \lambda \leq 0$, then $\text{Re } \lambda t \leq 0$. Combining this with (D.7), we obtain

$$\begin{aligned} \text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) &\leq \max\{0, \text{Re } \lambda\} |\xi - \zeta| (\nu_1 - 2\delta_0 \nu_2) - \nu_1 \text{Re } \lambda |\xi - \zeta| + \nu_2 (\text{Re } \lambda)^2 |\xi - \zeta| - \nu_2 \delta_1^2 |\xi - \zeta| \\ &\quad + O(|\lambda|^3) |\xi - \zeta| \end{aligned}$$

Since $|\text{Re } \lambda| \leq \max\{\delta_0, \delta_2\}$, and we are assuming $\delta_0 + \delta_2 \ll \delta_1^2$, we conclude

$$\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) \leq -C \delta_1^2 |\xi - \zeta| \leq -\mu |\xi - \zeta| - \mu t,$$

for some λ -, ξ -, ζ - and t -independent constants $C, \mu > 0$, using (D.7) to convert some of the $-|\xi - \zeta|$ localization into $-t$ decay.

Now assume $\lambda_{\min} < -\delta_0$, so that instead (D.8) holds. Note that in this case, the contours $\Gamma_{\xi, \zeta, t}^{1, \pm}$ are contained strictly in the left half-plane. Since also $|\text{Re } \lambda| \ll |\text{Im } \lambda|^2 = \delta_1^2$ for $\lambda \in \Gamma_{\xi, \zeta, t}^{1, \pm}$, we readily obtain

$$\text{Re}(\lambda t + \nu(\lambda)(\xi - \zeta)) \leq \text{Re } \lambda t - C \delta_1^2 |\xi - \zeta| \leq -\mu t - \mu |\xi - \zeta|,$$

for some λ -, ξ -, ζ - and t -independent constants $C, \mu > 0$, which leads to the desired estimate. $\square$

Having dealt with the contours $\Gamma_{\xi, \zeta, t}^0$ and $\Gamma_{\xi, \zeta, t}^{1, \pm}$ in both the case $|\lambda_{\min}| \leq \delta_0$ and $|\lambda_0| > \delta_0$, all that remains is to estimate the integral along the contours $\Gamma^{2, \pm}$, where it is not necessary to distinguish between these cases.

Lemma D.5. Fix non-negative integers ℓ and j , and let g satisfy the assumptions of Proposition 5.4 for some non-negative integer m . There exist constants $C, \mu_3 > 0$ such that

$$\int_{\Gamma^{2, \pm}} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_-(\xi - \zeta) |d\lambda| \leq C e^{-\mu_3 t} e^{-\mu_3 |\xi - \zeta|} \chi_-(\xi - \zeta),$$

for all $t > 1$ and $\xi, \zeta \in \mathbb{R}$.

Proof. The compact contours $\Gamma^{2, \pm}$ lie in the left half-plane but strictly to the right of $\Sigma_{\text{ess}}(\mathcal{L}_{\text{wt}})$, so $e^{\text{Re } \lambda t}$ contributes uniform exponential decay in time, while $e^{\text{Re } \nu(\lambda)(\xi - \zeta)} \chi_-(\xi - \zeta)$ contributes uniform spatial localization by Proposition 3.1, leading to the desired estimate. $\square$

With the obtained control from Lemmas D.1 through D.5 on the integral on the left-hand side of (5.13) along all the different portions of the contour $\Gamma_{\xi, \zeta, t}$ we are finally able to prove Proposition 5.4.

Proof of Proposition 5.4. Combining Lemmas D.1 through D.5, we obtain constants $C, \mu, M > 0$ such that the estimate

$$\begin{aligned} \int_{\Gamma_{\xi, \zeta, t}} |\lambda|^j |\nu_{\text{wt}}(\lambda)|^\ell e^{\text{Re}(\lambda t + \nu_{\text{wt}}(\lambda)(\xi - \zeta))} |g(\xi, \zeta, \lambda)| \chi_{-(\xi - \zeta)} |d\lambda| \\ \leq C \chi_{-(\xi - \zeta)} \left(\frac{1}{(1+t)^{\frac{1}{2} + \frac{\ell+j+m}{2}}} e^{-\frac{(\xi - \zeta + \nu_1^{-1}t)^2}{Mt}} + e^{-\mu t} e^{-\mu|\xi - \zeta|} \right) \end{aligned}$$

holds for all $t > 1$ and $\xi, \zeta \in \mathbb{R}$. By a basic scaling argument, one readily finds that the right hand side satisfies the desired L^p estimates of Proposition 5.4. $\square$

E Derivation of the equation for the inverse-modulated perturbation

Following the strategy in [46, Section 4.1], we derive an equation for the inverse-modulated perturbation $\mathbf{v}(t)$ given by (8.2). First, inserting the solution $\mathbf{u}_{\text{fr}}$ in (1.5) and taking spatial derivatives yields

$$0 = D\mathbf{u}_{\text{fr}}''' + c\mathbf{u}_{\text{fr}}'' + F'(\mathbf{u}_{\text{fr}})\mathbf{u}_{\text{fr}}' = \mathcal{A}_{\text{fr}}(\mathbf{u}_{\text{fr}}'). \quad (\text{E.1})$$

Next, we set $\tilde{\mathbf{u}}(\xi, t) = \mathbf{u}(\xi - \psi(\xi, t), t)$, $\tilde{\mathbf{u}}_1(\xi, t) = \mathbf{u}_\xi(\xi - \psi(\xi, t), t)$, $\tilde{\mathbf{u}}_2(\xi, t) = \mathbf{u}_t(\xi - \psi(\xi, t), t)$ and $\tilde{\mathbf{u}}_{11}(\xi, t) = \mathbf{u}_{\xi\xi}(\xi - \psi(\xi, t), t)$. Using that the perturbed solution $\mathbf{u}(t)$ solves (1.5), we obtain

$$\begin{aligned} \tilde{\mathbf{u}}_t - D\tilde{\mathbf{u}}_{\xi\xi} - c\tilde{\mathbf{u}}_\xi - F(\tilde{\mathbf{u}}) &= -\tilde{\mathbf{u}}_1\psi_t + D(\tilde{\mathbf{u}}_1\psi_{\xi\xi} + \tilde{\mathbf{u}}_{11}\psi_\xi(2 - \psi_\xi)) + c\tilde{\mathbf{u}}_1\psi_\xi \\ &= -\tilde{\mathbf{u}}_1\psi_t + \tilde{\mathbf{u}}_2\psi_\xi + D(\tilde{\mathbf{u}}_1\psi_{\xi\xi} + \tilde{\mathbf{u}}_1\psi_\xi(1 - \psi_\xi)) - F(\tilde{\mathbf{u}})\psi_\xi \\ &\quad + (D\tilde{\mathbf{u}}_{11} + c\tilde{\mathbf{u}}_1 + F(\tilde{\mathbf{u}}) - \tilde{\mathbf{u}}_2)\psi_\xi \\ &= -\tilde{\mathbf{u}}_1\psi_t + \tilde{\mathbf{u}}_2\psi_\xi + D(\tilde{\mathbf{u}}_1\psi_\xi)_\xi - F(\tilde{\mathbf{u}})\psi_\xi. \end{aligned} \quad (\text{E.2})$$

On the other hand, recalling that $\mathbf{u}_{\text{fr}}$ solves (1.5) and applying (E.1), we compute

$$\begin{aligned} (\partial_t - \mathcal{A}_{\text{fr}})[\mathbf{u}_{\text{fr}}'\psi] &= \mathbf{u}_{\text{fr}}'\psi_t - \psi\mathcal{A}[\mathbf{u}_{\text{fr}}'] - D(2\mathbf{u}_{\text{fr}}''\psi_\xi + \mathbf{u}_{\text{fr}}'\psi_{\xi\xi}) - c\mathbf{u}_{\text{fr}}'\psi_\xi \\ &= \mathbf{u}_{\text{fr}}'\psi_t - (D\mathbf{u}_{\text{fr}}'' + c\mathbf{u}_{\text{fr}}' + F(\mathbf{u}_{\text{fr}}))\psi_\xi + F(\mathbf{u}_{\text{fr}})\psi_\xi - D(\mathbf{u}_{\text{fr}}''\psi_\xi + \mathbf{u}_{\text{fr}}'\psi_{\xi\xi}) \\ &= \mathbf{u}_{\text{fr}}'\psi_t - D(\mathbf{u}_{\text{fr}}'\psi_\xi)_\xi + F(\mathbf{u}_{\text{fr}})\psi_\xi. \end{aligned} \quad (\text{E.3})$$

Set $\mathbf{z}(t) := \omega^{-1}\mathbf{v}(t) = \tilde{\mathbf{u}}(t) - \mathbf{u}_{\text{fr}}$. We establish

$$(\partial_t - \mathcal{A}_{\text{fr}})[\mathbf{z}\psi_\xi] = \mathbf{z}_t\psi_\xi + \mathbf{z}\psi_{\xi t} - D(\mathbf{z}\psi_\xi)_{\xi\xi} - c(\mathbf{z}\psi_\xi)_\xi - F'(\mathbf{u}_{\text{fr}})\mathbf{z}. \quad (\text{E.4})$$

Next, we express the temporal and spatial derivatives of $\mathbf{z}(t)$ as

$$\mathbf{z}_t = \tilde{\mathbf{u}}_2 - \tilde{\mathbf{u}}_1\psi_t, \quad \mathbf{z}_\xi = \tilde{\mathbf{u}}_1(1 - \psi_\xi) - \mathbf{u}_{\text{fr}}',$$

implying

$$\tilde{\mathbf{u}}_2 - \mathbf{u}_{\text{fr}}' - \mathbf{z}_t = \tilde{\mathbf{u}}_1\psi_t = \frac{(\mathbf{z}_\xi + \mathbf{u}_{\text{fr}}')\psi_t}{1 - \psi_\xi}, \quad \tilde{\mathbf{u}}_1 - \mathbf{u}_{\text{fr}}' - \mathbf{z}_\xi = \tilde{\mathbf{u}}_1\psi_\xi = \frac{(\mathbf{z}_\xi + \mathbf{u}_{\text{fr}}')\psi_\xi}{1 - \psi_\xi}.$$

Hence, with the aid of (E.2), (E.3) and (E.4) we obtain

$$\begin{aligned} \tilde{\mathbf{u}}_t - D\tilde{\mathbf{u}}_{\xi\xi} - c\tilde{\mathbf{u}}_\xi - F(\tilde{\mathbf{u}}) &= -(\partial_t - \mathcal{A}_{\text{fr}})[\mathbf{u}_{\text{fr}}'\psi] - (\tilde{\mathbf{u}}_1 - \mathbf{u}_{\text{fr}}')\psi_t + \tilde{\mathbf{u}}_2\psi_\xi + D((\tilde{\mathbf{u}}_1 - \mathbf{u}_{\text{fr}}')\psi_\xi)_\xi \\ &\quad - (F(\tilde{\mathbf{u}}) - F(\mathbf{u}_{\text{fr}}))\psi_\xi \\ &= -(\partial_t - \mathcal{A}_{\text{fr}})[\mathbf{u}_{\text{fr}}'\psi] - \mathbf{z}_\xi\psi_t - \tilde{\mathbf{u}}_1\psi_\xi\psi_t + \mathbf{z}_t\psi_\xi + \tilde{\mathbf{u}}_1\psi_\xi\psi_t \\ &\quad + D\left(\frac{(\mathbf{z}_\xi + \mathbf{u}_{\text{fr}}')\psi_\xi}{1 - \psi_\xi}\right)_\xi - (F(\mathbf{u}_{\text{fr}} + \mathbf{z}) - F(\mathbf{u}_{\text{fr}}))\psi_\xi \\ &= (\partial_t - \mathcal{A}_{\text{fr}})[\mathbf{z}\psi_\xi - \mathbf{u}_{\text{fr}}'\psi] + \left(\mathbf{z}(c\psi_\xi - \psi_t) + D\left(\frac{(\mathbf{z}_\xi + \mathbf{u}_{\text{fr}}')\psi_\xi}{1 - \psi_\xi} + (\mathbf{z}\psi_\xi)_\xi\right)\right)_\xi \\ &\quad - (F(\mathbf{u}_{\text{fr}} + \mathbf{z}) - F(\mathbf{u}_{\text{fr}}) - F'(\mathbf{u}_{\text{fr}})\mathbf{z})\psi_\xi. \end{aligned}$$

Therefore, using that $\mathbf{u}_{\text{fr}}$ solves (1.5), we establish

$$\begin{aligned} (\partial_t - \mathcal{A}_{\text{fr}}) \mathbf{z} &= (\partial_t - \mathcal{A}_{\text{fr}}) (\tilde{\mathbf{u}} - \mathbf{u}_{\text{fr}}) \\ &= (\tilde{\mathbf{u}}_t - D\tilde{\mathbf{u}}_{\xi\xi} - c\tilde{\mathbf{u}}_{\xi} - F(\tilde{\mathbf{u}})) + (D\mathbf{u}_{\text{fr}}'' + c\mathbf{u}_{\text{fr}}' + F(\mathbf{u}_{\text{fr}})) + F(\tilde{\mathbf{u}}) - F(\mathbf{u}_{\text{fr}}) - F'(\mathbf{u}_{\text{fr}})\mathbf{z} \\ &= (\partial_t - \mathcal{A}_{\text{fr}}) [\mathbf{z}\psi_{\xi} - \mathbf{u}_{\text{fr}}'\psi] + \mathcal{Q}(\mathbf{z}, \psi) + \partial_{\xi}\mathcal{R}(\mathbf{z}, \psi), \end{aligned} \quad (\text{E.5})$$

where the nonlinearities $\mathcal{Q}(\mathbf{z}, \psi)$ and $\mathcal{R}(\mathbf{z}, \psi)$ are defined in (8.5) and (8.6), respectively. Finally, to establish the equation (8.4) for the inverse-modulated perturbation we multiply (E.5) with ω^{-1} and recall $\mathbf{z}(t) = \omega\mathbf{v}(t)$.

F Local Well-Posedness of the Phase Modulation

We provide a proof for Proposition 8.3 yielding local well-posedness of the phase modulation. To this end, we first prove the following preliminary result.

Lemma F.1. *Let $\tilde{\mathbf{v}}(t)$ and T_{max} be as in Proposition 8.1. The map $V: H^2(\mathbb{R}) \times [0, T_{\text{max}}) \rightarrow H^2(\mathbb{R})$ given by*

$$V(\psi, t)[\xi] = \tilde{\mathbf{v}}(\xi - \psi(\xi), t) + \mathbf{u}_{\text{fr}}(\xi - \psi(\xi)) - \mathbf{u}_{\text{fr}}(\xi),$$

is well-defined, continuous in t , and locally Lipschitz continuous in ψ (uniformly in t on compact subintervals of $[0, T_{\text{max}})$).

Proof. Fix $R > 0$. The mean value theorem and the embedding $H^1(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$ yields

$$\|V(\psi_1, t) - V(\psi_2, t)\|_{H^2} \lesssim (\|\tilde{\mathbf{v}}_{\xi}(t)\|_{W^{2,\infty}} + \|\mathbf{u}_{\text{fr}}'\|_{W^{2,\infty}}) \|\psi_1 - \psi_2\|_{H^2}, \quad (\text{F.1})$$

for $\psi_{1,2} \in H^2(\mathbb{R})$ with $\|\psi_{1,2}\|_{H^2} \leq R$ and $t \in [0, T_{\text{max}})$. Hence, using Proposition 8.1, we conclude that V is locally Lipschitz continuous in ψ (uniformly in t on compact subintervals of $[0, T_{\text{max}})$). Moreover, setting $\psi_2 = 0$ in (F.1) and noting $V(0, t) = \tilde{\mathbf{v}}(t) \in H^2(\mathbb{R})$ by Proposition 8.1, shows that V is well-defined.

Using the mean value theorem and the embedding $H^1(\mathbb{R}) \hookrightarrow L^{\infty}(\mathbb{R})$ again, we obtain

$$\begin{aligned} \|V(\psi, t) - V(\psi, s)\|_{H^2} &\lesssim \sum_{j=0}^2 \left(\left\| \left[(\partial_{\xi}^j \tilde{\mathbf{v}})(\cdot - \psi(\cdot), t) - (\partial_{\xi}^j \tilde{\mathbf{v}})(\cdot - \psi(\cdot), s) \right] - \left[\partial_{\xi}^j \tilde{\mathbf{v}}(t) - \partial_{\xi}^j \tilde{\mathbf{v}}(s) \right] \right\|_{L^2 \cap L^{\infty}} \right. \\ &\quad \left. + \left\| \partial_{\xi}^j \tilde{\mathbf{v}}(t) - \partial_{\xi}^j \tilde{\mathbf{v}}(s) \right\|_{L^2 \cap L^{\infty}} \right) (1 + \|\psi\|_{H^2}) \\ &\lesssim (\|\tilde{\mathbf{v}}_{\xi}(t) - \tilde{\mathbf{v}}_{\xi}(s)\|_{W^{2,\infty}} + \|\tilde{\mathbf{v}}(t) - \tilde{\mathbf{v}}(s)\|_{H^2}) (1 + \|\psi\|_{H^2}), \end{aligned}$$

for $\psi \in H^2(\mathbb{R})$ with $\|\psi\|_{H^2} \leq R$ and $s, t \in [0, T_{\text{max}})$. Continuity of V with respect to t now follows from Proposition 8.1. $\square$

We are now in the position to establish local well-posedness of the phase modulation.

Proposition F.2. *Let $\tilde{\mathbf{v}}(t)$ and T_{max} be as in Proposition 8.1. Moreover, let $V: H^2(\mathbb{R}) \times [0, T_{\text{max}}) \rightarrow H^2(\mathbb{R})$ be as in Lemma F.1. Finally, denote by $Y_k(\mathbb{R})$ the closed subspace $\{f \in H^k(\mathbb{R}) : f(\xi) = 0 \text{ for } \xi \in [-1, \infty)\}$ of $H^k(\mathbb{R})$. Then, there exists a maximal time $\tau_{\text{max}} \in (0, T_{\text{max}}]$ such that the integral system*

$$\begin{aligned} \psi(t) &= s_p(t)\mathbf{v}_0 + \int_0^t s_p(t-s)\mathcal{N}(V(\psi(s), s), \psi(s), \psi_t(s))ds, \\ \psi_t(t) &= \partial_t s_p(t)\mathbf{v}_0 + \int_0^t \partial_t s_p(t-s)\mathcal{N}(V(\psi(s), s), \psi(s), \psi_t(s))ds, \end{aligned} \quad (\text{F.2})$$

admits a unique maximally defined solution

$$(\psi, \psi_t) \in C([0, \tau_{\max}), Y_4(\mathbb{R}) \times Y_2(\mathbb{R})). \quad (\text{F.3})$$

Moreover, $\tau_{\max} < T_{\max}$ yields

$$\limsup_{t \nearrow \tau_{\max}} \|(\psi, \psi_t)\|_{H^4 \times H^2} = \infty. \quad (\text{F.4})$$

Finally, $\psi \in C^1([0, \tau_{\max}), Y_2(\mathbb{R}))$ is continuously differentiable with $\partial_t \psi(t) = \psi_t(t)$ for $t \in [0, \tau_{\max})$.

Proof. The proof follows from a standard contraction mapping argument applied to the integral system (F.2). We collect the relevant details. First, we observe that the identities (6.27) and (7.6) and the estimates in Theorem 7.1 imply that the propagators $s_p(t): L^1_{0,1}(\mathbb{R}) \rightarrow Y_4(\mathbb{R})$ and $\partial_t s_p(t): L^1_{0,1}(\mathbb{R}) \rightarrow Y_2(\mathbb{R})$ are well-defined and t -uniformly bounded. Second, we note that all terms in the nonlinearity $\mathcal{N}(\mathbf{v}, \psi, \psi_t)$, which are nonlinear in $\mathbf{v}$ have the prefactor $\frac{1}{\omega}$, which is exponentially localized on $[0, \infty)$. Third, all terms in $\mathcal{N}(\mathbf{v}, \psi, \psi_t)$ which contain a factor ψ or ψ_t vanish on $[-1, \infty)$ for $(\psi, \psi_t) \in Y_4(\mathbb{R}) \times Y_2(\mathbb{R})$. Hence, using Lemma F.1, the continuous embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the inequality $\|wv\|_{L^1} \leq \|w\|_{L^2} \|v\|_{L^2}$ for $w, v \in L^2(\mathbb{R})$, we conclude that the nonlinear map $\tilde{\mathcal{N}}: Y_4(\mathbb{R}) \times Y_2(\mathbb{R}) \times [0, T_{\max}) \rightarrow L^1_0(\mathbb{R})$ given by

$$\tilde{\mathcal{N}}(\psi, \psi_t, t) = \mathcal{N}(V(\psi, t), \psi, \psi_t)$$

is continuous in t and locally Lipschitz continuous in (ψ, ψ_t) , uniformly in t on compact subintervals of $[0, T_{\max})$.

Using these observations, one readily obtains with standard arguments, see e.g. [19, Proposition 4.3.3] or [55, Theorem 6.1.4], that the right-hand side of the integral system (F.2) induces a contraction mapping on $C([0, \tau], B(R))$ for some $\tau, R > 0$, where $B(R)$ is the ball of radius R in $Y_4(\mathbb{R}) \times Y_2(\mathbb{R})$. By the Banach fixed point theorem this yields a unique solution $(\psi, \psi_t) \in C([0, \tau], Y_4(\mathbb{R}) \times Y_2(\mathbb{R}))$ of (F.2). Letting $\tau_{\max} \in (0, T_{\max}]$ be the supremum of all such τ , we obtain a maximally defined solution (F.3).

The blow-up condition (F.4) can be derived by arguing by contradiction, that is, one assumes $\tau_{\max} < T_{\max}$ and (F.4) does not hold. Using a standard procedure, see [36, Proposition B.2], one then finds a constant $\delta > 0$ such that the maximally defined solution can be extended to a larger interval $[0, \tau_{\max} + \delta]$, contradicting the maximality of $\tau_{\max}$.

Finally, by Theorem 7.1 we can differentiate the integral equation for $\psi(t)$ with respect to t . Using $s_p(0) = 0$ we obtain $\partial_t \psi(t) = \psi_t(t)$ for $t \in [0, \tau_{\max})$. This completes the proof. $\square$

References

- [1] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics*, 30(1):33–76, 1978.
- [2] M. Avery. Front selection in reaction-diffusion systems via diffusive normal forms. *arXiv:2211.11829*, 2022.
- [3] M. Avery, P. Carter, B. de Rijk, and A. Scheel. Spectral stability of pattern-forming invasion fronts in the FitzHugh-Nagumo system. *In preparation*.
- [4] M. Avery, P. Carter, B. de Rijk, and A. Scheel. Spectral stability of phase wave trains in the FitzHugh-Nagumo system. *In preparation*.
- [5] M. Avery, M. Holzer, and A. Scheel. Pushed and pulled fronts in a logistic Keller-Segel model with repulsive interaction. *arXiv:2308.01754*, 2023.

- [6] M. Avery, M. Holzer, and A. Scheel. Pushed-to-pulled front transitions: continuation, speed scalings, and hidden monotonicity. *J. Nonlinear Sci.*, 33:102, 2023.
- [7] M. Avery and A. Scheel. Asymptotic stability of critical pulled fronts via resolvent expansions near the essential spectrum. *SIAM J. Math. Anal.*, 53(2):2206–2242, 2021.
- [8] M. Avery and A. Scheel. Sharp decay rates for localized perturbations to the critical front in the Ginzburg-Landau equation. *J. Dyn. Diff. Equations*, 2021.
- [9] M. Avery and A. Scheel. Universal selection of pulled fronts. *Comm. Amer. Math. Soc.*, 2:172–231, 2022.
- [10] J. Bangsund, T. Fielitz, T. Steiner, K. Shi, J. V. Sambeek, C. Clark, and R. Holmes. Formation of aligned periodic patterns during the crystallization of organic semiconductor thin films. *Nature Mat.*, 18:725–731, 2019.
- [11] M. Bär, C. Zülicke, M. Eiswirth, and G. Ertl. Theoretical modeling of spatiotemporal self-organization in a surface catalyzed reaction exhibiting bistable kinetics. *J. Chem. Phys.*, 96:8595–8604, 1992.
- [12] D. Barkley. Simplifying the complexity of pipe flow. *Phys. Rev. E*, 84:016309, 2011.
- [13] M. Beck, T. Nguyen, B. Sandstede, and K. Zumbrun. Nonlinear stability of source defects in the complex Ginzburg-Landau equation. *Nonlinearity*, 27:739–786, 2014.
- [14] M. Beck, B. Sandstede, and K. Zumbrun. Nonlinear stability of time-periodic viscous shocks. *Arch. Rational Mech. Anal.*, 196:1011–1076, 2010.
- [15] A. Bers, M. Rosenbluth, and R. Sagdeev. Handbook of plasma physics. *MN Rosenbluth and RZ Sagdeev eds*, 1(3.2), 1983.
- [16] L. Brevdo. A dynamical system approach to the absolute instability of spatially developing localized open flows and media. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 458(2022):1375–1397, 2002.
- [17] J. Bricmont and A. Kupiainen. Stability of moving fronts in the Ginzburg-Landau equation. *Communications in Mathematical Physics*, 159(2):287–318, 1994.
- [18] P. Carter and A. Scheel. Wave train selection by invasion fronts in the Fitz-Hugh Nagumo equation. *Nonlinearity*, 31:5536–5572, 2018.
- [19] T. Cazenave and A. Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.
- [20] P. Collet and J.-P. Eckmann. The existence of dendritic fronts. *Commun. Math. Phys.*, 107:39–92, 1986.
- [21] P. Collet and J.-P. Eckmann. *Instabilities and fronts in extended systems*. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1990.
- [22] B. de Rijk. Nonlinear stability and asymptotic behavior of periodic wave trains in reaction-diffusion systems against C_{ub}^2 -perturbations. *arXiv:2205.04272*, 2022.
- [23] B. de Rijk and B. Sandstede. Diffusive stability against nonlocalized perturbations of planar wave trains in reaction-diffusion systems. *J. Differential Equations*, 268(7):3392–3448, 2020.
- [24] G. Dee and J. S. Langer. Propagating pattern selection. *Phys. Rev. Lett.*, 50:383–386, 1983.

- [25] A. Doelman, B. Sandstede, A. Scheel, and G. Schneider. Propagation of hexagonal patterns near onset. *European J. Appl. Math.*, 14(1):85–110, 2003.
- [26] A. Doelman, B. Sandstede, A. Scheel, and G. Schneider. The dynamics of modulated wave trains. *Mem. Amer. Math. Soc.*, 199(934):viii+105, 2009.
- [27] J.-P. Eckmann and G. Schneider. Nonlinear stability of bifurcating front solutions for the Taylor–Couette problem. *ZAMM Z. Angew. Math. Mech.*, 80(11–12):745–753, 2000. Special issue on the occasion of the 125th anniversary of the birth of Ludwig Prandtl.
- [28] J.-P. Eckmann and G. Schneider. Non-linear stability of modulated fronts for the Swift-Hohenberg equation. *Comm. Math. Phys.*, 225(2):361–397, 2002.
- [29] J.-P. Eckmann and C. E. Wayne. Propagating fronts and the center manifold theorem. *Commun. Math. Phys.*, 136:285–307, 1991.
- [30] J.-P. Eckmann and C. E. Wayne. The non-linear stability of front solutions for parabolic partial differential equations. *Commun. Math. Phys.*, 161(2):323–334, 1994.
- [31] G. Faye and M. Holzer. Asymptotic stability of the critical Fisher–KPP front using pointwise estimates. *Zeitschrift für angewandte Mathematik und Physik*, 70(1):13, 2018.
- [32] G. Faye and M. Holzer. Asymptotic stability of the critical pulled front in a Lotka-Volterra competition model. *J. Differential Equations*, 269:6559–6601, 2020.
- [33] B. Fiedler and A. Scheel. Spatio-temporal dynamics of reaction-diffusion patterns. In M. Kirkilionis, S. Krömker, R. Rannacher, and F. Tomi, editors, *Trends in Nonlinear Analysis*, pages 23–152, Berlin, Heidelberg, 2003. Springer Berlin Heidelberg.
- [34] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.
- [35] R. A. Gardner and K. Zumbrun. The gap lemma and geometric criteria for instability of viscous shock profiles. *Communications on Pure and Applied Mathematics*, 51(7):797–855, 1998.
- [36] M. Haragus, M. A. Johnson, W. R. Perkins, and B. de Rijk. Nonlinear modulational dynamics of spectrally stable Lugiato-Lefever periodic waves. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 40(4):769–802, 2022.
- [37] M. Haragus and G. Schneider. Bifurcating fronts for the Taylor-Couette problem in infinite cylinders. *Z. Angew. Math. Phys.*, 50:120–151, 1999.
- [38] D. Henry. *Geometric theory of semilinear parabolic equations*. Number 840 in Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1981.
- [39] M. Holzer and A. Scheel. Criteria for pointwise growth and their role in invasion processes. *J. Nonlinear Sci.*, 24(1):661–709, 2014.
- [40] P. Howard. Pointwise estimates and stability for degenerate viscous shock waves. *J. reine angew. Math.*, 545:19–65, 2002.
- [41] P. Howard. Pointwise Green’s function estimates towards stability for degenerate viscous shock waves. *Comm. Partial Differential Equations*, 31:73–121, 2006.
- [42] P. Howard. Asymptotic behavior near transition fronts for equations of generalized Cahn-Hilliard form. *Commun. Math. Phys.*, 269:765–808, 2007.

- [43] P. Howard. Nonlinear stability of degenerate shock profiles. *Differential Integral Equations*, 20(5):515–560, 2007.
- [44] M. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun. Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations. *Invent. Math.*, 197(1):115–213, 2014.
- [45] M. A. Johnson, P. Noble, L. M. Rodrigues, and K. Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: nonlinear stability. *Arch. Ration. Mech. Anal.*, 207(2):693–715, 2013.
- [46] M. A. Johnson and K. Zumbrun. Nonlinear stability of spatially-periodic traveling-wave solutions of systems of reaction-diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28(4):471–483, 2011.
- [47] T. Kapitula. On the stability of traveling waves in weighted L^∞ spaces. *J. Differential Equations*, 112:179–215, 1994.
- [48] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*. Applied Mathematical Sciences. Springer New York, NY, 2013.
- [49] T. Kapitula and B. Sandstede. Stability of bright solitary-wave solutions to perturbed nonlinear schrödinger equations. *Physica D: Nonlinear Phenomena*, 124(1):58–103, 1998.
- [50] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Progress in Nonlinear Differential equations and their Applications. Birkhäuser Basel, 1995.
- [51] S. Luther, F. Fenton, B. Kornreich, A. Squires, P. Bittihn, D. Hornung, M. Zabel, J. Flanders, A. Gladuli, L. Campoy, E. Cherry, G. Luther, G. Hasenfuss, V. Krinsky, A. Pumir, R. G. Jr, and E. Bodenschatz. Low-energy control of electrical turbulence in the heart. *Nature*, 475:235–239, 2011.
- [52] C. Mascia and K. Zumbrun. Pointwise Green function bounds for shock profiles of systems with real viscosity. *Arch. Rational Mech. Anal.*, 169:177–263, 2003.
- [53] A. Mikhailov and G. Ertl. *Chemical complexity: self-organization processes in molecular systems*. The Frontiers Collection. Springer Cham, 2017.
- [54] K. Palmer. Exponential dichotomies and Fredholm operators. *Proc. Amer. Math. Soc.*, 104:149–156, 1988.
- [55] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Applied Mathematical Sciences. Springer-Verlag New York, 1983.
- [56] D. Pearson, R. Bradley, F. Motta, and P. Shipman. Producing nanodot arrays with improved hexagonal order by patterning surfaces before ion sputtering. *Phys. Rev. B*, 92:062401, 2015.
- [57] A. Pogan and A. Scheel. Instability of spikes in the presence of conservation laws. *Z. Angew. Math. Phys.*, 61(6):979–998, 2010.
- [58] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV: Analysis of Operators*. Academic Press, Inc., 1978.
- [59] L. M. Rodrigues and K. Zumbrun. Periodic-coefficient damping estimates, and stability of large-amplitude roll waves in inclined thin film flow. *SIAM J. Math. Anal.*, 48(1):268–280, 2016.
- [60] B. Sandstede. Chapter 18 - stability of travelling waves. In B. Fiedler, editor, *Handbook of Dynamical Systems*, volume 2 of *Handbook of Dynamical Systems*, pages 983–1055. Elsevier Science, 2002.

- [61] B. Sandstede and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Physica D*, 145:233–277, 2000.
- [62] B. Sandstede and A. Scheel. On the structure of spectra of modulated traveling waves. *Math. Nachr.*, 232:91–93, 2001.
- [63] B. Sandstede and A. Scheel. Defects in oscillatory media: toward a classification. *SIAM J. Appl. Dyn. Sys.*, 3:1–68, 2004.
- [64] B. Sandstede and A. Scheel. Evans function and blow-up methods in critical eigenvalue problems. *Disc. Cont. Dyn. Sys.*, 10:941–964, 2004.
- [65] B. Sandstede and A. Scheel. Relative Morse indices, Fredholm indices, and group velocities. *Disc. Cont. Dyn. Sys. A*, 20:139–158, 2008.
- [66] B. Sandstede, A. Scheel, G. Schneider, and H. Uecker. Diffusive mixing of periodic wave trains in reaction-diffusion systems. *J. Differential Equations*, 252(5):3541–3574, 2012.
- [67] D. H. Sattinger. On the stability of waves of nonlinear parabolic systems. *Adv. Math.*, 22:312–355, 1976.
- [68] A. Scheel. Coarsening fronts. *Arch. Ration. Mech. Anal.*, 181(3):505–534, 2006.
- [69] A. Scheel. Spinodal decomposition and coarsening fronts in the Cahn-Hilliard equation. *J. Dynam. Differential Equations*, 29(2):431–464, 2017.
- [70] A. Scheel and Q. Wu. Diffusive stability of Turing patterns via normal forms. *J. Dynam. Differential Equations*, 27(3-4):1027–1076, 2015.
- [71] G. Schneider. Diffusive stability of spatial periodic solutions of the Swift-Hohenberg equation. *Comm. Math. Phys.*, 178:679–702, 1996.
- [72] G. Schneider. Nonlinear diffusive stability of spatially periodic solutions — abstract theorem and higher space dimensions. In *Proceedings of the International Conference on Asymptotics in Nonlinear Diffusive Systems*, volume 8 of *Tohoku Math. Publ.*, pages 159–167. Tohoku Univ., Sendai, 1998.
- [73] C. Storm, W. Spruijt, U. Ebert, and W. van Saarloos. Universal algebraic relaxation of velocity and phase in pulled fronts generating periodic or chaotic states. *Phys. Rev. E*, 61:R6063–R6066, Jun 2000.
- [74] W. van Saarloos. Front propagation into unstable states: marginal stability as a dynamical mechanism for velocity selection. *Phys. Rev. A*, 37(1):211–239, 1988.
- [75] W. van Saarloos. Front propagation into unstable states. *Phys. Rep.*, 386:29–222, 2003.
- [76] K. Zumbrun. Forward-modulated damping estimates and nonlocalized stability of periodic Lugiato-Lefever wave. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, accepted, 2023.
- [77] K. Zumbrun and P. Howard. Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.*, 47(3):741–871, 1998.