Lagrangian Submanifolds of Euclidean Space

Paul Carter
Magdalene College

Home Address:
1 Degas Drive
Monmouth Junction, NJ 08852
USA

Abstract

We give an exposition of the result that there is no closed exact Lagrangian submanifold L of $(\mathbb{C}^n, \omega_0)$ where ω_0 is the standard symplectic structure. We show that the assertion is equivalent to the statement that the perturbed Cauchy-Riemann equation $\bar{\partial}_{J_0} u = g$ for maps u from the unit disc D to $\mathbb{C}^n$ which map the boundary circle ∂D to L has no solution for some function g_0. To do this, we follow [1] and consider the universal moduli space $\mathcal{M} = \{(u, g) : \bar{\partial}_{J_0} u = g\}$ and show that if we assume L to be exact, the projection $(u, g) \mapsto g$ is surjective in suitable spaces. To obtain surjectivity, it is necessary to show that this projection is proper, a property which follows from Gromov's theorem of compactness for pseudoholomorphic curves. We provide a proof of this compactness theorem, following arguments in [12], by obtaining a subsequence which converges modulo bubbling and removing the bubble point singularities. A proof is given in the case of interior singularities and we give suggestions for how to modify the method for singularities on the boundary.

Contents

Introduction

The purpose of this essay is to provide an exposition of the proof of the statement that there are no closed exact Lagrangian submanifolds of $\mathbb{C}^n$ with respect to the standard symplectic structure. The original proof of this result is due to Gromov (in [5]) and involves a (much celebrated) theorem on the compactness of the moduli space of so called pseudoholomorphic curves. These objects, also introduced by Gromov in the same paper, have since sparked much interest in symplectic topology, most notably being responsible for the development of areas such as Floer theory and Gromov-Witten theory.

This essay is split into two main parts. Sections 1, 2, and 3 comprise the first part, and are concerned with obtaining a statement of this theorem in the context of pseudoholomorphic curves as well as proving the theorem under the assumption of the above-mentioned compactness theorem. In Section 1, we start by providing a short introduction to symplectic topology and introducing the concepts necessary in understanding the statement of the theorem. In Section 2, we introduce almost complex manifolds and Gromov's pseudoholomorphic curves, with a focus on the generalized Cauchy-Riemann equation describing these curves. Section 3 is concerned with the Fredholm setup of the problem and proving the result up to compactness.

Section 4 constitutes the second part of this essay, which is concerned with proving this compactness theorem. The main references used throughout the essay are [1] and [12].

1 Preliminaries

We provide a short introduction to symplectic topology, carefully defining symplectic manifolds and their Lagrangian submanifolds, as well as notions of exactness and weak exactness in these manifolds. As symplectic topology itself is not really the focus of this essay, only the minimum required to understand the statement of the main theorem (Theorem 1.2.3) is presented. References for the relevant symplectic topology include [11] and the early chapters of [1].

1.1 Some symplectic topology

Definition 1.1.1. *A skew symmetric bilinear form $\lambda : V \times V \to \mathbb{R}$ on a vector space V is called a* ***symplectic form*** *if the map $\Lambda : V \to V^*$ defined $\forall v, w \in V$ by $\Lambda(v)(w) = \lambda(v, w)$ is an isomorphism.*
The pair (V, λ) is then called a ***symplectic vector space****.*

Consider a symplectic vector space (V, λ) and let $U \subseteq V$ be a subspace of V. The subspace U is *Lagrangian* if $U = U^\perp$, where $U^\perp = \{v \in V : \lambda(v, u) = 0 \ \forall u \in U\}$ denotes the orthogonal complement of U.

Definition 1.1.2. *A* ***symplectic structure*** *on a C^∞-smooth manifold M is a closed 2-form $\omega \in \Omega^2(M)$ which is nondegenerate.*
The pair (M, ω) is then called a ***symplectic manifold****.*

The requirement for ω to be nondegenerate means that for each $p \in M$ the tangent space (T_pM, ω_p) is a symplectic vector space. Equivalently, ω is nondegenerate if the wedge product

$$\omega^n = \omega \wedge \ldots \wedge \omega$$

is nowhere vanishing.

Example 1.1.3. $\mathbb{R}^{2n}$ *with the 2-form*

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is a symplectic manifold of dimension $2n$*. Indeed* $d\omega_0 = 0$ *and the n-form*

$$\omega_0 \wedge \ldots \wedge \omega_0 = n! \; (dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n)$$

is just a multiple of the standard volume form on $\mathbb{R}^{2n}$*. Also note that* ω_0 *is exact, i.e.* $\omega_0 = d\lambda$ *where*

$$\lambda = -\sum_{i=1}^{n} y_i dx_i \; .$$

By identifying $\mathbb{C}^n$ with $\mathbb{R}^n$ in the usual way, we see that $(\mathbb{C}^n, \omega_0)$ is a symplectic manifold.

1.2 Lagrangian submanifolds

Consider a smooth map $f : L \to M$ from a smooth manifold L to a symplectic manifold (M, ω). The map f is *Lagrangian* if for each $p \in L$ the vector space $(df)_p(T_pL)$ is a Lagrangian subspace of the symplectic vector space $(T_{f(p)}M, \omega_{f(p)})$.

Definition 1.2.1. *A submanifold L of a symplectic manifold* (M, ω) *is called a* ***Lagrangian submanifold*** *if the corresponding embedding* $f : L \hookrightarrow M$ *is Lagrangian. (Note that this implies the form* $f^*\omega$ *on L is zero.)*

Example 1.2.2. *Lagrangian submanifolds in* $\mathbb{C}^n$

i. The circle S^1 *is Lagrangian under the inclusion map* $\iota : S^1 \hookrightarrow \mathbb{C}$ *where* $\mathbb{C}$ *carries the standard symplectic form* ω_0*.*
   The tangent space at any point $p \in S^1$ is the real line $\mathbb{R}$ which is a Lagrangian subspace of $\mathbb{C}$.

ii. The torus T^n *is Lagrangian under the inclusion* $T^n \hookrightarrow \mathbb{C}^n$*.*
   By expressing T^n as the product $T^n = S^1 \times \ldots \times S^1$, it follows that at any point $p = (p_1, \ldots, p_n) \in T^n$,

$$T_pT^n = T_{p_1}S^1 \times \ldots \times T_{p_n}S^1 \cong \mathbb{R}^n$$

   which is a Lagrangian subspace of $\mathbb{C}^n$.

A symplectic manifold (M, ω) is said to be **exact** if ω is exact. (Example 1.1.3 shows that $\mathbb{C}^n$ is exact.) Now if we consider a Lagrangian submanifold L of an exact symplectic manifold (M, ω), then L is said to be exact if $\lambda|_L$ is exact for each 1-form λ such that $d\lambda = \omega$. We are now ready to state the main theorem, conjectured by Arnold in the 60's and proved in 1985 by Gromov in [5]. The rest of this essay will be concerned with its proof.

Theorem 1.2.3. *If* $L \hookrightarrow (\mathbb{C}^n, \omega_0)$ *is a closed Lagrangian submanifold, then L is not exact.*[1]

[1] It is in fact possible to find closed exact Lagrangian manifolds of $(\mathbb{C}^n, \omega)$ under certain circumstances, namely only when ω is a so called **exotic** symplectic structure. This means that $(\mathbb{C}^n, \omega)$ does not embed into $\mathbb{C}^n$ with the standard form ω_0 as above. Gromov showed the existence of such a structure in [5]. Also see [11].

1.3 Weakly exact symplectic manifolds

Definition 1.3.1. *A symplectic manifold (M,ω) is said to be* ***weakly exact*** *if the map*

$$\int \omega : \pi_2(M) \to \mathbb{R}$$

is identically zero. (The second homotopy group $\pi_2(M)$ is defined by $\pi_2(M) = \{$homotopy classes of maps $f : S^2 \to M\}$.)
A Lagrangian submanifold L of a symplectic manifold (M,ω) is weakly exact if the map

$$\int \omega : \pi_2(M,L) \to \mathbb{R}$$

is zero, where the relative homotopy group $\pi_2(M,L) = \{$homotopy classes of maps $f : (D,\partial D) \to (M,L)$ sending the unit disc D to M and the boundary circle ∂D to $L\}$.

Proposition 1.3.2. *An exact Lagrangian submanifold L of an exact symplectic manifold (M,ω) is weakly exact.*

Proof. We need to show that the map

$$\int \omega : \pi_2(M,L) \to \mathbb{R}$$

is zero. So consider any map $f : (D,\partial D) \to (M,L)$ sending the disc D to M and the boundary ∂D to L. Then the integral

$$\int_{f(D)} \omega = \int_D f^*\omega = \int_D f^* d\lambda \ .$$

Then using the fact that $f^*d\lambda = d(f^*\lambda)$ and Stokes' theorem, we have

$$\int_{f(D)} \omega = \int_{\partial D} f^*\lambda \ .$$

Since by assumption $f(\partial D) \subseteq L$ and $\lambda|_L$ is exact, another application of Stokes' gives the result. □

2 Pseudoholomorphic Curves

To prove Theorem 1.2.3, we will need to look at Gromov's theory of pseudoholomorphic (or J-holomorphic) curves, introduced in [5]. Since their introduction, they have been widely studied and there are therefore many good references on the subject, such as [12], [1], and [7]. We begin by introducing almost complex manifolds and defining pseudoholomorphic curves. Then, in Section 2.2, we show how the standard Cauchy Riemann equations can be adapted to the more general setting of almost complex structures and describe some important properties of the resulting generalized Cauchy Riemann operator. Next, in Section 2.3, we encounter the perturbed Cauchy-Riemann equation and show how this can actually be reduced to the non-perturbed case. In the final section, we discuss the notion of the energy of a holomorphic curve and how this is a conformally invariant quantity. We also prove a "mean value estimate" for the energy which will be very useful in the proof of compactness in Section 4. Reference [1] is used as a guide for this most of this section, until the discussion on energy, where [12] is used.

2.1 Almost complex structures

An **almost complex structure** J on a symplectic manifold (M, ω) is an automorphism of the tangent bundle TM of M which satisfies $J^2 = -\mathrm{Id}$. The pair (M, J) is called an **almost complex manifold**.

Remark 2.1.1. *In the case where M is a complex manifold then the structure J is equivalent to multiplication by i and will be denoted by J_0. Locally, in a holomorphic coordinate chart with $z_k = x_k + iy_k$ we have that*

$$J_0\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k} \quad \text{and} \quad J_0\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k} \ .$$

If the following is satisfied

$$\omega(X, JX) > 0 \quad \forall X \in TM\backslash\{0\}$$

then ω is said to **tame** J.

Example 2.1.2. *It is not hard to show that the standard complex structure J_0 on $\mathbb{C}^n$ is tame. In fact for all $X, Y \in \mathbb{C}^n$, the following are satisfied:*

1. $\omega_0(X, J_0X) = \|X\|^2$
2. $\omega_0(X, Y) \leq \|X\|\|Y\|$

Consider two manifolds M, N with almost complex structures J, J' respectively and let $f : M \to N$ be a smooth map. Then f is said to be (J, J') holomorphic if for each x in the domain of f, the differential map $(df)_x : T_xM \to T_{f(x)}N$ is complex linear, i.e.

$$df_x \circ J = J' \circ df_x \ . \tag{1}$$

We will restrict our attention to the case where M is a Riemann surface (a one-dimensional complex manifold).

Definition 2.1.3. *A **J-holomorphic** (or **pseudoholomorphic**) **curve** is an (j_Σ, J)-holomorphic map $f : \Sigma \to M$ from a Riemann surface (Σ, j_Σ) to an almost complex manifold (M, J).*

For convenience, pseudoholomorphic curves from, for example, the open/closed disc or the sphere will be referred to as "holomorphic discs" and "holomorphic spheres."

Remark 2.1.4. *Our strategy to prove Theorem* 1.2.3 *will be to show that any closed Lagrangian submanifold L in $(\mathbb{C}^n, \omega_0)$ admits a non-constant holomorphic disc $u : (D, \partial D) \to (\mathbb{C}^n, L)$ which sends the boundary circle ∂D to L. Thus*

$$\int_{u(D)} \omega_0 \neq 0$$

so L is not weakly exact. Proposition 1.3.2 *then shows that L is not exact.*

2.2 The equation $\bar{\partial}_J u = 0$

Throughout this section we will consider maps from the unit disc D to $\mathbb{C}$. Everything then generalizes to maps from the unit disc to $\mathbb{C}^n$ by repeating the process for each of the n components of the map. For a differentiable function $f : U \to \mathbb{C}$ on a domain $U \subseteq \mathbb{C}$ we define the operators $\partial, \bar{\partial}$ by

$$\partial f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$
$$\bar{\partial} f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

The operator $\bar{\partial}$ is the Cauchy-Riemann operator which determines holomorphic f via the equation $\bar{\partial} f = 0$.

Having defined the notion of an almost complex structure J, we now want to generalize the $\bar{\partial}$ operator to J-holomorphic curves. Let (M, J, ω) be an almost complex symplectic manifold, and (Σ, j_Σ) a Riemann surface. From the definition of a J-holomorphic curve $u : (\Sigma, j_\Sigma) \to (M, J)$ and using the expression (1), we have that u is J-holomorphic if

$$du \circ j_\Sigma = J \circ du \ . \tag{2}$$

Using the property that for an almost complex structure J, we have $J^2 = -\mathrm{Id}$, we see that u is J-holomorphic if and only if u satisfies the equation

$$\bar{\partial}_J u = 0 \tag{3}$$

where

$$\bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j_\Sigma) \tag{4}$$

Choosing a local coordinate chart on M and a holomorphic coordinate chart on Σ and using Remark 2.1.1, it can be shown that u satisfies (3) if and only if

$$\frac{\partial u}{\partial y} = J(u)\frac{\partial u}{\partial x}$$

for coordinates $z = x + iy$. We can write this in terms of the $\partial, \bar{\partial}$ operators as

$$\frac{i}{2}\left(\partial u - \bar{\partial} u\right) = J(u)\frac{1}{2}\left(\partial u + \bar{\partial} u\right)$$

so that

$$\bar{\partial} u + L(u)\partial u = 0$$

where L is defined by

$$L(u) = (i + J(u))^{-1}(i - J(u)) \ .$$

This equation shows that locally $\bar{\partial}_J$ is a nonlinear partial differential operator of order 1. We want to study the linearized operator D_u which formally is defined as the differential of the $\bar{\partial}_J$ operator at u. In our local formulation, we can write it as

$$D_u(v) = \bar{\partial} v + L_u(z)\partial v$$

where $L_u = L(u)$. It turns out that this is a first order *elliptic* operator.

Definition 2.2.1. *An **elliptic operator** L of order m on a domain $\Omega \subseteq \mathbb{R}^n$ is of the form*

$$Lu = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha u$$

where the functions $a_\alpha(x)$ satisfy

$$\sum_{|\alpha| = m} a_\alpha(x) y^\alpha \neq 0$$

for all $x = (x^1, \ldots, x^n) \in \Omega$ and all non-zero $y = (y^1, \ldots, y^n) \in \mathbb{R}^n$. (Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. See Note A.2 in the Appendix.)

For example, the standard Cauchy-Riemann $\bar{\partial}$ operator is easily shown to be elliptic on the unit disc in $\mathbb{C} = \mathbb{R}^2$. Here $\bar{\partial} u = a_x \partial u/\partial x + a_y \partial u/\partial y$ with $a_x = 1$ and $a_y = i$. So we have that for any non-zero $z = (z_x, z_y) \in \mathbb{R}^2$

$$a_x z_x + a_y z_y = z_x + i z_y \neq 0 \ .$$

Another way to express the ellipticity of an operator is through various estimates which it satisfies. It can be shown that the linearized D_u operator above satisfies the following elliptic estimate for functions v belonging to the Sobolev space[2] $W^{1,p}$

$$\|v\|_{1,p} \leq c \left(\|D_u v\|_p + \|v\|_p \right)$$

which is derived from the Calderon-Zygmund inequality for the Laplace operator. Since the inclusion $W^{1,p} \hookrightarrow L^p$ is compact, it is straightforward to show that the operator D_u satisfies the conditions of Theorem $A.5$ and so has finite dimensional kernel and closed image. It can then be shown that the formal adjoint D_u^* of D_u is also a first order elliptic operator satisfying a similar inequality and so also has finite dimensional kernel. To get the result that D_u is Fredholm, the kernel of D_u^* is shown to be isomorphic to the cokernel of D_u. The details of the derivation of the above elliptic estimate as well as proving this Fredholm property of D_u can be found in [10], for instance.

We also have the following useful theorem on regularity from the theory of elliptic partial differential equations.

Theorem 2.2.2. *(Elliptic Regularity)*

1. *(for Sobolev spaces) Let $u \in W^{1,p}$ $(p > 2)$ be a J-holomorphic map from a Riemann surface (Σ, j_Σ) to an almost complex manifold (M, J) such that $u(\partial\Sigma)$ is contained in a Lagrangian submanifold L of M. If J is of class C^l for $l \geq 2$, then u is in $W^{l,p}$. If J is smooth, then so is u.*

2. *(for Hölder spaces[3]) Let $u \in C^1$ be a J-holomorphic map[4] from a Riemann surface (Σ, j_Σ) to an almost complex manifold (M, J) such that $u(\partial\Sigma)$ is contained in a Lagrangian submanifold L of M. If J is of class $C^{k+\mu}$, then u is of class $C^{k+1+\mu}$. If J is smooth, then so is u.*

[2] See Appendix for discussion of the Sobolev spaces $W^{k,p}$

[3] See Appendix for discussion of the Hölder spaces $C^{k+\mu}$

[4] In fact the C^1 condition on u is automatically satisfied if u is differentiable and J-holomorphic except on a discrete subset and continuous everywhere. This fact is shown, for instance, in [1] and [18] and will be very useful in the proof of Theorem 4.1.2 on removal of singularities.

The proof employs a "bootstrapping" argument, which involves a repeated use of the elliptic estimates to obtain bounds on higher derivatives of u, under the assumption that it already belongs to some class of differentiability or integrability (i.e. it belongs to the given Sobolev or Hölder space). The details for the Sobolev case can be found in [12] and in [1] for the Hölder case. As this is primarily a result of the theory of elliptic PDE, better places to find a full treatment of these topics would be in a reference on that subject, for instance in [6] or [2].

2.3 The equation $\bar{\partial}_J u = g$

As in the last section we are considering a map $u : (\Sigma, j_\Sigma) \to (M, J)$ where Σ is a Riemann surface. We drop the requirement that u is J-holomorphic and instead consider when u satisfies

$$\bar{\partial}_J u = g$$

for some function g where as before

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j_\Sigma) \ .$$

Formally, $\bar{\partial}_J u$ is a section of the bundle $u^*(TM)$ on Σ. If we instead consider the map $\mathrm{Id}\times u : \Sigma \to \Sigma \times M$ and the projection $P : \Sigma \times M \to M$, then $\bar{\partial}_J u$ can also be viewed as a section of the bundle $P^*(TM)$ over $\Sigma \times M$ defined over graph(u) where

$$\mathrm{graph}(u) = \{(s, u(s)) : s \in \Sigma\} \ .$$

So regarding g as a global section of this bundle, we have that

$$\bar{\partial}_J u = g|_{graph(u)}$$

Now we note that

$$\begin{aligned}
\bar{\partial}_J u \circ j_\Sigma &= \frac{1}{2}(du \circ j_\Sigma + J \circ du \circ (-\mathrm{Id})) \\
&= -J \circ \frac{1}{2}(J \circ du \circ j_\Sigma - du \circ (-\mathrm{Id})) \\
&= -J \circ \frac{1}{2}(J \circ du \circ j_\Sigma - du) \\
&= -J \circ \bar{\partial}_J u
\end{aligned}$$

which implies that $\bar{\partial}_J u = g|_{graph(u)}$ is *anti-complex*. This means we can define a complex structure J_g on $\Sigma \times M$ by

$$J_g(X, Y) = (j_\Sigma(X), J(Y) + 2g \circ j_\Sigma(X)) \tag{5}$$

since

$$\begin{aligned}
J_g^2(X, Y) &= (j_\Sigma^2 X, J^2(Y) + 2J \circ g \circ j_\Sigma(X) + 2g \circ j_\Sigma^2(X)) \\
&= (-X, -Y + 2(J \circ g \circ j_\Sigma - g)(X)) \\
&= -(X, Y)
\end{aligned}$$

where in the second line we used the fact that $J^2 = j_\Sigma^2 = -\mathrm{Id}$ and in the third we used the anti-complexity of g. We now note that $\mathrm{Id}\times u$ is J_g-holomorphic if

$$\begin{aligned}
0 = \bar{\partial}_{J_g}(\mathrm{Id}\times u) &= \frac{1}{2}(d(\mathrm{Id},u) + J_g \circ d(\mathrm{Id},u)\circ j_\Sigma) \\
&= \frac{1}{2}((\mathrm{Id},du) + J_g \circ (j_\Sigma, du\circ j_\Sigma)) \\
&= \frac{1}{2}((\mathrm{Id},du) + (-\mathrm{Id}, J\circ du \circ j_\Sigma - 2g)) \\
&= (0, \bar{\partial}_J u - g)\ .
\end{aligned}$$

i.e. if u satisfies the equation $\bar{\partial}_J u = g$.

2.4 Energy

Here we introduce the idea of energy, which turns out to be a topological invariant for holomorphic curves.

Definition 2.4.1. *Let Σ be a compact Riemann surface with volume form $dvol_\Sigma$ and (M,ω,J) almost complex symplectic manifold such that ω tames J. The energy of a differentiable map $u : \Sigma \to M$ is defined by*

$$E(u) = \frac{1}{2}\int_\Sigma |du|_J^2\ dvol_\Sigma$$

where the norm $|du|_J^2 = g_J(du,du)$ is taken with respect to the metric g_J given by

$$g_J(X,Y) := \frac{1}{2}\left(\omega(X,JY) + \omega(Y,JX)\right) \quad \forall X,Y\in TM\ .$$

We will denote the energy of u restricted to B for some $B\subseteq\Sigma$ by

$$E(u,B) = \frac{1}{2}\int_B |du|_J^2\ dvol_\Sigma\ .$$

Let Σ, M, J, and ω be as above; then we have the following proposition:

Proposition 2.4.2. *If $u : \Sigma\to M$ is J-holomorphic then*

$$E(u) = \int_\Sigma u^*\omega$$

Proof. By choosing coordinates $z = x+iy$, we can write the integrand of $E(u)$ as

$$\begin{aligned}
|du|_J^2\ \mathrm{dvol}_\Sigma &= \left(\left|\frac{\partial u}{\partial x}\right|_J^2 + \left|\frac{\partial u}{\partial y}\right|_J^2\right) dx\wedge dy \\
&= \left(\omega\left(\frac{\partial u}{\partial x}, J\frac{\partial u}{\partial x}\right) + \omega\left(\frac{\partial u}{\partial y}, J\frac{\partial u}{\partial y}\right)\right) dx\wedge dy \\
&= 2\cdot\omega\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dx\wedge dy
\end{aligned}$$

since $\partial_x u + J\partial_y u = 0$ for J-holomorphic u. It follows that

$$\begin{aligned} E(u) &= \frac{1}{2}\int_B 2\cdot\omega\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dx\wedge dy \\ &= \int_\Sigma u^*\omega \end{aligned}$$

as required. □

This identity demonstrates the "conformal invariance" of the energy. The expression derived for $E(u)$ above is a topological quantity which does not depend on a reparameterization or rescaling of coordinates. This idea will be useful when we come to bubbling (Section 4.6).

3 The Fredholm alternative

Throughout this section, we follow the setup given in [1] which comes from Gromov's original work in [5]. We will also call on results from [12].

3.1 The Fredholm setup

We now return to the setting of a closed Lagrangian submanifold L of $\mathbb{C}^n$ where we are trying to deduce the existence of a non-constant holomorphic disc $u : (D, \partial D) \to (\mathbb{C}^n, L)$ which sends the boundary circle ∂D to L. The goal here is to show that if we assume no such disc exists then for any g the equation

$$\bar{\partial}_{J_0} u = g\ .$$

has a solution u. It will then be shown that for L closed, we can find g such that there is no solution and achieve a contradiction. To do this, we will have to place the problem in a Banach manifold setting (see Appendix). So for non-negative integer k and $0 < \mu < 1$, let $r = k + \mu$ and define the spaces

$$\begin{aligned} U^{r+1} &= \{u : (D, \partial D) \to (\mathbb{C}^n, L) \mid u \in C^{k+1+\mu}(D),\ u(\partial D) \subseteq L, \\ &\qquad \text{and } u \text{ is homotopic to a point}\} \\ G^r &= \{g \mid g \in C^{k+\mu}(D)\} \\ \mathcal{M}^{r+1} &= \{(u, g) \in U^{r+1} \times G^r \mid \bar{\partial}_{J_0} u = g\} \end{aligned}$$

and the projection map

$$\pi^r : \mathcal{M}^{r+1} \to G^r, \quad \pi^r(u, g) = g\ .$$

It will also be useful to define for each $g \in G^r$ the moduli space

$$\mathcal{M}^{r+1}_g = \{(u, g) \in U^{r+1} \times \{g\} \mid \bar{\partial}_{J_0} u = g\} = (\pi^r)^{-1}(g)$$

($\mathcal{M}^{r+1}$ is sometimes referred to as the universal moduli space.) By construction the spaces U^{r+1} and G^r are Banach manifolds. However, we will need to prove the following

Theorem 3.1.1. *The universal moduli space $\mathcal{M}^{r+1}$ is a smooth Banach submanifold of $U^{r+1} \times G^r$.*

Sketch proof. [5] We return to the formalism of Section 2.3 where now $M = \mathbb{C}^n$ and G^r is viewed as the space of $C^{k+\mu}$ sections of the bundle $P^*TM \to D^2 \times M$. Take a point $u_0 \in U^{r+1}$ and a small neighborhood $B(u_0)$ of u_0, and for $u \in B(u_0)$, let $\Gamma^r(u)$ be the set of $C^{k+\mu}$ sections of P^*TM defined over graph(u). We then consider the parallel transport

$$\Pi_{u_0}(u) : \Gamma^r(u) \to \Gamma^r(u_0)$$

induced by the flat connection on $M = \mathbb{C}^n$ which defines an isomorphism for $u \in B(u_0)$. We now want to construct a map whose zero set is $\mathcal{M}^{r+1} \cap (B(u_0) \cup G^r)$ and use the implicit function theorem to show that this is a smooth manifold. So we define $\Phi : B(u_0) \cup G^r \to \Gamma^r(u_0)$ by

$$\Phi(u, g) = \Pi_{u_0}(u) \left(\bar{\partial}_{J_0} u - g|_{graph(u)}\right)$$

and for $u \in B(u_0)$ the map $\Phi_u : G^r \to \Gamma^r(u_0)$ is defined by $\Phi_u(g) = \Phi(u, g)$. The map Φ is smooth as it is the composition of parallel transport with a first order linear differential operator and a restriction map. The fact that for each $u \in B(u_0)$, Φ_u defines a surjective affine transformation means that Φ is regular at 0. The linearized operator $L\Phi_u$ has kernel given by $\mathrm{Ker}(L\Phi_u) = \{g : g|_{graph(u)} = 0\}$ and for each u the complement of $\mathrm{Ker}(L\Phi_u)$ is closed. Thus the implicit function theorem (Theorem A.10) applies and we have that $\mathcal{M}^{r+1} \cap (B(u_0) \cup G^r) = \Phi^{-1}(0)$ is a smooth submanifold of $U^{r+1} \times G^r$. Patching together these neighborhoods gives the final result. □

We note that due to elliptic regularity (Theorem 2.2.2 (ii)), if u satisfies $\bar{\partial}_{J_0} u = g$ for $g \in G^r$, then we must have $u \in U^{r+1}$, so that $(u, g) \in \mathcal{M}^{r+1}$. This means that to show that the equation has a solution for any $g \in G^r$ amounts to showing that the projection π^r is onto.

From the discussion in the last section, we can see that π^r is an elliptic operator and therefore Fredholm. The differential (or linearization) $(d\pi^r)_u$ of π^r at u is essentially the operator D_u which was discussed previously and so ellipticity of $(d\pi^r)_u$ follows (and therefore $(d\pi^r)_u$ is Fredholm for each u). So π^r is a Fredholm map and the following is now a direct application of the implicit function theorem (Theorem A.10).

Theorem 3.1.2. *Let $g^r_{reg} \subseteq G^r$ denote the set of regular values of π^r. Then for $g \in g^r_{reg}$, we have that $(\pi^r)^{-1}(g) = \mathcal{M}^{r+1}_g$ is a manifold of dimension equal to the index*[6] *of π^r.*

In particular, since by assumption the curves $u \in U^{r+1}$ are homotopic to a point and there are no holomorphic discs, we have that $(\pi^r)^{-1}(0)$ is just the set of constant maps defined by $u(z) = z_0$ for some $z_0 \in L$. So the space of curves given by the zero set of π^r has dimension n since $\dim(L) = 1/2 \dim(\mathbb{C}^n) = n$. Since this matches the index of π^r we have that π^r has 0 as a regular value (see [5]).

It is unclear how the manifolds $\mathcal{M}^{r+1}_g$ depend on the choice of regular value g. The following result gives an idea of what this dependence is. For the result to be meaningful, we must assume that the projection π^r is proper, which will be proven afterwards (Proposition 3.2.1).

[5] This argument comes from [1], though a more thorough discussion can be found in [12] or [9]. The argument in [12] is particularly good as it can be more easily generalized to prove Theorem 3.1.3.

[6] The index of π^r in this case happens to be n. This can be deduced from the Reimann-Roch theorem or the Atiyah-Singer index theorem. The index calculation using the Riemann-Roch theorem as well as a proof of this theorem can be found in [12]. Index calculations are also carried out in [5].

Theorem 3.1.3. *For a regular value g of π^r, let $\mathcal{M}_g^{r+1}(g_\lambda) = \{(u, \lambda) \in U^{r+1} \times [0, 1] | \ \bar{\partial}_{J_0} u = g_\lambda\}$ where $\lambda \mapsto g_\lambda$ is a smooth homotopy of functions from $g_0 = 0$ to $g_1 = g$. Then $\mathcal{M}_g^{r+1}(g_\lambda)$ is a compact manifold of dimension ind(π^r)+1 whose boundary is $\mathcal{M}_0^{r+1} \cup \mathcal{M}_g^{r+1} = (\pi^r)^{-1}(0) \cup (\pi^r)^{-1}(g)$.*

Idea of proof. The idea here is similar to that of Theorem 3.1.1 and more details on the proof can be found in [12]. We consider the space $G^r(0, g)$ of homotopies $[0, 1] \to G^r$ defined by $\lambda \mapsto g_\lambda$ from $g_0 = 0$ to $g_1 = g$ and the universal moduli space $\mathcal{W}^{r+1}(g) = \{(u, \lambda, g_\lambda) | (u, \lambda) \in \mathcal{M}_g^{r+1}(g_\lambda)$ for a homotopy $g_\lambda \in G^r(0, g)\}$. It can be shown that this universal moduli space is a C^k Banach manifold and the projection map onto $G^r(0, g)$ is Fredholm. Thus the inverse images of the regular values of this map are manifolds. By the Sard-Smale theorem (Theorem A.9), this set of regular values is dense in $G^r(0, g)$. □

The above theorem shows that for each regular value g of π^r, $(\pi^r)^{-1}(g)$ is in the same cobordism class as $(\pi^r)^{-1}(0)$, i.e. their disjoint union is the boundary of a higher dimensional manifold. The fact that π^r is proper (which in turn means the manifold $\mathcal{M}_g^{r+1}(g_\lambda) \cong (\pi^r)^{-1}\{g \mid g_\lambda = g$ for some $\lambda \in [0, 1]\}$ is compact) is crucial because, for example, every manifold is cobordant to the empty set via the cobordism $M \times [0, 1)$ (which is of course non compact). Because of this, the fact that $(\pi^r)^{-1}(0)$ is non-empty means that $(\pi^r)^{-1}(g)$ must also be non-empty for each regular g since they are related via a *compact* cobordism. This means that all regular values of π^r are in its image. Therefore, (π^r) is onto (see Note A.7 in Appendix A).

3.2 The projection π^r is proper

The goal here is to show the following:

Proposition 3.2.1. *π^r is proper*

Proof. We will argue by contradiction. Suppose that π^r is not proper, i.e. there exists a sequence $\{u_n, g_n\} \subseteq \mathcal{M}^{r+1}$ which has no convergent subsequence such that the sequence of projections $\{g_n = \pi^r(u_n, g_n)\}$ has a subsequence converging to an element $g \in G^r$. Passing to this subsequence, we thus have a convergent sequence $g_n \longrightarrow g$ in G^r and a sequence u_n in U^{r+1} with no convergent subsequence such that

$$\bar{\partial}_{J_0} u_n = g_n \quad \forall n \ .$$

From the discussion in Section 2.3, it is clear that this is equivalent to the maps $\mathrm{Id} \times u_n : D \to D \times \mathbb{C}^n$ being J_{g_n}-holomorphic for the almost complex structure J_{g_n} on $D \times \mathbb{C}^n$ induced by g_n, i.e.

$$\bar{\partial}_{J_{g_n}} u_n = 0 \quad \forall n \ .$$

In this formalism, we have a sequence of almost complex structures J_{g_n} which converge to the structure J_g and a sequence of J_{g_n}-holomorphic maps $\mathrm{Id} \times u_n$ which have no convergent subsequence. We now look to the following compactness theorem which will be proved in Section 4.7, which states that the only way in which the above convergence can fail is if there exists a holomorphic sphere or holomorphic disc which "bubbles" off in the limit. (This will discussed more explicitly in Sections 4.6 and 4.7.)

Theorem 3.2.2. *Let (M, ω) be a symplectic manifold with Lagrangian submanifold $\tilde{L}$ and Σ a Riemann surface. Let J_n be a sequence of almost complex structures of (M, ω) which are tamed by ω such that J_n $C^{k+\mu}$-converges to some almost complex structure J for integer $k \geq 1$ and $\mu \in (0,1)$. Consider a sequence $u_n : (\Sigma, \partial\Sigma) \to (M, \tilde{L})$ of J_n-holomorphic curves with uniformly bounded energy. Then there exists a subsequence (which will also be denoted by u_n) such that u_n converges to a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, \tilde{L})$ uniformly in all derivatives up to order $k+1$ on compact subsets of $\Sigma \setminus F$ where $F = \{z_1, \ldots, z_k\} \subset \Sigma$ is a finite set of points where bubbles occur. The limit map can then be extended to a $C^{k+1+\mu}$ map $u : \Sigma \to M$ with $u(\partial\Sigma \subseteq \tilde{L})$. (If the convergence of the J_n is C^∞ then u converges uniformly with all derivatives on compact subsets and can be extended to a smooth map over all of Σ.)*

There are of course no holomorphic spheres in $\mathbb{C}^n$, and by assumption there are no discs either. Providing we can show that the above theorem applies in this scenario, the contradiction is complete and we can conclude that π^r must be proper.

We set $M = D \times \mathbb{C}^n$, $\tilde{L} = \partial D \times L$, and $\omega = \alpha\omega_D \oplus \omega_{\mathbb{C}^n}$ for come constant α to be determined. (Here ω_D and $\omega_{\mathbb{C}^n}$ denote the standard symplectic structure ω_0 on D and $\mathbb{C}^n$ respectively.) To satisfy the conditions of the theorem, we need to show that each J_{g_n} is tamed by ω and that the energies of $\mathrm{Id} \times u_n$ are uniformly bounded. Also, there is a difficulty in applying the theorem directly because as it stands M is not compact. Fortunately, there is a way around this: a bound can be placed on the areas of the images of the maps u_n in $\mathbb{C}^n$, and we show that it will actually be sufficient to instead consider the manifold $\tilde{M} = D \times K$ where $K \subseteq \mathbb{C}^n$ is compact.

We begin by showing the tameness property of the structures J_{g_n}. Since the sequence g_n converges in G^r, we can place a bound on $\|g_n\|_\infty$. Define

$$\beta = \sup_n \|g_n\|_\infty \ .$$

Then for $X = X_D \oplus X_{\mathbb{C}^n} \in T(D \times \mathbb{C}^n)$,

$$\omega(X, J_{g_n} X) = \alpha\omega_D \oplus \omega_{\mathbb{C}^n}(X_D \oplus X_{\mathbb{C}^n}, J_0 X_D \oplus (J_0 X_{\mathbb{C}^n} + 2g_n(J_0 X_D)))$$

where we have used the formula (5) for J_{g_n}. Thus

$$\begin{aligned}\omega(X, J_{g_n} X) &= \alpha\|X_D\|^2 + \|X_{\mathbb{C}^n}\|^2 + 2\omega_{\mathbb{C}^n}(X_{\mathbb{C}^n}, g_n(J_0 X_D)) \\ &\geq \alpha\|X_D\|^2 + \|X_{\mathbb{C}^n}\|^2 - 2\beta\|X_D\|\|X_{\mathbb{C}^n}\|\end{aligned}$$

If we take $\alpha = 2\beta^2 + 1/2$, then using the identities in Example 2.1.2, we have

$$\begin{aligned}\omega(X, J_{g_n} X) &\geq \frac{1}{2}(\|X_D\|^2 + \|X_{\mathbb{C}^n}\|^2) + 2\beta^2\|X_D\|^2 + \frac{1}{2}\|X_{\mathbb{C}^n}\|^2 - 2\beta\|X_D\|\|X_{\mathbb{C}^n}\| \\ &= \frac{1}{2}(\|X_D \oplus X_{\mathbb{C}^n}\|^2) + \left(\sqrt{2}\beta\|X_D\| - \frac{1}{\sqrt{2}}\|X_{\mathbb{C}^n}\|\right)^2 \\ &\geq \frac{1}{2}\|X\|^2\end{aligned}$$

so ω tames $\{J_{g_n}\}_{n\in\mathbb{N}}$.

We now place a bound on the energies of $f_n = \mathrm{Id}\times u_n$. We have that

$$
\begin{aligned}
E(f_n) &= \frac{1}{2}\int_D |df_n|^2_{J_0} dxdy \\
&= \frac{1}{2}\int_D \left(|\partial_x f_n|^2_{J_0} + |\partial_y f_n|^2_{J_0}\right) dxdy \\
&= \frac{1}{2}\int_D \left(\left\|\frac{\partial f_n}{\partial x}\right\|^2 + \left\|\frac{\partial f_n}{\partial y}\right\|^2\right) dxdy \\
&\leq \frac{1}{2}\int_D 2\left(\omega\left(\frac{\partial f_n}{\partial x}, J_{g_n}\frac{\partial f_n}{\partial x}\right) + \omega\left(\frac{\partial f_n}{\partial y}, J_{g_n}\frac{\partial f_n}{\partial y}\right)\right) dxdy \\
&\leq \int_D \left(\omega\left(\frac{\partial f_n}{\partial x}, \frac{\partial f_n}{\partial y}\right) + \omega\left(\frac{\partial f_n}{\partial y}, -\frac{\partial f_n}{\partial x}\right)\right) dxdy \\
&\leq 2\int_D \left(\omega\left(\frac{\partial f_n}{\partial x}, \frac{\partial f_n}{\partial y}\right)\right) dxdy \\
&= 2\int_D f_n^*\omega
\end{aligned}
$$

where in the fourth line we used the taming property of ω and in the fifth line we used the fact that $\mathrm{Id}\times u_n$ is J_{g_n}-holomorphic. Now since by assumption all the u_n are contractible to a point, we have that

$$
\begin{aligned}
\int_D f_n^*\omega &= \alpha \int_D \omega_D + \int_D u_n^*\omega_{\mathbb{C}^n} \\
&= \alpha\pi + 0
\end{aligned}
$$

so that the energies of the functions f_n are uniformly bounded by the constant $2\alpha\pi$. We also note that the areas of f_n are uniformly bounded, i.e.

$$
\begin{aligned}
\mathrm{Area}(f_n) &= \int_D \left|\frac{\partial f_n}{\partial x} \wedge \frac{\partial f_n}{\partial y}\right|_{J_0} dxdy \\
&\leq \frac{1}{2}\int_D \left|\frac{\partial f_n}{\partial x}\right|^2_{J_0} + \left|\frac{\partial f_n}{\partial x}\right|^2_{J_0} dxdy \\
&= E(f_n) \leq 2\alpha\pi \ .
\end{aligned}
$$

So we now have a sequence of discs f_n with uniformly bounded areas in $D \times \mathbb{C}^n$ and whose boundaries are contained in the compact subset $S^1 \times L$. Since the maps are continuous, it follows that the discs u_n themselves must all lie inside a ball of sufficiently large size in $\mathbb{C}^n$. If we denote the closure of such a ball by K, then the subset K is compact and we can therefore apply the compactness theorem using the manifold $\tilde{M} = D \times K$. We therefore have a convergent subsequence and can conclude that π^r is proper. □

3.3 The Fredholm alternative

We have shown that under the assumption that there are no holomorphic discs with boundary in L, the map π^r is onto, i.e. for each $g \in G^r$, there exists a solution u to the equation $\bar{\partial}_{J_0} u = g$. We can sum up the discussion of the past two sections as a theorem:

Theorem 3.3.1. *Let L be a closed Lagrangian submanifold of $\mathbb{C}^n$. If there exists non-integral $r > 1$ and $g_0 \in G^r$ such that the equation $\bar{\partial}_{J_0} u = g_0$ has no solution $u : D \to \mathbb{C}^n$ homotopic to a point with $u(\partial D) \subseteq L$, then there exists a non-constant holomorphic disc with boundary in L.*

This is the *Fredholm alternative*: either the equation has a solution for every function g or there exists a non-constant holomorphic disc with boundary in L. Theorem 1.2.3 thus follows if we can find such a g_0.

Proof of Theorem 1.2.3. We write the equation $\bar{\partial}_{J_0} u = g$ in local coordinates

$$\frac{\partial u}{\partial x} + J_0 \frac{\partial u}{\partial y} = g \ .$$

The following proposition shows that we can't always get a solution to this equation which satisfies our constraints. The proof rests on the fact that we require $u(\partial D)$ to lie in L for any solution u. As $L \subseteq \mathbb{C}^n$ is closed and therefore compact, we can place a bound on its elements, namely there exists a constant C such that $sup_{z \in L} |z| \leq C$.

Proposition 3.3.2. *Let g_0 be a constant vector in $\mathbb{C}^n$ (then clearly $g_0 \in C^{k+\mu}$ for any k, μ). If $u : D \to \mathbb{C}^n$ satisfies*

$$\frac{\partial u}{\partial x} + J_0 \frac{\partial u}{\partial y} = g_0 \ .$$

with $u(\partial D) \subseteq L$, then $|g_0| \leq 4C$.

Proof. Integrating the above equation over D and taking the modulus, we have that

$$\begin{aligned} \pi |g_0| &= \left| \int_D \frac{\partial u}{\partial x} + J_0 \frac{\partial u}{\partial y} \ dydx \right| \\ &= \left| \int_0^{2\pi} (cos(\theta) + J_0 sin(\theta)) \ u(e^{i\theta}) \ d\theta \right| \\ &\leq 2 \int_0^{2\pi} \left| u(e^{i\theta}) \right| d\theta \\ &\leq 4\pi C \end{aligned}$$

where in the second line the divergence theorem was used, and in the last line we appealed to the fact that $u(\partial D) \subseteq L$. □

Thus the equation has no solution if we take g_0 to be a constant vector with large enough modulus. By Theorem 3.3.1, there exists a non-constant holomorphic disc with boundary in L. Therefore by Remark 2.1.4 and Proposition 1.3.2, L is not exact. This completes the proof. □

4 Compactness

The goal of this last section is to provide a proof of the compactness theorem from the previous section. We begin in Section 4.1 with a brief overview of the method as well as the statement of the main results. In Section 4.2, we justify the existence of a convergent subsequence when the first derivatives of the sequence are uniformly bounded. The next two sections are concerned with the proof of a theorem on removing singularities in the interior; Section 4.3 provides some preliminary results and the theorem is then proved in Section 4.4. In Section 4.5 we discuss the problem of singularities on the boundary and some ideas for how to modify the interior proof for this case. In Section 4.6, we introduce the concept of bubbling in the limit sequence. Finally, in Section 4.7, we aim to tie together the results of the previous sections to obtain the desired compactness theorem. For the most part, we follow [12] and [17], drawing on results from other authors where needed.

Ideally, we would give a complete proof of compactness, providing full details on the type of convergence and properties of the limit map. However, this is a very long and difficult problem, and is beyond the scope of this essay. It should also be made clear that for our purposes, these details are really beside the point. To achieve the argument by contradiction in Section 3, we really just need to show the existence of *any* disc. Whether this disc has attached bubbles or whether energy is lost in the limiting process is really not important here. We do however include a short discussion of these ideas in Section 4.8.

4.1 Overview

Throughout, we assume (M, ω) is a compact symplectic manifold, L is a compact Lagrangian submanifold of M, and Σ is a Riemann surface (with boundary) with fixed complex structure j_Σ. The goal is to prove Theorem 3.2.2 which we restate here for convenience.

Theorem 3.2.2. *Let (M, ω) be a symplectic manifold with Lagrangian submanifold L and Σ a Riemann surface. Let J_n be a sequence of almost complex structures of (M, ω) which are tamed by ω such that J_n $C^{k+\mu}$-converges to some almost complex structure J for integer $k \geq 1$ and $\mu \in (0, 1)$. Consider a sequence $u_n : (\Sigma, \partial\Sigma) \to (M, L)$ of J_n-holomorphic curves with uniformly bounded energy. Then there exists a subsequence (which will also be denoted by u_n) such that u_n converges to a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$ uniformly in all derivatives up to order $k + 1$ on compact subsets of $\Sigma \setminus F$ where $F = \{z_1, \ldots, z_k\} \subset \Sigma$ is a finite set of points where bubbles occur. The limit map can then be extended to a $C^{k+1+\mu}$ map $u : \Sigma \to M$ with $u(\partial\Sigma \subseteq L)$. (If the convergence of the J_n is C^∞ then u converges uniformly with all derivatives on compact subsets and can be extended to a smooth map over all of Σ.)*

The proof of this theorem, given in Section 4.7, is quite technical and relies heavily on the two theorems given below. The first states that when the first derivatives of the curves are uniformly bounded, we get compactness (without too much trouble) on compact subsets of Σ.

Theorem 4.1.1. *Let J_n be a sequence of $C^{k+\mu}$ almost complex structures of (M, ω) which are tamed by ω such that J_n converges in $C^{k+\mu}$ to some almost complex structure J. Consider a sequence $u_n : (U_n, U_n \cap \partial\Sigma) \to (M, L)$ of J_n-holomorphic curves with uniformly bounded energy defined on increasing open sets $U_n \subseteq \Sigma$ whose union[7] is Σ. Suppose also that*

$$\sup_n \|du_n\|_{L^\infty(K)} < \infty$$

for all compact subsets K of Σ. Then there exists a subsequence (which will also be denoted by u_n) such that u_n converges uniformly with all derivatives up to order $k + 1$ to a J-holomorphic curve $u : (\Sigma, \partial\Sigma) \to (M, L)$ on compact subsets of Σ. If the convergence of the J_n is C^∞, then $u_n \longrightarrow u$ uniformly with all derivatives on compact subsets of Σ.

[7]Equivalently: the sets U_n *exhaust* Σ

It turns out that due to the uniform energy bound, the first derivatives of the u_n can only blow up at a finite number of points, and at each of these points, a holomorphic sphere or disc "bubbles" off in the limit. This process is described in more detail in Section 4.6 as well as in the proof of Theorem 3.2.2. We thus get convergence on all compact subsets of Σ which do not include these "bubble points" (singularities).

Now the idea is to extend the limit map over these points to a holomorphic curve defined on all of Σ. To do this we need the following theorem on removable singularities, whose proof is again quite technical and involves numerous steps. Since we need compactness for the case when Σ has boundary, these singularities can occur either in the interior or on the boundary. The easier case of singularities in the interior is discussed at length. The boundary case is essentially a repeat of the interior case, but includes a few additional tricky concepts. In Section 4.5, there will be a brief discussion about how to go about getting the result for boundary singularities.

Theorem 4.1.2 (Removal of singularities)**.** *Let L be a compact Lagrangian submanifold of an almost complex compact symplectic manifold (M, J, ω) where ω tames J and J is of class $C^{k+\mu}$ for some non-negative integer k and $0 < \mu < 1$. Then*

1. *If $u : D \setminus \{0\} \to M$ is J-holomorphic with $E(u) < \infty$, then u extends to a $C^{k+1+\mu}$ map from D to M.*
2. *If $u : D \cap \mathbb{H} \setminus \{0\} \to (M, L)$ is J-holomorphic with $E(u) < \infty$ and $u(D \cap \mathbb{R}) \subseteq L$, then u extends to a $C^{k+1+\mu}$ map from $D \cap \mathbb{H}$ to M.*

(If J is smooth, then u extends to a smooth map)

Notation: Throughout, D will denote the unit disc in $\mathbb{C}$. The disc of radius r about a point $z \in \mathbb{C}$ will be denoted by $D_r(z)$. If $z = 0$, the disc will be denoted by D_r.

4.2 Compactness for curves with bounded first derivatives

In this section we provide a proof of Theorem 4.1.1 in the case $M = K \subset \mathbb{C}^n$ where K is compact.

Proof. We give a proof in this simpler case (which is really the one we care about). The result does hold in general, and though the general argument is considerably harder and more involved, it is based on the same ideas. In the simpler case we first note that the uniform bound on the first derivatives of the u_n immediately gives us a uniformly convergent (C^0 convergent) subsequence by the Arzelà-Ascoli theorem (Theorem A.11).
We also know that having a sequence u_n of J_n-holomorphic curves with $J_n \longrightarrow J$ in the $C^{k+\mu}$ topology is equivalent to having a sequence of curves u_n satisfying

$$\bar{\partial}_{J_0} u_n = g_n$$

where the functions g_n tend to a limit function g in the $C^{k+\mu}$ topology.

The method is as follows. If we consider the equation in local coordinates, we have that

$$\partial_x u_n + J_0 \partial_y u_n = g_n$$

and we note that the derivative $\partial_x u_n$ satisfies

$$\partial_x^2 u_n + J_0 \partial_y \partial_x u_n = \partial_x \left(\partial_x u_n + J_0 \partial_y u_n \right) = \partial_x g_n \tag{6}$$

with a similar equation

$$\partial_x \partial_y u_n + J_0 \partial_y^2 u_n = \partial_y g_n$$

holding for $\partial_y u_n$. So we see that the first derivatives of u_n satisfy the perturbed Cauchy-Riemann equation but now with the first derivatives of g_n on the right hand side. As the g_n converge in $C^{k+\mu}$, the first derivatives converge in $C^{k-1+\mu}$. We now refer to the following theorem, one of many so called interior Schauder estimates, from the theory of elliptic partial differential equations[8].

Theorem 4.2.1. *Let L be an elliptic partial differential operator of order m whose coefficients are of class $C^\mu(U)$ for some $\mu \in (0,1)$ and some domain U. Let u be a function in $C^{m+\mu}(U_0)$ for a bounded domain U_0 with $\overline{U_0} \subseteq U$ and suppose u satisfies the equation $Lu = g$ for some function g. Then for every compact subset $K \subset U_0$, we have the following estimate*

$$\|u\|_{C^{m+\mu}(K)} \leq const \cdot \left(\|g\|_{C^\mu(U_0)} + \|u\|_{C^0(U_0)} \right)$$

where the constant is independent of u.

We can apply this theorem to the current problem as follows. The above local equations imply that in an open subset U_0 of $\mathbb{C}$, the first derivatives of u_n satisfy a first order elliptic differential equation with the elliptic operator $\bar{\partial}_{J_0}$. Now the coefficients of the Cauchy-Riemann operator are constant and so are certainly of class C^μ in this coordinate chart. By elliptic regularity $u_n \in C^{k+1+\mu}$ and so its derivatives are in $C^{k+\mu} \subset C^{1+\mu}$ and we can therefore obtain the following bound for compact $K \subseteq U_0$

$$\begin{aligned} \|du_n\|_{C^{1+\mu}(K)} &\leq c \left(\|dg_n\|_{C^\mu(U_0)} + \|du_n\|_{C^0(U_0)} \right) \\ &\leq c' \left(\|dg_n\|_{C^{k-1+\mu}(U_0)} + \|du_n\|_{C^0(U_0)} \right) \end{aligned}$$

where the constants c, c' are independent of u_n. As the first derivatives of the u_n are uniformly bounded by assumption and the $C^{k-1+\mu}$ norms of the derivatives of the g_n are uniformly bounded (since g_n converges in $C^{k+\mu}$), we have that

$$\|du_n\|_{C^{1+\mu}(K)} \leq C$$

for a constant C now independent of n. This uniform bound on the $C^{1+\mu}$ norms of the derivatives of the u_n now implies that the derivatives have a convergent subsequence in C^1 by Arzelà-Ascoli. Thus u_n converges in C^2 by taking a subsequence of this subsequence. Since the sets U_n are increasing and exhaust Σ and K is compact, we have $K \subseteq U_n$ for all $n \geq N$ for some N. So without loss of generality, we can assume u_n is defined over all of K for each n.

We can now differentiate the u_n again and repeat this argument gaining uniform bounds on the higher derivatives of the u_n until we run out of differentiability in the g_n. This process

[8] This particular estimate comes from [2], though there are many other good references on obtaining such results, for example [8].

is a form of "bootstrapping". Thus we can get a subsequence whose derivatives up to order $k+1$ converge uniformly on any given compact subset K of this coordinate chart. (In the case where the g_n are smooth, the bootstrap argument can continue indefinitely, and we thus get uniform convergence in all derivatives of u_n.) Given any compact subset of Σ, we can extend this argument by patching together charts and thus get a convergent subsequence on a given compact subset of Σ.

To get a subsequence which converges on *all* compact subsets of Σ, we consider an exhausting sequence of (increasing) compact subsets. We can find a convergent subsequence for each of these, so by taking a diagonal subsequence of these convergent subsequences, we can then get a sequence converging on all compact subsets of Σ as required.

□

In the general case, the bootstrapping argument is more complicated, mostly because the partial differential equations solved by the u_n and their derivatives do not have constant coefficients and so when commuting the derivatives as in (6), we pick up extra terms.

4.3 Energy estimates and quantization

We first note the following a priori energy estimate.

Lemma 4.3.1 (Mean value estimate)**.** *Consider a compact almost complex manifold (M, J). Then there exists a constant $\delta > 0$ such that for any $\epsilon > 0$ and any J-holomorphic curve $u : D_\epsilon(0) \to M$ satisfying*

$$\int_{D_\epsilon(0)} |du|^2 < \delta$$

we have the estimate

$$|du(0)|^2 \leq \frac{8}{\pi\epsilon^2} \int_{D_\epsilon(0)} |du|^2 \ .$$

This important result is a consequence of the following estimate which is derived from properties of the Laplace operator $\Delta := \partial_x^2 + \partial_y^2$ on $\mathbb{R}^2$.

Lemma 4.3.2. *Suppose we have a non-negative C^2 function $\phi : D_r \to \mathbb{R}$ which satisfies*

$$\Delta\phi \geq -c\phi^2 \quad \textit{and} \quad \int_{D_r} \phi < \frac{\pi}{8a}$$

for constants $\epsilon > 0$ and $a \geq 0$. Then

$$\phi(0) \leq \frac{8}{\pi\epsilon^2} \int_{D_r} \phi \ .$$

The proof of this lemma can be found in [16]. The mean value estimate then follows by setting $\phi = \frac{1}{2}|du|^2$ and checking that the conditions are satisfied. The details are in [12].

We also have a corresponding boundary energy estimate.

Lemma 4.3.3. *(Boundary mean value estimate) Let L be a compact Lagrangian submanifold of a compact almost complex manifold (M, J). Then there exists a constant $\delta > 0$ such that for any $\epsilon > 0$ and any J-holomorphic curve $u : D_{2\epsilon}(0) \cap \mathbb{H} \to M$ satisfying $u(D_{2\epsilon}(0) \cap \mathbb{R}) \subseteq L$ and*

$$\int_{D_{2\epsilon}(0)\cap\mathbb{H}} |du|^2 < \delta$$

we have the estimate[9]

$$\sup_{D_\epsilon(0)\cap\mathbb{H}} |du(0)|^2 \leq \frac{8}{\pi\epsilon^2}\int_{D_{2\epsilon}(0)\cap\mathbb{H}} |du|^2 \ .$$

From these estimates, we can now prove the following lemma on quantization of energy. It states that the energy of any (non-constant) holomorphic sphere or disc is bounded below. This will be useful when we discuss bubbling in Section 4.6: we cannot have bubbles with arbitrarily small energy!

Lemma 4.3.4. *Let L be a Lagrangian submanifold of an almost complex manifold (M, J). Then there exists $h > 0$ such that for any non constant J-holomorphic sphere $u : S^2 \to M$ or disc $u : (D, \partial D) \to (M, L)$*

$$E(u) \geq h \ .$$

Proof. For a holomorphic sphere $u : S^2 \to M$, since $S^2 \cong \mathbb{C}\cup\{\infty\}$, we have that u is holomorphic on $B_\epsilon(z) \subset \mathbb{C}$ for each $\epsilon > 0$ and $z \in \mathbb{C}$. Lemma 4.3.1 then states that there exists δ such that

$$|du(z)|^2 \leq \frac{8}{\pi\epsilon^2}\int_{D_\epsilon(z)} |du|^2 \quad \text{whenever} \quad \int_{D_\epsilon(z)} |du|^2 < \delta \ .$$

Similarly, for a holomorphic disc $u : (D, \partial D) \to (M, L)$, since $D \cong \mathbb{H} \cup \{\infty\}$, u is holomorphic on $D_{2\epsilon}(z) \cap \mathbb{H}$ for each ϵ and Lemma 4.3.3 says

$$\sup_{D_\epsilon(z)\cap\mathbb{H}} |du(z)|^2 \leq \frac{8}{\pi\epsilon^2}\int_{D_{2\epsilon}(z)\cap\mathbb{H}} |du|^2 \quad \text{whenever} \quad \int_{D_{2\epsilon}(z)\cap\mathbb{H}} |du|^2 < \delta \ .$$

If $E(u) < \delta$ then the above inequalities hold for any $\epsilon > 0$. Letting $\epsilon \longrightarrow \infty$ we get in both cases that $|du(z)| = 0$ for each $z \in \mathbb{C}$(or $\mathbb{H}$ for discs), contradicting the assumption that u is non constant. Taking h to be the value of δ above gives the result.

□

4.4 Removal of interior singularities

We start by stating the following "isoperimetric inequality" which will be essential to the proof regarding removal of singularities. A sketch proof of this inequality is included at the end of the section.

Lemma 4.4.1. *(Isoperimetric inequality) Let J be an almost complex structure tamed by ω on a compact symplectic manifold (M, ω), and let $u : D \setminus \{0\} \to M$ be J-holomorphic on the punctured disc with energy $E(u) \leq C_E < \infty$. Let $\gamma_r(\theta) = u(re^{i\theta})$ and let D_r denote the disc of radius r about zero. Then there exist $c_I > 0$ and $r_0 > 0$ such that for $0 < r < r_0$*

$$E(u, D_r) \leq c_I l(\gamma_r)^2$$

where $l(\gamma_r)$ is the length of the loop γ_r defined by

$$l(\gamma) := \int_0^{2\pi} |\dot\gamma(\theta)|_J \ d\theta$$

[9]There is a caveat: more work is needed before we can apply Lemma 4.3.2 to obtain this estimate (see Section 4.5).

Proof of Theorem 4.1.2. We first note that $|\dot{\gamma}(\theta)| = |ire^{i\theta}du(re^{i\theta}| = r|du(re^{i\theta}|$. Then for $0 < r < r_0$ (as in Lemma 4.4.1), define

$$E_u(r) := E(u, D_r) = \int_{D_r} |du|^2 \ .$$

Lemma 4.4.1 then implies

$$\begin{aligned} E_u(r) &\leq c_I l(\gamma_r)^2 \\ &= c_I \int_0^{2\pi} |\dot{\gamma_r}(\theta)| \ d\theta \\ &= c_I \left(\int_0^{2\pi} r|du(re^{i\theta})| \ d\theta \right)^2 \end{aligned}$$

We can then use Cauchy-Schwarz so that

$$\begin{aligned} E_u(r) &\leq c_I r^2 \left(2\pi \int_0^{2\pi} |du(re^{i\theta})|^2 \ d\theta \right) \\ &= 2\pi c_I r \frac{d}{dr} \left(\int_0^r r' \int_0^{2\pi} |du(re^{i\theta})|^2 \ d\theta dr' \right) \\ &= 2\pi c_I r \frac{d}{dr} \left(\int_{D_r} |du|^2 \right) \ . \end{aligned}$$

Rearranging gives the following equation

$$\frac{dr}{2\pi c_I r} \leq \frac{dE_u(r)}{E_u(r)}$$

which we can integrate from r to ρ for some $\rho_0 < r_0$ to obtain

$$\left(\frac{\rho_0}{r}\right)^\mu \leq \frac{E_u(\rho_0)}{E_u(r)}$$

where $\mu = \frac{1}{2\pi c_I}$ (we can assume μ is less than 1). Thus

$$E_u(r) \leq \left(\frac{r}{\rho_0} \right)^\mu E_u(\rho_0) \ .$$

For r small enough, we can now use Lemma 4.3.1 to obtain the following estimate

$$\begin{aligned} |du(re^{i\theta})|^2 &\leq \frac{8}{\pi r^2} \int_{D_r(re^{i\theta})} |du|^2 \\ &\leq \frac{8}{\pi r^2} E_u(2r) \\ &\leq K \frac{1}{r^{2-\mu}}, \qquad \text{with } K = \left(\frac{2}{\rho_0} \right)^\mu E_u(\rho_0) \ . \end{aligned} \tag{7}$$

We can use this estimate to show that u is Hölder continuous with exponent $\mu/2$ in a neighborhood D_r of 0. We need to find a constant c_H such that

$$\text{dist}(u(z), u(w)) \leq c_H |z - w|^\mu \quad \forall z, w \in D_r \ .$$

We take arbitrary $z, w \in D_r$ where r is small enough for the estimate (7) to hold. Then say $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ and assume without loss of generality that $r_2 \geq r_1$. In the case where $\theta_1 = \theta_2$ we have that $|z - w| = r_2 - r_1$ and

$$\begin{aligned}
\text{dist}(u(z), u(w)) &\leq \int_{r_1}^{r_2} |du(re^{i\theta_1})e^{i\theta_1}| \ dr \\
&\leq \int_{r_1}^{r_2} K \frac{1}{r^{1-\mu/2}} \ dr \\
&\leq \frac{2K}{\mu}(r_2^{\mu/2} - r_1^{\mu/2}) \\
&\leq \frac{2K}{\mu}(r_2 - r_1)^{\mu/2}
\end{aligned}$$

where in the last line we used the fact that for $x, y \in (0, 1)$ and $\alpha \in (0, 1)$, $(x - y)^\alpha \geq (x^\alpha - y^\alpha)$ whenever $x > y$.

In the general case, assume without loss of generality that $\theta_1 = 0$ and $r_2 > r_1$ (the case where $r_2 = r_1$ is similar and a bit simpler), then we can parameterize a path from $u(z)$ to $u(w)$ by $u(r(\theta)e^{i\theta})$ where

$$r(\theta) = r_1 + \frac{\theta}{\theta_2}(r_2 - r_1) \ .$$

Then

$$\begin{aligned}
\text{dist}(u(z), u(w)) &\leq \int_0^{\theta_2} |du(r(\theta)e^{i\theta})||\dot{r}(\theta + ir(\theta)| \ d\theta \\
&\leq \int_0^{\theta_2} K \frac{1}{r(\theta)^{1-\mu/2}} \left(\frac{r_2 - r_1}{\theta_2} + r(\theta) \right) \ d\theta \\
&\leq \int_{r_1}^{r_2} K \frac{1}{r^{1-\mu/2}} \left(\frac{r_2 - r_1}{\theta_2} + r \right) \ \frac{\theta_2}{r_2 - r_1} dr \\
&< \frac{2K}{\mu}(r_2 - r_1)^{\mu/2} \\
&\leq \frac{2K}{\mu}|z - w|^{\mu/2} \ .
\end{aligned}$$

We conclude that u is Hölder continuous and so can be continuously extended across 0. The fact that u can then be extended across 0 to a $C^{k+1+\mu}$ J-holomorphic map follows[10] from elliptic regularity (Theorem 2.2.2). In the case where J is smooth, then smoothness of u also follows from elliptic regularity.

□

[10]To get that $u \in C^1$, see [18] for example.

Proof of isoperimetric inequality (Lemma 4.4.1*).* We start by noting the most common isoperimetric iequality, that of a closed curve in $\mathbb{R}^2$. We have that

$$a \leq \frac{1}{4\pi} l$$

where a is the area of the curve and l is its length. Equality holds when the curve is a circle.

We aim to prove a corresponding result in our current setting. We take r_0 small enough so that $l(\gamma_r)$ is less than the injectivity radius of M for $0 < r < r_0$. This means that we can then use the exponential map to define what is called a "unique local extension" $u_{\gamma_r} : D \to M$ for each r. We construct this as follows: define

$$u_{\gamma_r}(\rho e^{i\theta}) := \exp_{\gamma_r(0)}(\rho X_r(\theta))$$

where $X_r(\theta) \in T_{\gamma_r(0)}M$ satisfies

$$\exp_{\gamma_r(0)}(X_r(\theta)) = \gamma_r(\theta) \ .$$

We thus have a "geodesic disc" on which we can define a chart using the exponential map. When mapped to Euclidean space, the curve γ_r satisfies the isoperimetric inequality in $\mathbb{R}^2$ and without directly calculating the constant, we can translate this back into M where we can now assume that

$$a(\gamma_r) \leq c_I l(\gamma_r)^2$$

where $a(\gamma_r)$ is the area

$$a(\gamma_r) = \int_{D_r} u_{\gamma_r}^* \omega$$

and c_I is a constant which depends on the metric. This area is sometimes referred to as the *local symplectic action.*

Next take $0 < \rho < r < r_0$ and consider the cylinder $u|_{D_r \setminus D_\rho}$. We are going to "cap" the top and bottom of this cylinder with the geodesic discs (local extensions) u_{γ_r} and u_{γ_ρ} obtained as above for the boundary circles γ_r and γ_ρ. This forms a sphere which is the boundary of the union of the discs u_{γ_s} for $\rho \leq s \leq r$ and is therefore contractible. Hence the energy (or area) of the cylinder (without caps) is given by the difference between the area of the top cap and the bottom cap, i.e.

$$E(u, D_r \setminus D_\rho) = \int_D u_{\gamma_r}^* \omega - \int_D u_{\gamma_\rho}^* \omega$$

Letting $\rho \longrightarrow 0$ gives the isoperimetric inequality as claimed. □

4.5 Removal of boundary singularities

In this section, we give a brief discussion of the problem of singularities on the boundary. These can be removed in a similar fashion to those in the interior, but with a few modifications.

Firstly, we recall the boundary mean value estimate (Lemma 4.3.3). Unfortunately, this estimate does not directly follow from Lemma 4.3.2 about the Laplace operator as in the interior case. In fact, some more work must be done; the idea is to extend the map u to the whole disc by reflection over the boundary, also known as "doubling". This then reduces the problem to the interior case. The problem is that this process does not work for general metrics, so a suitable one must be constructed. Specifically, a metric g is needed which makes the Lagrangian submanifold L *totally geodesic*[11] in M with respect to g, so that the normal derivative of u vanishes across the boundary. In turns out this (with a few other conditions) is sufficient for the reflection argument. The proof of existence of a metric g with the necessary conditions for reflection is given in [4].

Once we have the mean value inequality, we can then prove an isoperimetric inequality for the boundary in the following form

Lemma 4.5.1 (Boundary isoperimetric inequality)**.** *Let J be an almost complex structure tamed by ω on a compact symplectic manifold (M, ω), and let $u : D \cap \mathbb{H} \setminus \{0\} \to M$ be J-holomorphic on the punctured half-disc such that $u(D \cap \mathbb{R}) \subseteq L$ for a Lagrangian submanifold L of (M, ω). Suppose the energy of u is bounded: $E(u) \leq C_E < \infty$. Let $\gamma_r : [0, \pi] \to M$ be defined by $\gamma_r(\theta) = u(re^{i\theta})$ and let D_r denote the disc of radius r about zero. Then there exist $c_I > 0$ and $r_0 > 0$ such that for $0 < r < r_0$*

$$E(u, D_r \cap \mathbb{H}) \leq c_I l(\gamma_r)^2$$

where $l(\gamma_r)$ is the length of the arc γ_r defined by

$$l(\gamma) := \int_0^\pi |\dot{\gamma}(\theta)|_J \, d\theta$$

The proof of this inequality is similar to the interior case. In the interior case, we constructed geodesic discs via the loops γ_r and proved the inequality by considering the local symplectic action of these loops. In the boundary case the process can be repeated, but instead geodesic half discs u_{γ_r} are constructed from the arcs γ_r (whose boundary points lie in L) and we define

$$a(\gamma_r) = \int_{D_r \cap \mathbb{H}} u_{\gamma_r}^* \omega$$

to be local symplectic action for an arc γ_r.

With the above results, the proof of part (i) of Theorem 4.1.2 can be adapted to accommodate the boundary case by instead defining

$$E_u(r) := E(u, D_r \cap \mathbb{H}) = \int_{D_r \cap \mathbb{H}} |du|^2$$

and then repeating the calculations with this new definition of $E_u(r)$.

The removable singularities theorem for points on the boundary can also be proved using a different method due to Oh in [13]. He deliberately avoids the doubling argument in the aims of simplifying the proof. He achieves the result through a series of estimates, resting on results such as the Courant-Lebesgue Lemma (Lemma 4.2 in [13]) from minimal surface theory as well as a theorem which states that the image of a holomorphic curve is contained in a Darboux neighborhood[12]. The result is a L^p bound on the derivatives of u for $p > 2$ which he then bootstraps to get smoothness. The drawback to his argument is that it only works for

[11] This means all geodesics in L are also geodesics in M.

[12] An open set U with $L \subseteq U$ on which the symplectic form $\omega = -d\lambda$ where $\lambda = 0$ identically on the Lagrangian submanifold L.

suitable *calibrations* which are triples (M, ω, J) such that the almost complex structure J on M is ω-calibrated[13].

Yet another method of proof is given by Ye in [18], which does not have the calibration restriction.

4.6 Bubbling

In this section, bubbling is introduced, a concept first discovered in [15]. The treatment here comes from [12]. We know that when the first derivatives of u_n are uniformly bounded, we can find a subsequence, so we are interested in cases where du_n is unbounded where it is possible for compactness to fail. Here we will restrict our attention to maps $u : \Sigma \to M$ where Σ is a *closed* Riemann surface. So let J_n be a sequence of almost complex structures converging in $C^{k+\mu}$ to an almost complex structure J. Consider a sequence of J_n-holomorphic curves u_n with uniformly bounded energy and unbounded first derivatives, i.e.

$$\sup_n \|du_n\|_\infty = \infty$$

and there exists $C_E > 0$ such that

$$\sup_n E(u_n) \leq C_E < \infty \ .$$

We then consider the points of Σ where $|du_n|$ attains its maximum. Denoting these by z_n and the values $|du_n(z_n)| = \|du_n\|_\infty$ by a_n, we can assume without loss of generality that

$$z_n \longrightarrow z_\infty \quad \text{for some } z_\infty \in \Sigma$$

and

$$a_n \longrightarrow \infty \quad \text{as } n \longrightarrow \infty \ .$$

We now switch to local holomorphic coordinates and choose a coordinate chart $\phi : W \to \phi(W) \subset \Sigma$ where $W \subset \mathbb{C}$ is open and constants $C_1, C_2 > 0$ such that

1. for some $\epsilon > 0$, $D_\epsilon(0) \subseteq W$ and $\phi(0) = z_\infty$
2. the pullback volume form on W is given by $\phi^* dvol_\Sigma = \lambda^2 dx \wedge dy$ where $\lambda : W \to \mathbb{R}$ is bounded by C_1, C_2, i.e.
$$C_1 \leq \lambda(w) \leq C_2 \quad \forall w \in \phi(W) \ .$$

We therefore have that $z_n \in \phi(W)$ for $n \geq N$ for some N, so after removing a finite number of elements, we can assume the entire sequence $\{z_n\}$ is contained in $\phi(W)$. We can now consider the sequence locally in this chart

$$\tilde{u}_n := u_n \circ \phi : W \to M, \quad w_n := \phi^{-1}(z_n) \in W$$

Then

$$a_n = |du_n(z_n)| = \frac{|d\tilde{u}_n(w_n)|}{\lambda(w_n)} = \sup_{w \in W} \frac{|d\tilde{u}_n(w)|}{\lambda(w)}$$

[13] An almost complex structure J is ω-*compatible* (or ω-calibrated) on a symplectic manifold (M, ω) if $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric on M.

so that

$$C_1 a_n \leq \|d\tilde{u}_n(w)\|_{L^\infty(W)} \leq C_2 a_n \ . \tag{8}$$

Also, we have

$$w_n \longrightarrow 0 = \phi^{-1}(z_\infty) \quad \text{as } n \longrightarrow \infty \ .$$

Now since the sequence $w_n \to 0$ is contained in the open set V we can find $\delta > 0$ such that $D_\delta(w_n) \subseteq W$ for all n. If we define $\delta_n = \delta a_n$ and pass to a subsequence such that a_n is monotonic increasing, then the sets D_{δ_n} are increasing with D_{δ_n} exhausting $\mathbb{C}$ as $n \longrightarrow \infty$. Now consider the sequence $v_n : D_{\delta_n} \to M$ where

$$v_n(w) = \tilde{u}_n\left(w_n + \frac{w}{a_n}\right) \ .$$

With this new sequence we have by (8)

$$\|dv_n\|_{L^\infty(D_{\delta_n})} = \frac{\|d\tilde{u}_n\|_{L^\infty(D_{\delta_n})}}{a_n} \leq C_2$$

and

$$|dv_n(0)| = \frac{|d\tilde{u}_n(0)|}{a_n} \geq C_1 \ .$$

We also have a bound on the energy of v_n

$$E(v_n) = E(\tilde{u}_n, D_\delta(w_n)) \leq E(u_n) \ .$$

By Theorem 4.1.1, v_n has a subsequence which converges uniformly with all derivatives up to order $k+1$ on compact sets to a limit function $v : \mathbb{C} \to M$ which satisfies

$$|dv(0)| \geq C_1 \ .$$

Also, the energy of v is bounded by

$$E(v) \leq C_E \ .$$

The conformal invariance of the energy implies that the energy of the map from $\mathbb{C} \setminus \{0\}$ to M given by

$$z \mapsto v(1/z)$$

also has bounded energy and so by Theorem 4.1.2 the singularity at 0 can be removed and the map extends to a $C^{k+1+\mu}$ J-holomorphic map $\mathbb{C} \to M$ which means v extends to a $C^{k+1+\mu}$ J-holomorphic map from the sphere $\mathbb{C} \cup \{\infty\} = S^2$ to M. The map v constructed here is called a *bubble.*

4.7 Proof of Theorem 3.2.2

We assume for this section that $(M, \omega), L, J_n$, and J are as in the statement of Theorem 3.2.2. (The method below can be trivially extended to the case where J_n, J are smooth.)The proof of the theorem can be deduced from the following theorem which states that near each point at which the first derivatives are unbounded, there is a small concentration of energy. At each of these points, we get either a sphere or disc bubble.

Theorem 4.7.1. *We consider an open set W of the upper half plane $\mathbb{H}$ and a sequence of J_n-holomorphic functions $u_n : (W, W \cap \mathbb{R}) \to (M, L)$ with bounded energy*

$$\sup_n E(u, W) \leq C < \infty \quad \textit{for some } C > 0$$

Suppose there is a sequence $\{z_n\} \subset W$ with $z_n \longrightarrow z_\infty \in W$ and $|du_n(z_n)|$ unbounded as $n \longrightarrow \infty$. Then for any $\epsilon > 0$, we have that

$$\liminf_{n \longrightarrow \infty} E(u_n, D_\epsilon(z_\infty) \cap W) \geq h \tag{9}$$

and either a non-constant holomorphic sphere $v : S^2 \to M$ or non-constant holomorphic disc $v : (D, \partial D) \to (M, L)$ bubbles off in the limit.

From this theorem, Theorem 3.2.2 follows easily.

Proof of Theorem 3.2.2. We denote the **singular set** of the sequence u_n by

$$Z = \{z \in \Sigma : \text{there exists a sequence } z_n \longrightarrow z \text{ such that } |du_n(z_n)| \longrightarrow \infty \} .$$

We show that Z is finite and that u_n converges uniformly with all derivatives up to order $k+1$ on compact subsets of $\Sigma \backslash Z$. Let $Z_0 = \emptyset$ and let $V_0 = \Sigma \backslash Z_0 = \Sigma$. Then if $\sup_n \|du_n\|_{L^\infty(K)} < \infty$ for all compact $K \subset V_0$, we are done by Theorem 4.1.1. Otherwise take a compact set K and a sequence $\{z_n^1\} \subset K$ with

$$|du_n(z_n^1)| = \sup_{z \in K} |du_n(z)| = \|du_n\|_{L^\infty(K)} .$$

We can assume without loss of generality that $z_n^1 \longrightarrow z^1 \in V_0$ and $|du_n(z_n^1)| \longrightarrow \infty$ so that $Z_1 = \{z^1\} \subseteq Z$. Continuing in this way, we get sets $Z_k = \{z^1, \ldots, z^k\}$ and $V_k = \Sigma \backslash Z_k$ such that Z_k is contained in the singular set Z of u_n. However, this process cannot continue forever. By Lemma 4.7.1, at each point in Z_k, a disc or sphere of energy greater than or equal to h bubbles off in the limit. As the energy of u_n is bounded by C_E, only a finite number of these bubbles can exist (namely at most $\frac{C_E}{h}$ of them). Therefore $Z = Z_k$ for some k and $\sup_n \|du_n\|_{L^\infty(K)} < \infty$ for all compact $K \subset V_k = \Sigma \setminus Z_k$.

Now by Theorem 4.1.1, there is a subsequence converging on all compact subsets of $\Sigma \setminus Z$ to a J-holomorphic curve $u : (\Sigma \setminus Z, \partial\Sigma \setminus Z) \to (M, L)$. By Theorem 4.1.2, u can be extended to a $C^{k+1+\mu}$ J-holomorphic curve u from all of Σ to M with $u(\partial\Sigma) \subseteq L$ by removing each singularity. □

We now need to prove Theorem 4.7.1. The following lemma will be useful (see [12] for a proof):

Lemma 4.7.2. *Let $f : X \to [0, \infty)$ be a continuous function on a metric space X with metric d. For $z \in X$ and $\delta > 0$, if the closed ball $\overline{B}_{2\delta}(z)$ is complete, then there exists $\epsilon \leq \delta$ and $w \in B_{2\delta}(z)$ (here $B_r(x)$ denotes the open ball of radius r about x) such that*

$$\sup_{x \in B_\epsilon(w)} f(x) \leq 2f(w) \quad \textit{and} \quad \epsilon f(w) \geq \delta f(z) .$$

Proof of Theorem 4.7.1. Fix $\epsilon > 0$ such that $D_\epsilon(z_\infty) \subseteq W$ and let $\delta_n := (|du_n(z_n)|)^{-\frac{1}{2}}$.
Step 1:
We claim the following: There exists a sequence $w_n \in \mathbb{C}$ and $\epsilon_n > 0$ such that $w_n \longrightarrow z_\infty$, $\epsilon_n \longrightarrow 0$ with

$$\sup_{z \in D_{\epsilon_n}(w_n)} |du_n(z)| \leq 2a_n \quad \text{and} \quad \epsilon_n a_n \longrightarrow \infty \tag{10}$$

where $a_n := |du_n(w_n)|$.
To do this, fix n and consider the function $f_n : D_\epsilon(z_\infty) \cap \mathbb{H} \to \mathbb{R}$ defined by

$$f_n(z) = |du_n(z)| \ .$$

Then by Lemma 4.7.2 there exists a point $w_n \in D_{2\delta_n}(z_n)$ and a constant $0 < \epsilon_n \leq \delta_n$ such that

$$\sup_{z \in D_{\epsilon_n}(w_n)} f_n(z) \leq 2f_n(w_n) \quad \text{and} \quad \epsilon_n f_n(w_n) \geq \delta_n f_n(z_n) \ . \tag{11}$$

If we do this for each n, we obtain sequences w_n and ϵ_n. As $0 < \epsilon_n \leq \delta_n$ and $\delta_n \longrightarrow 0$, we have that $\epsilon \longrightarrow 0$ as well. Also, since $w_n \in D_{2\delta_n}(z_n)$, and $\delta_n \longrightarrow 0$, $z_n \longrightarrow z_\infty$, we also have that $w_n \longrightarrow z_\infty$. Since $\delta_n a_n = |du_n(z_n)|^{1/2} \longrightarrow \infty$, the desired conditions (10) follow directly from (11).

We now turn to the sequence $a_n \mathrm{Im}(w_n)$. This sequence is either bounded or unbounded. These two cases are discussed in Steps 2 and 3, respectively.

Step 2 :
We claim the following:

If the sequence $a_n Im(w_n)$ is bounded, a holomorphic disc bubbles off.

As a_n diverges, we must have that $\mathrm{Im}(w_n) \longrightarrow 0$ which means that z_∞ lies in the boundary $W \cap \mathbb{R}$. As $a_n \mathrm{Im}(w_n)$ is bounded, passing to a subsequence if necessary, we can assume that

$$l = \lim_{n \longrightarrow \infty} a_n \mathrm{Im}(w_n)$$

exists. Also, since $\epsilon_n \longrightarrow 0$ and $|\mathrm{Re}(w_n) - z_\infty| \longrightarrow 0$, by considering the sequence for large enough n we can assume without loss of generality that

$$\epsilon_n + |\mathrm{Re}(w_n) - z_\infty| < \epsilon \ .$$

We now consider the sequence $v_n : D_{\epsilon_n a_n}(0) \cap \mathbb{H} \to M$ defined by

$$v_n(z) = u_n \left(\mathrm{Re}(w_n) + \frac{z}{a_n} \right) \ .$$

We have that

$$\sup_{z \in D_{\epsilon_n a_n}(0) \cap \mathbb{H}} |dv_n(z)| \leq \sup_{z \in D_{\epsilon_n}(w_n) \cap \mathbb{H}} \frac{|du_n(z)|}{a_n} \leq 2$$

and

$$|dv_n(i a_n \mathrm{Im}(w_n))| = \frac{|du_n(w_n)|}{a_n} = 1 \ .$$

The energy of v_n is bounded as follows:

$$E(v_n, D_{\epsilon_n a_n}(0) \cap \mathbb{H}) \leq E(u_n, D_\epsilon(z_\infty) \cap \mathbb{H}) \leq C \ . \tag{12}$$

As v_n has uniformly bounded first derivatives on increasingly large half discs $D_{\epsilon_n a_n}(0) \cap \mathbb{H}$ in $\mathbb{H}$, Theorem 4.1.1 implies that v_n converges uniformly with all derivatives up to order $k+1$ on compact subsets of $\mathbb{H}$ to a holomorphic curve $v : \mathbb{H} \to M$ such that $v(\mathbb{R}) \subseteq L$. Since

$$|dv_n(ia_n \mathrm{Im}(w_n))| = 1 \quad \forall n$$

we have that $|dv(il)| = 1$ and so v is non constant. There is also an energy bound on v from 12. So by the conformal invariance of the energy, we can define the map $\tilde{v} : D \setminus \{-1\} \to M$ by

$$\tilde{v}(z) = v\left(\frac{i(1-z)}{1+z}\right)$$

which has bounded energy $E(\tilde{v}) = E(v)$. By Theorem 4.1.2, this map can be extended to a $C^{k+1+\mu}$ map over all of D. Noticing that for $z \in \partial D$

$$\overline{\left(\frac{i(1-z)}{1+z}\right)} + \left(\frac{i(1-z)}{1+z}\right) = 0$$

so that $\frac{i(1-z)}{1+z} \in \mathbb{R}$. So $\tilde{v}(\partial D) \subseteq v(\mathbb{R}) \subseteq L$. Thus we have shown that an appropriate disc $\tilde{v}$ bubbles off as claimed.

We still need to show in this case that (9) is satisfied. To do this, we note that by Lemma 4.3.4,

$$E(v) = E(\tilde{v}) \geq h$$

since $\tilde{v}$ is a holomorphic disc. Then for any $0 < \delta < 1$ we can find a constant r such that

$$\delta h < E(v, D_r(0) \cap \mathbb{H}) = \lim_{n \to \infty} E(v_n, D_r(0) \cap \mathbb{H}) \ .$$

Since

$$\begin{aligned} E(v_n, D_r(0) \cap \mathbb{H}) &= E(u_n, D_{r/a_n}(\mathrm{Re}(w_n)) \cap \mathbb{H}) \\ &\leq E(u_n, D_\epsilon(\mathrm{Re}(z_\infty)) \cap \mathbb{H}) \qquad \text{for large } n, \end{aligned}$$

it follows that for all $\delta < 1$

$$\delta h < E(u_n, D_\epsilon(\mathrm{Re}(z_\infty)) \cap \mathbb{H}) \ .$$

From this we can deduce (9).

Step 3:
We claim the following:

If the sequence $a_n Im(w_n)$ is unbounded, a holomorphic sphere bubbles off.

Here we assume that $a_n \mathrm{Im}(w_n) \longrightarrow \infty$. There are now two possibilities: we either have that z_∞ lies on the boundary $W \cap \mathbb{R}$ or in the interior of W. If z_∞ lies in the interior, we can place a ball around z_∞ and follow the procedure of Section 4.6 to see that a sphere bubbles off as claimed.

When z_∞ lies on the boundary, we follow a similar argument to the one above. The key here is that $\mathrm{Im}(w_n) \longrightarrow 0$ so we can take $\epsilon_n \leq \mathrm{Im}(w_n)$ but still ensure that $\epsilon_n a_n \longrightarrow \infty$ since $a_n \mathrm{Im}(w_n)$ diverges. We can therefore ensure that $D_{\epsilon_n}(w_n) \subseteq W$. Again, as $\epsilon_n \longrightarrow 0$ and $|w_n - z_\infty| \longrightarrow 0$, we can take the sequence for large enough n and assume that

$$\epsilon_n + |w_n - z_\infty| < \epsilon \ .$$

Now the sequence $v_n : D_{\epsilon_n a_n}(0) \to M$ defined by

$$v_n(z) = u_n \left(w_n + \frac{z}{a_n} \right)$$

satisfies

$$\sup_{z \in D_{\epsilon_n a_n}(0)} |dv_n(z)| \leq \sup_{z \in D_{\epsilon_n}(w_n)} \frac{|du_n(z)|}{a_n} \leq 2$$

and

$$|dv_n(0)| = \frac{|du_n(w_n)|}{a_n} = 1 \ .$$

Again we have bounded energy

$$E(v_n, D_{\epsilon_n a_n}(0)) \leq E(u_n, D_\epsilon(z_\infty)) \leq C \ . \tag{13}$$

Since the radii of the discs $D_{\epsilon_n a_n}$ tend to infinity as $n \longrightarrow \infty$, by Theorem 4.1.1, v_n converges uniformly with all derivatives up to order $k+1$ to a holomorphic curve $v : \mathbb{C} \to M$. The limit curve v is non constant as

$$|dv_n(0)| = 1 \quad \forall n \implies |dv(0)| = 1$$

and has bounded energy by (13). So by conformal invariance of energy, the map from $\mathbb{C} \setminus \{0\}$ to M defined by

$$z \mapsto v(1/z)$$

also has bounded energy and so extends to a $C^{k+1+\mu}$ map over all of $\mathbb{C}$ by Theorem 4.1.2. This means that v extends to a $C^{k+1+\mu}$ map from the sphere $\mathbb{C} \cup \{\infty\} = S^2$ to M as claimed.

Again, we need to show in this case that (9) is satisfied. By a similar argument to the first case, since $E(v) \geq h$ by 4.3.4, for $0 < \delta < 1$ there exists r such that

$$\delta h < E(v, D_r(0)) = \lim_{n \to \infty} E(v_n, D_r(0)) \leq \lim_{n \to \infty} E(u_n, D_{r/\epsilon}(z_\infty))$$

since

$$E(v_n, D_r(0)) = E(u_n, D_{r/a_n}(w_n)) \leq E(u_n, D_\epsilon(z_\infty)) \quad \text{for large } n \ .$$

The result (9) follows since the above holds for any $\delta < 1$.

□

4.8 Concluding remarks

This "convergence modulo bubbling" argument is sufficient for the purposes of this essay, namely in being able to prove Theorem 1.2.3. In the proof from [12] given above, we ignored the bubbles which developed in the limiting process and extended the limit map over the bubble point singularities to a smooth map on the whole disc. However, this is not exactly the same as the compactness theorem originally proved by Gromov in [5]. His result was of a more general nature, and he was able to describe the limit map in more detail with the concept of cusp-curves.

In fact it is possible to obtain the limit map as a cusp-curve, crudely: a disc with bubbles attached at each of the singularities. In this approach it is possible to gain some information on the exact nature of the convergence in terms of the number of bubbles which develop, how bubbles may form on top of bubbles (e.g. bubble towers or trees, see [14]), and how much energy is lost through bubbling in the limiting process. A discussion of these cusp-curves is carried out briefly in Gromov's original work [5] and in more detail in [1] and [7].

This is still not, however, the most careful description of the nature of the convergence. In [12], the notion of *stable maps*, a carefully and precisely defined description of Gromov's cusp-curves are considered. Existence and uniqueness of limits of these so called stable maps are obtained through an equally precise definition of how the convergence should be defined, so-called *Gromov convergence*. The case for spheres is proved in [12], and the case for discs is tackled in [4]. These proofs are very technical (and very hard). There are many other proofs of compactness which use different methods, for example in [3] or Ye's proof in [18].

A Appendix: Analysis Background

There are a number of concepts in analysis which are essential to the discussions in Sections 2 and 3, namely those of Banach manifolds and Fredholm maps. The notion of a smooth Banach manifold generalizes the standard notion of a manifold, the most important consequence of which is that infinite dimensional structures can then be considered. The definition of a smooth Banach manifold differs only from that of a smooth manifold in that charts on a Banach manifold map to open sets in some real or complex (possibly infinite dimensional) Banach space X, rather than $\mathbb{R}^n$ or $\mathbb{C}^n$. If the transition maps on a Banach manifold M are only C^k rather than C^∞, then M is said to be a C^k Banach manifold.

The Banach spaces considered in the essay are the **Sobolev spaces** $W^{k,p}$ and the **Hölder spaces** $C^{k+\alpha}$. The reason that we need to resort to working in these spaces is because the space C^∞ of smooth functions is not complete. To get useful analytic results, we thus need to move to the Sobolev and Hölder spaces which are completions of C^∞ with respect to the Sobolev and Hölder norms respectively. To define Sobolev spaces, we first need the following definition

Definition A.1. *Suppose $u : \Omega \to \mathbb{R}$ is a function defined on an open subset $\Omega \subset \mathbb{R}^n$ which is locally integrable. and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index. Then a locally integrable function u_α is called a **weak derivative** of u corresponding to α if the following is satisfied*

$$\int_\Omega u(x)\partial^\alpha \phi(x)dx = (-1)^{|\alpha|}\int_\Omega u_\alpha(x)\phi(x)dx$$

for every $\phi \in C_0^\infty(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is smooth with compact support}\}$. The order of the weak derivative is $|\alpha|$.

Note A.2. *Here the multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *for non-negative integers* α_i *with* $|\alpha| = \sum_{i=1}^n \alpha_i$ *satisfies*

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \quad and \quad \partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \ .$$

For a non-negative integer k and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is defined to be the space of all equivalence classes (under the relation $f \sim g$ if $f = g$ almost everywhere) of functions whose weak derivatives exist up to order k and are in $L^p(\Omega)$. For $1 \leq p < \infty$ the norm for a function $u \in W^{k,p}(\Omega)$ is denoted[14] by

$$\|u\|_{k,p} := \left(\int_\Omega \sum_{|\alpha| \leq k} |\partial^\alpha u(x)|^p \, dx \right)^{\frac{1}{p}} \ .$$

For the case $p = \infty$ the norm for a function $u \in W^{k,\infty}(\Omega)$ is defined as the maximum of the L^∞ norms of the weak derivatives $\partial^\alpha u$ for any $|\alpha| \leq k$. Note that when $k = 0$, the Sobolev space $W^{0,p}$ is just the space L^p. Under the above norms, the $W^{k,p}$ spaces are complete and therefore Banach spaces. These spaces can easily be generalized to functions u from $\mathbb{R}^n$ to $\mathbb{R}^m$ by summing the norms of each of the m components of u. When this is the case we still refer to these spaces as $W^{k,p}(\Omega)$ if it is clear what the target space is.

We now introduce the notion of Hölder continuity. A function $u : \Omega \to \mathbb{R}$ is Hölder continuous of exponent $\mu \in (0, 1)$ if there exists a constant C such that

$$\sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu} \leq C \ .$$

For $0 < \mu < 1$ and non-negative integer k, we define the norms[15]

$$\|u\|_\mu = \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu} + \sup_{x \in \Omega} |u(x)|$$

and

$$\|u\|_{k+\mu} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_\mu \ .$$

A function $u : \Omega \to \mathbb{R}$ is an element of the Hölder space $C^{k+\mu}(\Omega)$ if u is a C^k functions with finite norm $\|u\|_{k+\mu}$. Equivalently $C^{k+\mu}(\Omega)$ is the space of all functions $u : \Omega \to \mathbb{R}$ whose derivatives exist up to order k and whose k-th order derivatives are Hölder continuous. These spaces are complete and so are Banach spaces and, by the same argument as with Sobolev spaces, the definition can be generalized to spaces of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

The argument in Section 3 also relies on some Fredholm theory, so it will be useful to include the relevant results here.

Definition A.3. *A bounded linear operator* $T : X \to Y$ *between Banach spaces* X, Y *is a* ***Fredholm operator*** *if* $Ker(T)$ *and* $Coker(T)$ *are finite dimensional. The* ***index*** *of a Fredholm operator is defined as*

$$index(T) = dim \ Ker(T) - dim \ Coker(T) \ .$$

[14] When it is ambiguous over which set Ω the norm is taken, it will be denoted by $\|u\|_{W^{k,p}(\Omega)}$

[15] When it is ambiguous over which set Ω the norm is taken, it will be denoted by $\|u\|_{C^{k+\mu}(\Omega)}$

From the above definition, we can immediately deduce the following useful property of Fredholm operators.

Proposition A.4. *Let $T : X \to Y$ be Fredholm, then Ker(T) has closed complement $C = X \backslash Ker(T)$.*

Proof. As Ker(T) finite dimensional, pick a basis $\{v_i\}$, then we can find a dual basis $\{v_i^*\}$ of X^*. We then have that $C = \cap \mathrm{Ker}(v_i^*)$ is the intersection of finitely many closed sets. $\square$

Lemma A.5. *Let $L : X \to Y$ be a bounded linear operator and $K : X \to Z$ be a compact operator between Banach spaces U, V, W. If there exists a positive constant C such that*

$$\|u\|_U \leq C\left(\|Lu\|_V + \|Ku\|_W\right)$$

for all $u \in U$, then Im(L) is closed and Ker(L) is finite dimensional.

Definition A.6. *A map $f : M \to N$ between Banach manifolds M, N is a **Fredholm map** if $d_x f : T_p M \to T_{f(x)} N$ is a Fredholm operator for each $x \in M$.*

Consider a map $f : M \to N$ between manifolds M, N. The points in the domain of f can be divided into *regular* and *critical* points. We say that f is regular at a point $p \in M$ if the differential map $d_x f : T_p M \to T_{f(x)} N$ is surjective and is critical at p otherwise. Points in N can similarly be divided into regular and critical *values* of f as follows. A point $q \in N$ is a regular value of f if f is regular at p for all $p \in M$ such that $f(p) = q$ and is a critical value of f if $q = f(p)$ where $p \in M$ is a critical point of f.

Note A.7. *It is important to note what is contained in the image of f. By definition, all critical values of f are in Im(f). However, not all regular values of f are necessarily contained in Im(f). It is easy to see then that a map $f : M \to N$ is onto if $f^{-1}(q) \neq \emptyset$ for all regular values $q \in N$ of f.*

Definition A.8. *A set V is a **Baire set** (or generic set) if V a countable intersection of open dense sets.*

The *Baire category theorem* states that a Baire set in a complete metric space is dense. We now state a few (powerful) theorems of analysis.

Theorem A.9. *(Sard-Smale) If $f : M \to N$ is a smooth Fredholm map between Banach manifolds M, N, then the set $V \subseteq N$ of regular values of f is a Baire set.*

Theorem A.10. *(Implicit function theorem) Let $f : M \to N$ be a C^k Fredholm[16] map between Banach manifolds and let $q \in N$ be a regular value of f. Then*

(i). $P = f^{-1}(q)$ is a C^k submanifold of M of dimension index(f) (so P is finite dimensional)

(ii). $T_p P =$Ker($d_x f$) for all $p \in P$

We also have the classical Arzelà-Ascoli theorem:

[16]The requirement on f being Fredholm is a bit strong; in fact, the necessary condition is that f has closed kernel which is satisfied by Fredholm maps (Proposition A.4).

Theorem A.11. *(Arzelà-Ascoli) A sequence of functions $f_k : K \subset \mathbb{R}^m \to \mathbb{R}^n$ for a compact set K is called* equicontinuous *if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in K$ satisfy $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Suppose f_k is a sequence of equicontinuous functions which are also uniformly bounded, i.e. $\sup_n \|f_n\|_\infty \leq M$ for some constant M. Then f_k has a uniformly convergent subsequence.*

Note that if the sequence of functions f_k are differentiable on K with uniformly bounded first derivatives, the conditions of the Arzelà-Ascoli theorem are satisfied.

References

[1] M. Audin and F. Lafontaine, editors. *Holomorphic Curves in Symplectic Geometry*, volume 117 of *Progress in Mathematics.* Birkhäuser, 1994.

[2] L. Bers, F. John, and M. Schechter. *Partial Differential Equations*, volume 3 of *Lectures in Appl. Math.* John Wiley & Sons, Inc., 1964.

[3] K. Cieliebak and K. Mohnke. Compactness for punctured holomorphic curves. *Journal of Symplectic Geometry*, 3(4):589–654, 2005.

[4] U. Frauenfelder. Gromov convergence of pseudoholomorphic discs. *J. fixed point theory appl.*, 3:215–271, 2008.

[5] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. math.*, 82:307–247, 1985.

[6] L. Hörmander. *The Analysis of Linear Partial Differential Operators III.* Springer-Verlag, 1985.

[7] C. Hummel. *Gromov's Compactness Theorem for Pseudo-holomorphic Curves*, volume 151 of *Progress in Mathematics.* Birkhäuser, Basel, Switzerland, 1997.

[8] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*, volume 12 of *Graduate Studies in Mathematics.* Amer. Math. Soc, 1996.

[9] D. McDuff. Examples of symplectic structures. *Invent. math.*, 89:13–36, 1987.

[10] D. McDuff and D. Salamon. *J-Holomorphic Curves and Quantum Cohomology.* Amer. Math. Soc., 1994.

[11] D. McDuff and D. Salamon. *Introduction to Symplectic Topology.* Oxford Math. Monogr., Oxford Univ. Press, 1995.

[12] D. McDuff and D. Salamon. *J-Holomorphic Curves and Symplectic Topology*, volume 52 of *Colloq. Publ.* Amer. Math. Soc., Providence, RI, 2004.

[13] Y.G. Oh. Removal of boundary singularities of pseudo-holomorphic curves with lagrangian boundary conditions. *Comm. Pure. Appl. Math.*, 45:121–139, 1992.

[14] T.H. Parker and J.G. Wolfson. Pseudo-holomorphic maps and bubble trees. *Journal of Geometric Analysis*, 3(1), 1993.

[15] J. Sacks and K. Uhlenbeck. The existence on minimal immersions of 2-spheres. *Ann. Math.*, 113(1):1–24, 1981.

[16] D. Salamon. Morse theory, the conley index and floer homology. *Bull. London Math. Soc.*, 22(2):113–140, 1990.

[17] J.G. Wolfson. Gromov's compactness of pseudo-holomorphic curves and symplectic geometry. *J. Diff. Geom.*, 28:383–405, 1988.

[18] R. Ye. Gromov's compactness theorem for pseudo holomorphic curves. *Trans. Amer. Math. Soc.*, 342(2):671–694, 1994.