# Fast pulses with oscillatory tails in the FitzHugh-Nagumo system 

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#### Abstract

Numerical studies indicate that the FitzHugh-Nagumo system exhibits stable traveling pulses with oscillatory tails. In this paper, the existence of such pulses is proved analytically in the singular perturbation limit near parameter values where the FitzHugh-Nagumo system exhibits folds. In addition, the stability of these pulses is investigated numerically, and a mechanism is proposed that explains the transition from single to double pulses that was observed in earlier numerical studies. The existence proof utilizes geometric blow-up techniques combined with the Exchange Lemma: the main challenge is to understand the passage near two fold points on the slow manifold where normal hyperbolicity fails.


## 1 Introduction

In this paper we consider the FitzHugh-Nagumo equations in the form

$$
\begin{align*}
u_{t} & =u_{x x}+f(u)-w  \tag{1.1}\\
w_{t} & =\delta(u-\gamma w)
\end{align*}
$$

where the nonlinearity $f(u)=u(u-a)(1-u), 0<a<1 / 2, \gamma>0$, and $0<\delta \ll 1$. The PDE (1.1) was originally proposed $([6,20])$ as a simplification of the Hodgkin-Huxley model of nerve axon dynamics.

We are interested in traveling wave solutions, that is, solutions of the form $(u, w)(x, t)=(u, w)(x+c t)$ for wavespeed $c>0$. Finding such solutions to (1.1) is equivalent to finding bounded solutions of the following system of ODEs

$$
\begin{align*}
\frac{d u}{d \xi} & =v  \tag{1.2}\\
\frac{d v}{d \xi} & =c v-f(u)+w \\
\frac{d w}{d \xi} & =\epsilon(u-\gamma w)
\end{align*}
$$

where $\xi=x+c t$ is the traveling wave variable, and $0<\epsilon=\delta / c$. We assume $\epsilon \ll 1$ so that we may view (1.2) as a singular perturbation problem in the parameter $\epsilon$. In addition, we take $\gamma>0$ sufficiently small so that $(u, v, w)=(0,0,0)$ is the only equilibrium of the system.

It is well known that for each $0<a<1 / 2$ and each sufficiently small $\epsilon>0,(1.1)$ admits a localized traveling pulse solution. Equivalently, in (1.2) this corresponds to the existence of an orbit homoclinic to the only equilibrium $(u, v, w)=(0,0,0)$ with positive wavespeed $c=\mathcal{O}(1)$. This existence result has been obtained using a number of different techniques: classical singular perturbation theory ([9]), Conley index ([2]), and geometric singular perturbation theory ([15]). Numerics suggest that as $a \rightarrow 0$, the tails of the pulses develop small amplitude oscillations, evidence of a Shilnikov saddle-focus homoclinic and thus the bifurcation of $N$-homoclinic orbits ([10,


Figure 1: Shown is a schematic bifurcation diagram depicting the branch of pulses guaranteed by Theorem 1.1. The monotone pulse and oscillatory pulse shown were computed numerically for the parameter values $(c, a, \epsilon)=$ $(0.593,0.069,0.0036)$ and $(c, a, \epsilon)=(0.689,0.002,0.0036)$, respectively.
§5.1.2]). The goal of this work is to understand this phenomenon analytically and prove an extension of this existence theorem which also encompasses the onset of oscillations in the tails of the pulses.

Specifically, in this work we prove the following existence theorem
Theorem 1.1. There exists $K^{*}, \mu>0$ such that the following holds. For each $K>K^{*}$, there exists $a_{0}, \epsilon_{0}>0$ such that for each $(a, \epsilon) \in\left(0, a_{0}\right) \times\left(0, \epsilon_{0}\right)$ satisfying $\epsilon<K a^{2}$, there exists $c=c(a, \epsilon)$ given by

$$
c(a, \epsilon)=\sqrt{2}\left(\frac{1}{2}-a\right)-\mu \epsilon+\mathcal{O}(\epsilon(|a|+\epsilon))
$$

such that (1.1) admits a traveling pulse solution. Furthermore, for $\epsilon>K^{*} a^{2}$, the tail of the pulse is oscillatory.

The statement and implications of Theorem 1.1 will be presented in more detail in $\S 3$. We note here that this result extends the classical existence result by guaranteeing, at least near the point $(c, a, \epsilon) \approx(1 / \sqrt{2}, 0,0)$, a surface of solutions which contains both pulses with monotone tails and pulses with oscillatory tails (see Figure 1).
The general strategy behind the proof of Theorem 1.1 presented in this paper is similar to that of the classical existence result for fast pulses using geometric singular perturbation theory and the Exchange Lemma, albeit with a number of additional technical challenges due to the nature of the $(c, a, \epsilon) \approx(1 / \sqrt{2}, 0,0)$ limit in which normal hyperbolicity is lost at two points on the critical manifold: these challenges will be described more precisely in $\S 2$. Related difficulties have also been encountered in other constructions of traveling wave solutions, e.g. in $[1,4]$, and we will discuss in $\S 2.2$ below how these results differ from ours.

The geometric framework of the proof developed here provides also insight into a possible mechanism for the termination of the branch of fast pulses that was previously studied in, for example, $[3,7,8]$. We propose a geometric explanation, supported by a numerical analysis, for the termination of this branch of fast pulses and the transition of a single fast pulse into a double pulse.
The remainder of the paper is structured as follows. In $\S 2$, we describe the classical existence result for pulses and the difficulties that arise in proving the above extension. In $\S 3$, we outline the proof of Theorem 1.1 and the relation to oscillations in the tails of the pulses; $\S 4-6$ are devoted to the proof of Theorem 1.1. Section 7 contains a numerical analysis relating the above result to previous numerical analyses of (1.2), and in particular, we describe a termination mechanism for the branch of pulses guaranteed by the theorem as well as a brief numerical stability analysis of the pulse solutions. Finally, $\S 8$ contains a discussion of the results.

## 2 Background

### 2.1 Known existence results for pulses

It is known that for each $0<a<1 / 2$ and each sufficiently small $\epsilon>0$, there exists $c>0$ such that (1.2) admits an orbit homoclinic to $(u, v, w)=(0,0,0)$, the only equilibrium of the full system. In this section, we describe a proof of this result using geometric singular perturbation theory ([5]) and the Exchange Lemma ([14]), in the spirit of [15]. Many of the arguments carry over to the proof of Theorem 1.1, and we indicate where these arguments fail and more work is needed to establish this extension.
To keep similar notation as in the relevant literature for geometric singular perturbation theory results, we abuse notation and denote the independent variable in (1.2) by $t$ and write the system as

$$
\begin{align*}
\dot{u} & =v  \tag{2.1}\\
\dot{v} & =c v-f(u)+w \\
\dot{w} & =\epsilon(u-\gamma w)
\end{align*}
$$

where $=\frac{d}{d t}$. We separately consider (2.1) above, which we call the fast system, and the system below obtained by rescaling time as $\tau=\epsilon t$, which we call the slow system:

$$
\begin{align*}
\epsilon u^{\prime} & =v  \tag{2.2}\\
\epsilon v^{\prime} & =c v-f(u)+w \\
w^{\prime} & =(u-\gamma w)
\end{align*}
$$

where ' denotes $\frac{d}{d \tau}$. The two systems (2.1) and (2.2) are equivalent for any $\epsilon>0$. The idea of geometric singular perturbation theory is to determine properties of the $\epsilon>0$ system by piecing together information from the simpler equations obtained by separately considering the fast and slow systems in the singular limit $\epsilon=0$.

We first set $\epsilon=0$ in (2.1), and we obtain the layer problem

$$
\begin{align*}
\dot{u} & =v  \tag{2.3}\\
\dot{v} & =c v-f(u)+w \\
\dot{w} & =0
\end{align*}
$$

so that $w$ becomes a parameter for the flow and $\mathcal{M}_{0}(c, a)=\{(u, v, w): v=0, w=f(u)\}$ is a set of equilibria (though the critical manifold does not depend on $c$, we keep track of this anyways for convenience later). Considering this system in the plane $w=0$, we obtain the Nagumo system

$$
\begin{align*}
\dot{u} & =v  \tag{2.4}\\
\dot{v} & =c v-f(u)
\end{align*}
$$

It can be shown that for each $0 \leq a \leq 1 / 2$, for $c=c^{*}(a)=\sqrt{2}(1 / 2-a)$, this system possesses a heteroclinic connection $\phi_{f}$ (the Nagumo front) between the critical points $(u, v)=(0,0)$ and $(u, v)=(1,0)$. In (2.3), this manifests as a connection between the left and right branches of $\mathcal{M}_{0}(c, a)$ in the plane $w=0$. By symmetry, there exists $w^{*}(a)$ such that there is a connection $\phi_{b}$ (which we call the Nagumo back) in the plane $w=w^{*}(a)$ between the right and left branches of $\mathcal{M}_{0}(c, a)$ traveling with the same speed $c=c^{*}(a)$. The layer problem is shown in Figure 2. We will use the notation $\mathcal{M}_{0}^{r}(c, a)$ and $\mathcal{M}_{0}^{\ell}(c, a)$ to denote the right and left branches of $\mathcal{M}_{0}(c, a)$, respectively.

Similarly, by setting $\epsilon=0$ in (2.2), we obtain the reduced problem

$$
\begin{align*}
0 & =v  \tag{2.5}\\
0 & =c v-f(u)+w \\
w^{\prime} & =(u-\gamma w)
\end{align*}
$$



Figure 2: Shown is the fast subsystem for $\epsilon=0$ and $0<a<1 / 2$.


Figure 3: Shown is the slow subsystem for $\epsilon=0$ and $0<a<1 / 2$.
where the flow is now restricted to the set $\mathcal{M}_{0}(c, a)$ with flow determined by the equation for $w$. This is shown in Figure 3.

Combining elements of both the fast and slow subsystems, we see that there is a singular $\epsilon=0$ "pulse" obtained by following $\phi_{f}$, then up $\mathcal{M}_{0}^{r}(c, a)$, back across $\phi_{b}$, then down $\mathcal{M}_{0}^{\ell}(c, a)$. This exists purely as a formal object as the two subsystems are not equivalent to (2.1) for $\epsilon=0$. This singular structure is shown in Figure 4.

We now use Fenichel theory and the Exchange Lemma to construct a pulse for $\epsilon>0$ as a perturbation of this singular structure. The first thing to note is that for any $0<a<1 / 2$ the Nagumo front $\phi_{f}$ and Nagumo back $\phi_{b}$ leave and arrive at points on segments of $\mathcal{M}_{0}^{r}(c, a)$ and $\mathcal{M}_{0}^{\ell}(c, a)$ which are normally hyperbolic. Therefore such segments persist for $\epsilon>0$ as locally invariant manifolds $\mathcal{M}_{\epsilon}^{r}(c, a)$ and $\mathcal{M}_{\epsilon}^{\ell}(c, a)$. Also, the stable manifold $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$, consisting of the union of the stable fibers of the equilibria lying on $\mathcal{M}_{0}^{\ell}(c, a)$, also persists for $\epsilon>0$ as a two-dimensional manifold $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$. By Fenichel fibering, we in fact have that $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)=\mathcal{W}_{\epsilon}^{s}(0 ; c, a)$, the stable manifold of the origin.

In addition, the origin has a one-dimensional unstable manifold $\mathcal{W}_{0}^{u}(0 ; c, a)$ which persists for $\epsilon>0$ as $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$. The idea is to track $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ forwards and track $\mathcal{W}_{\epsilon}^{s}(0 ; c, a)$ backwards and show that there is an intersection provided we adjust $c \approx c^{*}(a)$ appropriately. The difficulty in this procedure comes from trying to track these manifolds in a neighborhood of the right branch $\mathcal{M}_{\epsilon}^{r}(c, a)$, where the flow spends time of order $\epsilon^{-1}$. The Exchange Lemma is used to describe the flow in this region.


Figure 4: Shown is the singular pulse $\epsilon=0$.


Figure 5: Shown is the set up for the Exchange lemma.

Since we are only concerned with a normally hyperbolic segment of $\mathcal{M}_{0}^{r}(c, a)$, as stated before it perturbs to a manifold $\mathcal{M}_{\epsilon}^{r}(c, a)$. In addition its stable and unstable manifolds, $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ also perturb to locally invariant manifolds $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$. Also, in a neighborhood of $\mathcal{M}_{\epsilon}^{r}(c, a)$, there exists a smooth change of coordinates in which the flow takes a very simple form, the Fenichel normal form ([5, 14]):

$$
\begin{align*}
X^{\prime} & =-A(X, Y, Z, c, a, \epsilon) X  \tag{2.6}\\
Y^{\prime} & =B(X, Y, Z, c, a, \epsilon) Y \\
Z^{\prime} & =\epsilon(1+E(X, Y, Z, c, a, \epsilon) X Y),
\end{align*}
$$

where $\mathcal{M}_{\epsilon}^{r}(c, a)$ is given by $X=Y=0$, and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ and $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ are given by $X=0$ and $Y=0$, respectively, and the functions $A$ and $B$ are bounded below by some constant $\eta>0$. The Exchange Lemma ([14]) then states that for sufficiently small $\Delta>0$ and $\epsilon>0$, any sufficiently large $T$, and any $Z_{0}$, there exists a solution to (2.6) satisfying $X(0)=\Delta, Z(0)=Z_{0}$, and $Y(T)=\Delta$ and the norms $|X(T)|,|Y(0)|$, and $\left|Z(T)-Z_{0}-\epsilon T\right|$ are of order $e^{-\eta T}$. The result is shown in Figure 5.
The idea is now to follow $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ and $\mathcal{W}_{\epsilon}^{s}(0 ; c, a)$ up to this neighborhood of $\mathcal{M}_{\epsilon}^{r}(c, a)$ and determine how they behave at $X=\Delta$ and $Y=\Delta$. This gives a system of equations in $c, T, \epsilon$ which we can now solve to connect


Figure 6: Shown is the construction of the pulse solution.


Figure 7: Shown is the bifurcation diagram indicating the known regions of existence for pulses in (2.1). Pulses on the upper branch are referred to as "fast" pulses, while those along the lower branch are called "slow" pulses. These two branches coalesce near the point $(c, a, \epsilon)=(0,1 / 2,0)$.
$\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ and $\mathcal{W}_{\epsilon}^{s}(0 ; c, a)$ using the solution given by the Exchange lemma, completing the construction of the pulse which is shown in Figure 6.

The existence results for pulses in the FitzHugh-Nagumo system are collected in the bifurcation diagram in Figure 7 where the green surface denotes the existence region for pulses. The pulses constructed above for $c \approx c^{*}(a)>0$ are called "fast" pulses and the region of existence is given by the upper branch. For each $0<a<1 / 2$, there are also "slow" pulses which bifurcate for small $c, \epsilon>0$, and the region of existence of such pulses is given by the lower branch. It is also known ([16]) that near the point $(c, a, \epsilon)=(0,1 / 2,0)$, these two branches coalesce and form a surface as shown.

### 2.2 Motivation and complications for $a \approx 0$

Numerical evidence (see, for instance, $\S 7$ ) suggests that when one of the fast pulses constructed above is continued in $(c, a)$ for fixed $\epsilon$, the tail of the pulse becomes oscillatory as $a \rightarrow 0$, i.e. as one moves towards the upper left corner of the bifurcation diagram of Figure 7. Pulses with oscillatory tails correspond to homoclinic orbits of the travelling wave ODE (2.1) for which the origin is a saddle-focus with one strongly unstable eigenvalue and two weakly stable complex conjugate (non-real) eigenvalues: such homoclinic orbits are often referred to as Shilnikov saddle-focus homoclinic orbits. The numerical observation that pulses with oscillatory tails exist is of interest, because such pulses are typically accompanied by infinitely many distinct $N$-pulses for each given $N \geq 2$ ([10,


Figure 8: Shown is the singular pulse for $\epsilon=0$ in the case of $(c, a)=(1 / \sqrt{2}, 0)$.
§5.1.2]): here, an $N$-pulse is a travelling pulse that resembles $N$ well separated copies of the original pulse. The goal of this current work is to prove the existence of pulses with oscillatory tails analytically by studying the branch of fast pulses in the regime near the singular point $(c, a, \epsilon)=(1 / \sqrt{2}, 0,0)$ in the bifurcation diagram. We will accomplish this by looking for pulses which arise as perturbations from the singular $\epsilon=0$ structure for the case of $(c, a)=(1 / \sqrt{2}, 0)$, which is shown in Figure 8.

Proceeding as in the case of fast waves, we wish to find an intersection between the stable and unstable manifolds of the origin. Let $I_{a}=\left[-a_{0}, a_{0}\right]$ for some small $a_{0}>0$. In the plane $w=0$, the fast system for $a \in I_{a}$ reduces to

$$
\begin{align*}
\dot{u} & =v  \tag{2.7}\\
\dot{v} & =c v-u(u-a)(1-u) .
\end{align*}
$$

As stated previously, for $a>0$ this system possesses Nagumo front type solutions connecting $u=0$ to $u=1$ for any $c=c^{*}(a)$. For $-a_{0}<a<0$ with $a_{0}$ sufficiently small, this system possesses front type solutions for any $c>1 / \sqrt{2}(1+a)$ connecting $u=0$ to $u=1$. For the critical value $c=c^{*}(a)=\sqrt{2}(1 / 2-a)$ the front leaves the origin along the strong unstable manifold of the origin, and for all other values of $c$, the front leaves the origin along a weak unstable direction. Our primary concern is the case of $a=0$, in which (2.7) reduces to a Fisher-KPP type equation

$$
\begin{align*}
\dot{u} & =v  \tag{2.8}\\
\dot{v} & =c v-u^{2}(1-u) .
\end{align*}
$$

Again, it is known that this system possesses front type solutions connecting $u=0$ to $u=1$ for any $c \geq 1 / \sqrt{2}$. For the critical value $c=1 / \sqrt{2}$ the front leaves the origin along the strong unstable manifold of the origin, and for $c>1 / \sqrt{2}$, the front leaves the origin along a center manifold. We are concerned with the case of $(c, a)=(1 / \sqrt{2}, 0)$ in which, as is the case with the Nagumo front, the singular fast front solution leaves the origin along the strong unstable manifold; here the solution is given explicitly by

$$
\begin{align*}
& u_{f}(t)=\frac{1}{2}\left(\tanh \left(\frac{1}{2 \sqrt{2}} t\right)+1\right)  \tag{2.9}\\
& v_{f}(t)=\frac{1}{\sqrt{2}} u_{f}(t)\left(1-u_{f}(t)\right) .
\end{align*}
$$

Note that by symmetry, for $(c, a)=(1 / \sqrt{2}, 0)$, the fast singular back solution also leaves the upper right fold point along the strong unstable direction.

Thus from Fenichel theory, the origin has a strong unstable manifold $\mathcal{W}_{0}^{u}(0 ; c, a)$ for $c \in I_{c}, a \in I_{a}$, and $\epsilon=0$ which persists as an invariant manifold $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ for $a, c$ in the same range and $\epsilon \in\left[0, \epsilon_{0}\right]$, some $\epsilon_{0}$. Here $I_{c}$ is a fixed closed interval which contains the set $\left\{c^{*}(a): a \in I_{a}\right\}$ in its interior. Recall $c^{*}(a)$ is the wavespeed for which the front solution in the strong unstable manifold exists for this choice of $a$, and $c^{*}(0)=1 / \sqrt{2}$. We note that for $-a_{0}<a<0$ with $a_{0}$ sufficiently small, though the origin sits on the unstable middle branch of the critical manifold, it still has a well defined strong unstable manifold.

Taking any piece of $\mathcal{M}_{0}^{r}(c, a)$ which is normally hyperbolic, i.e. away from the fold point, Fenichel theory again ensures that this persists a locally invariant manifold $\mathcal{M}_{\epsilon}^{r}(c, a)$ for $\epsilon \in\left(0, \epsilon_{0}\right]$. Similarly outside of a small fixed neighborhood of the fold, $\mathcal{M}_{0}^{r}(c, a)$ has stable and unstable manifolds $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ which persist as locally invariant manifolds $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$.
We follow $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ along the front into a neighborhood of the right branch $\mathcal{M}_{\epsilon}^{r}(c, a)$, and using the Exchange Lemma, we can follow $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ along $\mathcal{M}_{\epsilon}^{r}(c, a)$, but only up to a fixed neighborhood of the fold point. Here the Exchange Lemma breaks down.
Another issue is that the origin does not have a well defined stable manifold as in the case of $0<a<1 / 2$. For $a=0$, the origin sits on the fold of the critical manifold $\mathcal{M}_{0}(c, a)$ and thus does not lie in the region where the branch $\mathcal{M}_{0}^{\ell}(c, a)$ is normally hyperbolic. Therefore, we cannot use the results of Fenichel as before to deduce that any section of $\mathcal{M}_{0}^{\ell}(c, a)$ containing the origin persists as an invariant manifold for $\epsilon>0$. In the same vein, we cannot deduce that $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)=\mathcal{W}_{\epsilon}^{s}(0 ; c, a)$.
However, outside any small fixed neighborhood of the origin, Fenichel theory applies, and we know that $\mathcal{M}_{0}^{\ell}(c, a)$ and its stable manifold $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ perturb to invariant manifolds $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ which enter this small fixed neighborhood of the origin. In addition, the origin remains an equilibrium for $\epsilon>0$, so it remains to find conditions which ensure that $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and nearby trajectories on $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ in fact converge to zero. This is discussed in $\S 6$. It is important to note in this case that the manifolds $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ are not unique and are only defined up to errors exponentially small in $1 / \epsilon$. The forthcoming analysis is valid for any such choice of these manifolds, and in $\S 6$, we show that under certain conditions it is possible to choose $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ so that they in fact lie on $\mathcal{W}_{\epsilon}^{s}(0 ; c, a)$.
We now follow the manifold $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ backwards along the back up to a small neighborhood of the fold point, where again the theory breaks down. Thus we may be able to find a connection between $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ and $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ up to understanding the flow near the fold point. The flow in this region and the interaction with the exchange lemma is discussed in Sections 4 and 5.

Figure 9 summarizes what is given by the usual Fenichel theory arguments, which apply outside of small neighborhoods of the two fold points at which the critical manifold is not normally hyperbolic.

We note that there have been other studies of constructing singular solutions passing near non-hyperbolic fold points. In [1], for instance, a pulse solution was constructed in a model of cardiac tissue: in this model, the fast 'back' portion of the pulse also originated from a non-hyperbolic fold point as in the case for FitzHugh-Nagumo above. Both models exhibit a Fisher-KPP type equation as described above when viewing the layer problem in the plane containing the singular fast 'back' solution. One difference between these two cases is that, in [1], only wavespeeds $c>1 / \sqrt{2}$ are considered, which means that the back solution leaves the fold point along the center manifold: in particular, the desired pulse solution can be constructed by following a continuation of the slow manifold in the center manifold of the fold point. A second difference is that the origin of the model considered in [1] is hyperbolic, instead of being a second fold point as in the situation discussed in our paper. The setup discussed in [4] is similar to the one studied in [1] in that a condition is imposed on the wavespeed that ensures that the singular back solution leaves the fold along a center manifold rather than a strong unstable fiber.

In our case, we consider the critical wavespeed $c=1 / \sqrt{2}$ in which the back leaves along a strong unstable fiber. As in [1], we will use the blow up techniques of [17] to construct the desired pulse solution. However, a number of refinements of the results of [17] are needed to track the solution in a neighborhood of the fold point as the


Figure 9: Shown are the regions of difficulty for the case of $a \approx 0$.
solution exits this neighborhood along a strong unstable fiber as opposed to remaining on the center manifold. This will be described in more detail in Sections 4 and 5 .

## 3 Statement of the main result

We start by collecting a few results which follow from Fenichel theory. Define the closed intervals $I_{a}=\left[-a_{0}, a_{0}\right]$ for some small $a_{0}>0$ and $I_{c}=\left\{c^{*}(a): a \in I_{a}\right\}$; recall $c^{*}(a)$ is the wavespeed for which the Nagumo front exists for this choice of $a$. Then for sufficiently small $\epsilon_{0}$, standard geometric singular perturbation theory gives the following:

1. The origin has a strong unstable manifold $\mathcal{W}_{0}^{u}(0 ; c, a)$ for $c \in I_{c}, a \in I_{a}$, and $\epsilon=0$ which persists for $a, c$ in the same range and $\epsilon \in\left[0, \epsilon_{0}\right]$.
2. We consider the critical manifold defined by $\{(u, v, w): v=0, w=f(u)\}$. For each $a \in I_{a}$, we consider the right branch of the critical manifold $\mathcal{M}_{0}^{r}(c, a)$ up to a neighborhood of the knee for $\epsilon=0$. This manifold persists as a slow manifold $\mathcal{M}_{\epsilon}^{r}(c, a)$ for $\epsilon \in\left[0, \epsilon_{0}\right]$. In addition, $\mathcal{M}_{0}^{r}(c, a)$ possesses stable and unstable manifolds $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ which also persist for $\epsilon \in\left[0, \epsilon_{0}\right]$ as invariant manifolds which we denote by $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$.
3. In addition, we consider the left branch of the critical manifold $\mathcal{M}_{0}^{\ell}(c, a)$ up to a neighborhood of the origin for $\epsilon=0$. This manifold persists as a slow manifold $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ for $\epsilon \in\left[0, \epsilon_{0}\right]$. In addition, $\mathcal{M}_{0}^{\ell}(c, a)$ possesses a stable manifold $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ which also persists for $\epsilon \in\left[0, \epsilon_{0}\right]$ as an invariant manifold which we denote by $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$.

The goal of this paper is to prove Theorem 1.1 which we restate here for convenience:
Theorem 1.1. There exists $K^{*}, \mu>0$ such that the following holds. For each $K>K^{*}$, there exists $a_{0}, \epsilon_{0}>0$ such that for each $(a, \epsilon) \in\left(0, a_{0}\right) \times\left(0, \epsilon_{0}\right)$ satisfying $\epsilon<K a^{2}$, there exists $c=c(a, \epsilon)$ given by

$$
c(a, \epsilon)=\sqrt{2}\left(\frac{1}{2}-a\right)-\mu \epsilon+\mathcal{O}(\epsilon(|a|+\epsilon))
$$

such that (1.1) admits a traveling pulse solution. Furthermore, for $\epsilon>K^{*} a^{2}$, the tail of the pulse is oscillatory.


Figure 10: Schematic bifurcation diagram for the parameters $(c, a, \epsilon)$. Here the green surface is the region of existence of pulses as in Theorem 1.1; in the grey region, Theorem 1.1 does not apply. The red curve denotes the location of the Belyakov transition which occurs at the origin.

In $\S 5.5$, the wave speed of the pulse is computed as

$$
\begin{equation*}
c(a, \epsilon)=c^{*}(a)-\mu \epsilon+\mathcal{O}(\epsilon(|a|+\epsilon)) \tag{3.1}
\end{equation*}
$$

where $\mu>0$. Figure 10 shows the location of the surface of solutions guaranteed by the theorem in the bifurcation diagram for the parameters $(c, a, \epsilon)$. We emphasize that this theorem does indeed guarantee the existence of the desired branch of pulses with oscillatory tails. The onset of the oscillations in the tail of the pulse is due to a transition occurring in the linearization of (2.1) about the origin in which the two stable real eigenvalues collide and emerge as a complex conjugate pair as $a$ decreases for fixed $\epsilon$. If a pulse/homoclinic orbit is present when eigenvalues changes in this fashion, then this situation is referred to as a Belyakov transition ([10, §5.1.4]): all $N$-pulses that accompany a Shilnikov homoclinic orbit terminate near the Belyakov transition point. The linearization of (2.1) about the origin is given by

$$
J=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{3.2}\\
a & c & 1 \\
\epsilon & 0 & -\epsilon \gamma
\end{array}\right)
$$

We can compute the location of the Belyakov transition for small ( $a, \epsilon$ ) by finding real eigenvalues which are double roots of the characteristic polynomial of $J$. Thus, we determine for which $(\epsilon, a)$ both the characteristic polynomial and its derivative vanish, and find that this holds when

$$
\begin{equation*}
\epsilon=\frac{a^{2}}{4 c}+\mathcal{O}\left(a^{3}\right) \tag{3.3}
\end{equation*}
$$

This gives the location of the transition and allows us to choose the quantity $K^{*}>\frac{1}{4 c^{*}(0)}$ for which the statement in Theorem 1.1 holds for all sufficiently small $(a, \epsilon)$. Then by taking $K$ sufficiently large in Theorem 1.1, we see that the surface of pulses in cat-space which are given by the theorem encompasses both sides of this Belyakov transition and therefore captures both the monotone and oscillatory tails (see Figure 10).
The proof of Theorem 1.1 is presented in three parts:

1. In $\S 4$, we present an analysis of the flow in a small neighborhood of the upper right fold point.
2. In $\S 5$, using the Exchange Lemma together with the analysis of $\S 4$, we show that for each $a \in I_{a}$ and $\epsilon \in\left(0, \epsilon_{0}\right)$, there exists $c=c(a, \epsilon)$ such that $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ connects to $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ after passing near the upper right fold point.
3. In $\S 6$, we show that for each $(a, \epsilon)$ satisfying the relation in the statement of Theorem 1.1 , the manifold $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and nearby solutions on $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ in fact converge to the equilibrium, completing the construction of the pulse.

## 4 Tracking around the fold

### 4.1 Preparation of equations

We append an equation for the parameter $\epsilon$ to (2.1) and arrive at the system

$$
\begin{align*}
\dot{u} & =v  \tag{4.1}\\
\dot{v} & =c v-f(u)+w \\
\dot{w} & =\epsilon(u-\gamma w) \\
\dot{\epsilon} & =0 .
\end{align*}
$$

For $(c, a) \in I_{c} \times I_{a}$, the fold point is given by the fixed point $(u, v, w, \epsilon)=\left(u^{*}, 0, w^{*}, 0\right)$ of (4.1) where

$$
\begin{equation*}
u^{*}=\frac{1}{3}\left(a+1+\sqrt{a^{2}-a+1}\right) \tag{4.2}
\end{equation*}
$$

and $w^{*}=f\left(u^{*}\right)$. The linearization of (4.1) about this point is

$$
J=\left(\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{4.3}\\
0 & c & 1 & 0 \\
0 & 0 & 0 & u^{*}-\gamma w^{*} \\
0 & 0 & 0 & 0
\end{array}\right)
$$

This matrix has one positive eigenvalue $\lambda=c$ with eigenvector ( $1, c, 0,0$ ) as well as an eigenvalue $\lambda=0$ with algebraic multiplicity three and geometric multiplicity one. The associated eigenvector is $(1,0,0,0)$ and generalized eigenvectors are $(0,1,-c, 0)$ and $\left(0,0, u^{*}-\gamma w^{*},-c\right)$. By making the coordinate transformation

$$
\begin{align*}
& z_{1}=u-u^{*}-\frac{v}{c}-\frac{w-w^{*}}{c^{2}}  \tag{4.4}\\
& z_{2}=-\frac{w-w^{*}}{c} \\
& z_{3}=\frac{v}{c}+\frac{w-w^{*}}{c^{2}}
\end{align*}
$$

we arrive at the system

$$
\begin{align*}
\dot{z_{1}} & =z_{2}+\frac{-1}{c}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right)\left(z_{1}+z_{3}\right)^{2}-\frac{1}{c}\left(z_{1}+z_{3}\right)^{3}-\frac{\epsilon}{c^{2}}\left(z_{1}+z_{3}+c \gamma z_{2}+u^{*}-\gamma w^{*}\right)  \tag{4.5}\\
\dot{z_{2}} & =-\frac{\epsilon}{c}\left(z_{1}+z_{3}+c \gamma z_{2}+u^{*}-\gamma w^{*}\right) \\
\dot{z_{3}} & =c z_{3}+\frac{1}{c}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right)\left(z_{1}+z_{3}\right)^{2}+\frac{1}{c}\left(z_{1}+z_{3}\right)^{3}+\frac{\epsilon}{c^{2}}\left(z_{1}+z_{3}+c \gamma z_{2}+u^{*}-\gamma w^{*}\right) \\
\dot{\epsilon} & =0
\end{align*}
$$

which, for $\epsilon=0$, is in Jordan normal form for the three dynamic variables $\left(z_{1}, z_{2}, z_{3}\right)$. To understand the dynamics near the fold point, we separate the nonhyperbolic dynamics which occur on a three-dimensional center manifold. In a small neighborhood of the fold point, this manifold can be represented as a graph

$$
\begin{align*}
z_{3} & =F\left(z_{1}, z_{2}, \epsilon\right)  \tag{4.6}\\
& =\beta_{0} z_{1}+\beta_{1} z_{2}+\beta_{2} z_{1}^{2}+\mathcal{O}\left[\epsilon, z_{1} z_{2}, z_{2}^{2}, z_{1}^{3}\right]
\end{align*}
$$

We can directly compute the coefficients $\beta_{i}$, and we find that

$$
\begin{equation*}
\beta_{0}=\beta_{1}=0, \quad \beta_{2}=\frac{-1}{c^{2}}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right) . \tag{4.7}
\end{equation*}
$$

We now make the following change of coordinates

$$
\begin{align*}
& x=\frac{-1}{c}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right) z_{1}  \tag{4.8}\\
& y=\frac{-1}{c}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right) z_{2}
\end{align*}
$$

which gives the flow on the center manifold in the coordinates $(x, y, \epsilon)$ as

$$
\begin{align*}
\dot{x} & =y+x^{2}+\mathcal{O}\left(\epsilon, x y, y^{2}, x^{3}\right)  \tag{4.9}\\
\dot{y} & =\epsilon\left[\frac{1}{c^{2}}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right)\left(u^{*}-\gamma w^{*}\right)+\mathcal{O}(x, y, \epsilon)\right] \\
\dot{\epsilon} & =0 .
\end{align*}
$$

Making one further coordinate transformation in the variable $z_{3}$ to straighten out the unstable fibers, we arrive at the full system

$$
\begin{align*}
\dot{x} & =y+x^{2}+\mathcal{O}\left(\epsilon, x y, y^{2}, x^{3}\right)  \tag{4.10}\\
\dot{y} & =\epsilon\left[\frac{1}{c^{2}}\left(\sqrt{a^{2}-a+1}+\frac{1}{2}\right)\left(u^{*}-\gamma w^{*}\right)+\mathcal{O}(x, y, \epsilon)\right] \\
\dot{z} & =z(c+\mathcal{O}(x, y, z, \epsilon)) \\
\dot{\epsilon} & =0 .
\end{align*}
$$

Let $V_{f} \subset \mathbb{R}^{3}$ be a small fixed neighborhood of $(x, y, z)=(0,0,0)$ where the above computations are valid. Define the neighborhood $U_{f}$ by

$$
\begin{equation*}
U_{f}=\left\{(x, y, z, c, a) \in V_{f} \times I_{c} \times I_{a}\right\}, \tag{4.11}
\end{equation*}
$$

and denote the change of coordinates from $(x, y, z, c, a)$ to the original $(u, v, w, c, a)$ coordinates by $\Phi_{f}: U_{f} \rightarrow O_{f}$ where $O_{f}$ is the corresponding neighborhood of the fold in $(u, v, w)$-coordinates for $(c, a) \in I_{c} \times I_{a}$. We note that in the neighborhood $U_{f}$ the equations for the variables $(x, y)$ are in the canonical form for a fold point as in [17], that is, we have

$$
\begin{align*}
\dot{x} & =y+x^{2}+h(x, y, \epsilon, c, a)  \tag{4.12}\\
\dot{y} & =\epsilon g(x, y, \epsilon, c, a) \\
\dot{z} & =z(c+\mathcal{O}(x, y, z, \epsilon)) \\
\dot{\epsilon} & =0,
\end{align*}
$$

where

$$
\begin{align*}
h(x, y, \epsilon, c, a) & =\mathcal{O}\left(\epsilon, x y, y^{2}, x^{3}\right)  \tag{4.13}\\
g(x, y, \epsilon, c, a) & =1+\mathcal{O}(x, y, \epsilon) .
\end{align*}
$$

We assume that the neighborhood $V_{f}$ has been chosen small enough so that $g(x, y, \epsilon, c, a)$ is bounded away from zero, say $g_{m}<g(x, y, \epsilon, c, a)<g_{M}$ with $g_{m}>0$. We have thus factored out the one hyperbolic direction (given by $z$ ) and the flow consists of the flow on a three-dimensional center manifold, parametrized by ( $x, y, \epsilon$ ) and the one-dimensional flow along the fast unstable fibers (the $z$-direction).


Figure 11: Shown is the set up of (4.14) for the passage near the fold point as in [17]: note that the positive $x$-axis points to the left, so that the attracting branch $S_{0}^{+}(c, a)$ corresponding to $x<0$ is on the right.

### 4.2 Tracking solutions around the fold point: existing theory

Here we describe the existing theory for extending geometric singular perturbation theory to a fold point. Consider the 2D system

$$
\begin{align*}
\dot{x} & =y+x^{2}+h(x, y, \epsilon, c, a)  \tag{4.14}\\
\dot{y} & =\epsilon g(x, y, \epsilon, c, a),
\end{align*}
$$

with parameters $(\epsilon, c, a)$. We collect a few relevant results from [17]. For $\epsilon=0$, this system possesses a critical manifold given by $\left\{(x, y): y+x^{2}+h(x, y, 0, c, a)=0\right\}$, which in a sufficiently small neighborhood of the origin is shaped as a parabola opening downwards. The branch of this parabola corresponding to $x<0$, which we denote by $S_{0}^{+}(c, a)$, is attracting and normally hyperbolic away from the fold point. Thus by Fenichel theory, this critical manifold persists as an attracting slow manifold $S_{\epsilon}^{+}(c, a)$ for sufficiently small $\epsilon>0$ and consists of a single solution. This slow manifold is unique up to exponentially small errors. In [17], this slow manifold is tracked around the knee where normal hyperbolicity is lost. The set up is shown in Figure 11; note that the orientation is chosen so that the positive $x$-axis points to the left.
For sufficiently small $\rho>0$ (to be chosen) and an appropriate interval $J$, define the following sections $\Delta^{\text {in }}(\rho)=$ $\left\{\left(x,-\rho^{2}\right): x \in J\right\}$ and $\Delta^{\text {out }}(\rho)=\{(\rho, y): y \in \mathbb{R}\}$. Then we have the following
Theorem 4.1 ([17, Theorem 2.1]). For each sufficiently small $\rho>0$, there exists $\epsilon_{0}>0$ such that for each $(c, a) \in I_{c} \times I_{a}$ and $\epsilon \in\left(0, \epsilon_{0}\right)$, the manifold $S_{\epsilon}^{+}(c, a)$ passes through $\Delta^{\text {out }}(\rho)$ at a point $\left(\rho, \tilde{y}_{\epsilon}(c, a)\right)$ where $\tilde{y}_{\epsilon}(c, a)=\mathcal{O}\left(\epsilon^{2 / 3}\right)$.

This theorem describes how the slow manifold exits a neighborhood of the fold point but not the nature of the passage near the fold point. Since the solution we are trying to construct will leave the neighborhood $U_{f}$ along a strong unstable fiber before reaching $\Delta^{o u t}$, we need to extend the results of $[17]$ to derive estimates which hold throughout this neighborhood, not just at the entry/exit sections.

### 4.3 Tracking solutions in a neighborhood of the fold point

For our purposes, we actually need to be able to say a bit more about the nature of $S_{\epsilon}^{+}(c, a)$ as well as nearby solutions between the two sections $\Delta^{i n}(\rho)$ and $\Delta^{\text {out }}(\rho)$. We can think of the slow manifold $S_{\epsilon}^{+}(c, a)$ as being a
one-dimensional slice of a two-dimensional critical manifold $M^{+}(c, a)=\cup_{\epsilon<\epsilon_{0}} S_{\epsilon}^{+}(c, a)$ of the three-dimensional $(x, y, \epsilon)$ subsystem of (4.12). It will sometimes be useful to consider the manifold $M^{+}(c, a)$ instead as we utilize a number of different coordinate systems in the analysis below.
Let $\tilde{x}_{\epsilon}(c, a)$ denote the $x$-value at which the manifold $S_{\epsilon}^{+}(c, a)$ intersects the section $\Delta^{i n}(\rho)$ and define the following set for small $\sigma, \rho, \delta$ to be chosen later:

$$
\begin{equation*}
\Sigma_{i}^{+}=\left\{\left(\tilde{x}_{\epsilon}(c, a)+x_{0},-\rho^{2}, \epsilon, c, a\right): 0 \leq\left|x_{0}\right|<\sigma \rho \epsilon, \epsilon \in\left(0, \rho^{3} \delta\right),(c, a) \in I_{c} \times I_{a}\right\} \tag{4.15}
\end{equation*}
$$

We also define the exit set

$$
\begin{equation*}
\Sigma_{o}^{+}=\left\{(\rho, y, \epsilon, c, a): y \in \mathbb{R}, \epsilon \in\left(0, \rho^{3} \delta\right),(c, a) \in I_{c} \times I_{a}\right\} \tag{4.16}
\end{equation*}
$$

Between the two sections $\Sigma_{i}^{+}$and $\Sigma_{o}^{+}$, the slow manifold $S_{\epsilon}^{+}(c, a)$ consists of a single solution $\gamma_{\epsilon}(t ; c, a)$ which can be written as

$$
\begin{equation*}
\gamma_{\epsilon}(t ; c, a)=\left(x_{\epsilon}(t ; c, a), y_{\epsilon}(t ; c, a), \epsilon, c, a\right), \tag{4.17}
\end{equation*}
$$

with $\gamma_{\epsilon}(0 ; c, a) \in \Sigma_{i}^{+}$and $\gamma_{\epsilon}\left(\tau_{\epsilon} ; c, a\right) \in \Sigma_{o}^{+}$for some time $\tau_{\epsilon}=\tau_{\epsilon}(c, a)$.
We define the $\mathcal{C}^{1}$ function $s_{0}(x ; c, a)$ so that between $\Delta^{i n}(\rho)$ and $\Delta^{o u t}(\rho), y=s_{0}(x ; c, a)$ is the graph of the singular solution obtained by following $S_{0}^{+}(c, a)$ to $(x, y)=(0,0)$ then continuing on the fast fiber defined by $y=0$.

The following Proposition 4.1 and Corollary 4.1, which will be proved in Sections 4.7 and 4.8 below, are the main results of this section. Proposition 4.1 gives estimates on the flow of (4.12) in the center manifold $z=0$ between the sections $\Sigma_{i}^{+}$and $\Sigma_{o}^{+}$. Corollary 4.1 then describes the implications for the full four dimensional flow of (4.12) where the dynamics of the basepoints of the unstable fibers are given by the flow on the center manifold.

Proposition 4.1. Consider the flow of (4.12) in the three dimensional center manifold $z=0$. There exists $\delta>0$ such that for all sufficiently small choices of $\sigma, \rho$, all solutions starting in $\Sigma_{i}^{+}$cross $\Sigma_{o}^{+}$. Furthermore, there exists $\tilde{k}>0$ such that the following holds. Given a solution $\gamma(t)=(x(t), y(t), \epsilon, c, a)$ with $\gamma(0) \in \Sigma_{i}^{+}$, let $\tau$ denote the first time at which $\gamma(\tau) \in \Sigma_{o}^{+}$. Then
(i) $\dot{x}(t)>\tilde{k} \epsilon$ for $t \in[0, \tau]$

In addition (see Remark 4.1 below), for each $(c, a) \in I_{c} \times I_{a}$, we can represent the manifold $S_{\epsilon}^{+}(c, a)$ as a graph $\left(x, s_{\epsilon}(x ; c, a), \epsilon\right)$ for $x \in\left[x_{\epsilon}(0 ; c, a), \rho\right]$ where $s_{\epsilon}(x ; c, a)$ is an invertible function of $x$ and
(ii) $\left|s_{\epsilon}(x ; c, a)-s_{0}(x ; c, a)\right|=\mathcal{O}\left(\epsilon^{2 / 3}\right)$
(iii) $\left|\frac{d s_{\epsilon}}{d x}(x ; c, a)-\frac{d s_{0}}{d x}(x ; c, a)\right|=\mathcal{O}\left(\epsilon^{1 / 3}\right)$
on the interval $\left[x_{\epsilon}(0 ; c, a), \rho\right]$.
Remark 4.1. The above result shows that there exists $\tilde{k}>0$ such that for each $(c, a) \in I_{c} \times I_{a}$, we have $\dot{x}_{\epsilon}(t ; c, a)>\tilde{k} \epsilon$ for $t \in\left[0, \tau_{\epsilon}(c, a)\right]$. Note that due to the bounds on the function $g(x, y, \epsilon, c, a)$ in System (4.12), there is a similar lower bound $\dot{y}_{\epsilon}(t ; c, a) \geq g_{m} \epsilon$. Thus we can represent the manifold $S_{\epsilon}^{+}(c, a)$ as a graph $\left(x, s_{\epsilon}(x ; c, a), \epsilon\right)$ for $x \in\left[x_{\epsilon}(0 ; c, a), \rho\right]$ where $s_{\epsilon}(x ; c, a)$ is an invertible function of $x$ on the interval $\left[x_{\epsilon}(0 ; c, a), \rho\right]$. Since this trajectory is contained in the neighborhood $V_{f}$, there exists an upper bound for the derivative

$$
\begin{equation*}
\dot{x}=y+x^{2}+h(x, y, \epsilon, c, a) \leq \tilde{K} \tag{4.18}
\end{equation*}
$$

We therefore have the following bounds on the derivatives $\frac{d s_{\epsilon}}{d x}(x ; c, a)$ and $\frac{d\left(s_{\epsilon}^{-1}\right)}{d y}(y ; c, a)$ :

$$
\begin{array}{r}
\frac{g_{m} \epsilon}{\tilde{K}} \leq \frac{d s_{\epsilon}}{d x}(x ; c, a) \leq \frac{g_{M}}{\tilde{k}} \\
\frac{\tilde{k}}{g_{M}} \leq \frac{d\left(s_{\epsilon}^{-1}\right)}{d y}(y ; c, a) \leq \frac{\tilde{K}}{g_{m} \epsilon} \tag{4.20}
\end{array}
$$

We now fix $\rho, \sigma$ small enough to satisfy Proposition 4.1. We have the following
Corollary 4.1. There exists $\bar{x}_{f}, \epsilon_{0}>0$ such that for each sufficiently small $\Delta_{z}$, each

$$
\begin{equation*}
\left(\epsilon, c, a, x_{f}\right) \in\left(0, \epsilon_{0}\right) \times I_{c} \times I_{a} \times\left[-\bar{x}_{f}, \bar{x}_{f}\right] \tag{4.21}
\end{equation*}
$$

and each $0 \leq\left|x_{i}\right|<\sigma \rho \epsilon$ there exists $z_{i}=z_{i}\left(\Delta_{z}, \epsilon, x_{i}, x_{f}, c, a\right)$ and $y_{f}=y_{f}\left(\epsilon, x_{i}, x_{f}, c, a\right)$, time $T=T\left(\epsilon, x_{i}, x_{f}, c, a\right)$, and a solution $\phi\left(t ; \epsilon, x_{i}, x_{f}, c, a\right)$ to (4.12) satisfying

1. $\phi\left(0 ; \epsilon, x_{i}, x_{f}, c, a\right)=\left(\tilde{x}_{\epsilon}(c, a)+x_{i},-\rho^{2}, z_{i}, \epsilon\right)$
2. $\phi\left(T ; \epsilon, x_{i}, x_{f}, c, a\right)=\left(x_{f}, s_{\epsilon}\left(x_{f} ; c, a\right)-y_{f}, \Delta_{z}, \epsilon\right)$
where $\left|y_{f}\right|=\mathcal{O}\left(x_{i}\right),\left|D_{\lambda_{0}} y_{f}\right|=\mathcal{O}\left(x_{i} / \epsilon\right),\left|D_{\lambda_{1} \ldots \lambda_{n}} T\right|=\mathcal{O}\left(\epsilon^{-(n+1)}\right)$ and $z_{i}=\mathcal{O}\left(e^{-\eta T}\right)$, some $\eta>0$, for $\lambda_{j}=$ $\left\{x_{i}, x_{f}, c, a\right\}, j=0, \ldots, n$.

Remark 4.2. Corollary 4.1 solves a boundary value problem for (4.12) in the following sense. For each sufficiently small $x_{i}, \Delta_{z}, x_{f}$, the result guarantees the existence of a solution to (4.12) whose basepoint in the center manifold is distance $x_{i}$ in the $x$-direction from $S_{\epsilon}^{+}(c, a)$ in $\Sigma_{i}^{+}$and whose strong unstable $z$ component reaches $\Delta_{z}$ at $x=x_{f}$. Also, the result gives estimates on the derivatives of the initial unstable component $z_{i}$ in $\Sigma_{i}^{+}$, the time $T$ spent until $z=\Delta_{z}$, and the distance $y=y_{f}$ in the $y$-direction from $S_{\epsilon}^{+}(c, a)$ when $(x, z)=\left(x_{f}, \Delta_{z}\right)$.

To prove these results we will use blow up techniques as in [17], and the proofs are given in Sections 4.7 and 4.8, respectively. The blow up is essentially a rescaling which "blows up" the degenerate point $(x, y, \epsilon)=(0,0,0)$ to a 2 -sphere. The blow up transformation is given by

$$
\begin{equation*}
x=\bar{r} \bar{x}, \quad y=-\bar{r}^{2} \bar{y}, \quad \epsilon=\bar{r}^{3} \bar{\epsilon} \tag{4.22}
\end{equation*}
$$

Defining $B_{f}=S^{2} \times\left[0, \bar{r}_{0}\right]$ for some sufficiently small $\bar{r}_{0}$, we consider the blow up as a mapping $B \rightarrow \mathbb{R}^{3}$ with $(\bar{x}, \bar{y}, \bar{\epsilon}) \in S^{2}$ and $\bar{r} \in\left[0, \bar{r}_{0}\right]$. The point $(x, y, \epsilon)=(0,0,0)$ is now represented as a copy of $S^{2}$ (i.e. $\left.\bar{r}=0\right)$ in the blow up transformation. To study the flow on the manifold $B_{f}$ and track solutions near $S_{\epsilon}^{+}(c, a)$ around the fold, there are three relevant coordinate charts. Keeping the same notation as in [17], the first is the chart $\mathcal{K}_{1}$ which uses the coordinates

$$
\begin{equation*}
x=r_{1} x_{1}, \quad y=-r_{1}^{2}, \quad \epsilon=r_{1}^{3} \epsilon_{1} \tag{4.23}
\end{equation*}
$$

the second chart $\mathcal{K}_{2}$ uses the coordinates

$$
\begin{equation*}
x=r_{2} x_{2}, \quad y=-r_{2}^{2} y_{2}, \quad \epsilon=r_{2}^{3} \tag{4.24}
\end{equation*}
$$

and the third chart $\mathcal{K}_{3}$ uses the coordinates

$$
\begin{equation*}
x=r_{3}, \quad y=-r_{3}^{2} y_{3}, \quad \epsilon=r_{3}^{3} \epsilon_{3} \tag{4.25}
\end{equation*}
$$

The setup for the coordinate charts is shown in Figure 12. With these three sets of coordinates, a short calculation gives the following

Lemma 4.1. The transition map $\kappa_{12}: K_{1} \rightarrow K_{2}$ between the coordinates in $\mathcal{K}_{1}$ and $\mathcal{K}_{2}$ is given by

$$
\begin{equation*}
x_{2}=\frac{x_{1}}{\epsilon_{1}^{1 / 3}}, \quad y_{2}=\frac{1}{\epsilon_{1}^{2 / 3}}, \quad r_{2}=r_{1} \epsilon_{1}^{1 / 3}, \quad \text { for } \epsilon_{1}>0 \tag{4.26}
\end{equation*}
$$

and the transition map $\kappa_{23}: K_{2} \rightarrow K_{3}$ between the coordinates in $\mathcal{K}_{2}$ and $\mathcal{K}_{3}$ is given by

$$
\begin{equation*}
r_{3}=r_{2} x_{2}, \quad y_{3}=\frac{y_{2}}{x_{2}^{2}}, \quad \epsilon_{3}=\frac{1}{x_{2}^{3}}, \quad \text { for } x_{2}>0 \tag{4.27}
\end{equation*}
$$



Figure 12: Shown is the set up for the coordinate charts $\mathcal{K}_{i}, i=1,2,3$.

### 4.4 Dynamics in $\mathcal{K}_{1}$

The desingularized equations in the new variables are given by

$$
\begin{align*}
x_{1}^{\prime} & =-1+x_{1}^{2}+\frac{1}{2} \epsilon_{1} x_{1}+\mathcal{O}\left(r_{1}\right)  \tag{4.28}\\
r_{1}^{\prime} & =\frac{1}{2} r_{1} \epsilon_{1}\left(-1+\mathcal{O}\left(r_{1}\right)\right) \\
\epsilon_{1}^{\prime} & =\frac{3}{2} \epsilon_{1}^{2}\left(1+\mathcal{O}\left(r_{1}\right)\right)
\end{align*}
$$

where $^{\prime}=\frac{d}{d t_{1}}=\frac{1}{r_{1}} \frac{d}{d t}$ denotes differentiation with respect to a rescaled time variable $t_{1}$. Here we collect a few results from [17]. Firstly, there are two invariant subspaces for the dynamics of (4.28): the plane $r_{1}=0$ and the plane $\epsilon_{1}=0$. Their intersection is the invariant line $l_{1}=\left\{\left(x_{1}, 0,0\right): x_{1} \in \mathbb{R}\right\}$, and the dynamics on $l_{1}$ evolve according to $x_{1}^{\prime}=-1+x_{1}^{2}$. There are two equilibria $p_{a}=(-1,0,0)$ and $p_{r}=(1,0,0)$. The equilibrium we are interested in, $p_{a}$ has eigenvalue -2 for the flow along $l_{1}$. In the plane $\epsilon_{1}=0$, the dynamics are given by

$$
\begin{align*}
x_{1}^{\prime} & =-1+x_{1}^{2}+\mathcal{O}\left(r_{1}\right)  \tag{4.29}\\
r_{1}^{\prime} & =0
\end{align*}
$$

This system has a normally hyperbolic curve of equilibria $S_{0,1}^{+}(c, a)$ emanating from $p_{a}$ which exactly corresponds to the branch $S_{0}^{+}(c, a)$ of the critical manifold $S$ in the original coordinates. Along $S_{0,1}^{+}(c, a)$ the linearization of (4.29) has one zero eigenvalue and one eigenvalue close to -2 for small $r_{1}$.
In the invariant plane $r_{1}=0$, the dynamics are given by

$$
\begin{align*}
x_{1}^{\prime} & =-1+x_{1}^{2}+\frac{1}{2} \epsilon_{1} x_{1}  \tag{4.30}\\
\epsilon_{1}^{\prime} & =\frac{3}{2} \epsilon_{1}^{2} .
\end{align*}
$$

Here we still have the equilibrium $p_{a}$ which now has an additional zero eigenvalue due to the second equation. The corresponding eigenvector is $(-1,4)$ and hence there exists a one-dimensional center manifold $N_{1}^{+}(c, a)$ at $p_{a}$ along which $\epsilon_{1}$ increases. Note that the branch of $N_{1}^{+}(c, a)$ in the half space $\epsilon_{1}>0$ is unique.
Restricting attention to the set

$$
\begin{equation*}
D_{1}=\left\{\left(x_{1}, r_{1}, \epsilon_{1}\right): x_{1} \in \mathbb{R}, 0 \leq r_{1} \leq \rho, 0 \leq \epsilon_{1} \leq \delta\right\} \tag{4.31}
\end{equation*}
$$

we have the following result from [17]


Figure 13: Shown is the set up in the chart $\mathcal{K}_{1}$.

Proposition 4.2 ([17, Proposition 2.6]). For any $(c, a) \in I_{c} \times I_{a}$ and any sufficiently small $\rho, \delta>0$, the following assertions hold for the dynamics of (4.28):

1. There exists an attracting center manifold $M_{1}^{+}(c, a)$ at $p_{a}$ which contains the line of equilibria $S_{0,1}^{+}(c, a)$ and the center manifold $N_{1}^{+}(c, a)$. In $D_{1}, M_{1}^{+}(c, a)$ is given as a graph $x_{1}=h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)=-1+\mathcal{O}\left(r_{1}, \epsilon_{1}\right)$ with

$$
\begin{equation*}
-3 / 2<h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)<-1 / 2 \quad \text { on } D_{1} \tag{4.32}
\end{equation*}
$$

The branch of $N_{1}^{+}(c, a)$ in $r_{1}=0, \epsilon>0$ is unique. (Note that the manifold $M_{1}^{+}(c, a)$ is precisely the manifold $M^{+}(c, a)$ in the $\mathcal{K}_{1}$ coordinates.)
2. There exists a stable invariant foliation $\mathcal{F}^{s}(c, a)$ with base $M_{1}^{+}(c, a)$ and one-dimensional fibers. For any $\eta>-2$, for any sufficiently small $\rho, \delta$, the contraction along $\mathcal{F}^{s}(c, a)$ during a time interval $[0, T]$ is stronger than $e^{\eta T}$.

Making the change of variables $\tilde{x}_{1}=x_{1}-h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)$, we arrive at the system

$$
\begin{align*}
\tilde{x}_{1}^{\prime} & =\tilde{x}_{1}\left(-2+\tilde{x}_{1}+\mathcal{O}\left(r_{1}, \epsilon_{1}\right)\right)  \tag{4.33}\\
r_{1}^{\prime} & =\frac{1}{2} r_{1} \epsilon_{1}\left(-1+\mathcal{O}\left(r_{1}\right)\right) \\
\epsilon_{1}^{\prime} & =\frac{3}{2} \epsilon_{1}^{2}\left(1+\mathcal{O}\left(r_{1}\right)\right)
\end{align*}
$$

In the chart $\mathcal{K}_{1}$, the section $\Sigma_{i}^{+}$is given by

$$
\begin{equation*}
\Sigma_{1}^{i n}=\left\{\left(x_{1}, r_{1}, \epsilon_{1}\right): 0<\epsilon_{1}<\delta, 0 \leq\left|\tilde{x}_{1}\right|<\sigma \rho^{3} \epsilon_{1}, r_{1}=\rho\right\} \tag{4.34}
\end{equation*}
$$

We define the exit section

$$
\begin{equation*}
\Sigma_{1}^{o u t}=\left\{\left(x_{1}, r_{1}, \epsilon_{1}\right): \epsilon_{1}=\delta, 0 \leq\left|\tilde{x}_{1}\right|<\sigma r_{1}^{3} \delta, 0<r_{1} \leq \rho\right\} \tag{4.35}
\end{equation*}
$$

The set up is shown in Figure 13. We have the following
Lemma 4.2. Consider the system (4.33). There exists $k_{1}>0$ such that for all $(c, a) \in I_{c} \times I_{a}$ and all sufficiently small $\rho, \delta, \sigma>0$, the following holds. Let $\gamma_{1}(t)=\left(x_{1}(t), r_{1}(t), \epsilon_{1}(t)\right)$ denote a solution with $\gamma_{1}(0) \in \Sigma_{1}^{i n}$. Then $\gamma_{1}$ reaches $\Sigma_{1}^{o u t}$. In addition, letting $\tau_{1}$ denote the first time at which $\gamma_{1}\left(\tau_{1}\right) \in \Sigma_{1}^{o u t}$, we have

$$
\begin{align*}
\frac{d x}{d t_{1}} & =r_{1} \frac{d x_{1}}{d t_{1}}+x_{1} \frac{d r_{1}}{d t_{1}}  \tag{4.36}\\
& =r_{1}\left(\tilde{x}_{1}+h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)\right)^{\prime}+r_{1}^{\prime}\left(\tilde{x}_{1}+h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)\right) \\
& >k_{1} \rho r_{1} \epsilon_{1} \quad \text { for } t \in\left[0, \tau_{1}\right]
\end{align*}
$$

Proof. Consider a solution $\gamma_{1}(t)=\left(x_{1}(t), r_{1}(t), \epsilon_{1}(t)\right)$ with $\gamma_{1}(0) \in \Sigma_{1}^{i n}$. We note that for sufficiently small $\rho, \delta$, since $r_{1}^{3} \epsilon_{1}$ is a constant of the motion, $\left|\tilde{x}_{1}\right|$ is decreasing, and $\gamma_{1}$ does indeed exit $\Sigma_{1}^{o u t}$.

To prove (4.36), we compute

$$
\begin{equation*}
r_{1}\left(\tilde{x}_{1}+h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)\right)^{\prime}+r_{1}^{\prime}\left(\tilde{x}_{1}+h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)\right)=r_{1} \tilde{x}_{1}^{\prime}+\frac{1}{2} r_{1} \epsilon_{1}\left(1-\tilde{x}_{1}\right)+\mathcal{O}\left(r_{1}^{2} \epsilon_{1}, r_{1} \epsilon_{1}^{2}\right) \tag{4.37}
\end{equation*}
$$

Since $r_{1}^{3} \epsilon_{1}$ is a constant of the motion and $\left|\tilde{x}_{1}\right|$ is decreasing, $\left|\tilde{x}_{1}\right|<\sigma r_{1}^{3} \epsilon_{1}$. Also, from (4.33), we have $\tilde{x}_{1}^{\prime}=$ $\tilde{x}_{1}\left(-2+\tilde{x}_{1}+\mathcal{O}\left(r_{1}, \epsilon_{1}\right)\right)$ so that

$$
\begin{align*}
r_{1}\left(\tilde{x}_{1}+h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)\right)^{\prime}+r_{1}^{\prime}\left(\tilde{x}_{1}+h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)\right) & =r_{1} \tilde{x}_{1}\left(-2+\tilde{x}_{1}\right)+\frac{1}{2} r_{1} \epsilon_{1}\left(1-\tilde{x}_{1}\right)+\mathcal{O}\left(r_{1}^{2} \epsilon_{1}, r_{1} \epsilon_{1}^{2}\right) \\
& =\frac{1}{2} r_{1} \epsilon_{1}\left(1+\mathcal{O}\left(r_{1}, \epsilon_{1}\right)\right) \tag{4.38}
\end{align*}
$$

Thus there exists $k_{1}>0$ such that for all sufficiently small $\rho, \delta$, the relation (4.36) holds.

### 4.5 Dynamics in $\mathcal{K}_{3}$

In the chart $\mathcal{K}_{3}$, the equations in the new variables are given by

$$
\begin{align*}
\dot{r}_{3} & =r_{3}^{2} F\left(r_{3}, y_{3}, \epsilon_{3}, c, a\right)  \tag{4.39}\\
\dot{y}_{3} & =r_{3}\left[\epsilon_{3}\left(-1+\mathcal{O}\left(r_{3}\right)\right)-2 y_{3} F\left(r_{3}, y_{3}, \epsilon_{3}, c, a\right)\right] \\
\dot{\epsilon}_{3} & =-3 r_{3} \epsilon_{3} F\left(r_{3}, y_{3}, \epsilon_{3}, c, a\right)
\end{align*}
$$

where $F\left(r_{3}, y_{3}, \epsilon_{3}, c, a\right)=1-y_{3}+\mathcal{O}\left(r_{3}\right)$. For small $\beta>0$, consider the set

$$
\begin{equation*}
\Sigma_{3}^{i n}=\left\{\left(r_{3}, y_{3}, \epsilon_{3}\right): 0<r_{3}<\rho, y_{3} \in[-\beta, \beta], \epsilon_{3}=\delta\right\} \tag{4.40}
\end{equation*}
$$

The analysis in [17] shows that $\Sigma_{3}^{i n}$ is carried by the flow of (4.39) to the set

$$
\begin{equation*}
\Sigma_{3}^{o u t}=\left\{\left(r_{3}, y_{3}, \epsilon_{3}\right): r_{3}=\rho, y_{3} \in[-\beta, \beta], \epsilon_{3} \in(0, \delta)\right\} \tag{4.41}
\end{equation*}
$$

What we take from this is that for some fixed $k_{3} \ll 1$, for all sufficiently small $\beta, \rho, \delta$, between the sections $\Sigma_{3}^{i n}$ and $\Sigma_{3}^{o u t}$ we have $F\left(r_{3}, y_{3}, \epsilon_{3}\right)>k_{3} \delta^{2 / 3}$ and thus $\dot{r}_{3}>r_{3}^{2} k_{3} \delta^{2 / 3}$. So for a trajectory starting at $t=0$ in $\Sigma_{3}^{i n}$ with initial $r_{3}(0)=r_{0}$, we have $\dot{r}_{3}>r_{0}^{2} k_{3} \delta^{2 / 3}$. Since $\epsilon_{3}=\delta$ in $\Sigma_{3}^{i n}$ and $\epsilon=r_{3}^{3} \epsilon_{3}$ is a constant of the flow, this implies $\dot{r}_{3}>\epsilon^{2 / 3} k_{3}$.
We can also compute an upper bound for the time spent between $\Sigma_{3}^{i n}$ and $\Sigma_{3}^{o u t}$. By integrating the estimate $\dot{r}_{3}>r_{3}^{2} k_{3} \delta^{2 / 3}$ from 0 to $t$ and using the relation $\epsilon=r_{0}^{3} \delta$, we obtain that $r_{3}(t)>\frac{\epsilon^{1 / 3}}{\delta^{1 / 3}-\epsilon^{1 / 3} \delta^{2 / 3} k_{3} t}$. Thus any trajectory crossing $\Sigma_{3}^{i n}$ reaches $\Sigma_{3}^{o u t}$ in time $t<\frac{1}{k_{3}}\left(\frac{1}{\epsilon^{1 / 3} \delta^{1 / 3}}-\frac{1}{\rho}\right)$. We sum this up in the following
Lemma 4.3. For any $(c, a) \in I_{c} \times I_{a}$ and all sufficiently small $\rho, \delta, \beta$, any trajectory entering $\Sigma_{3}^{i n}$ exits $\Sigma_{3}^{o u t}$ in time $t<\frac{1}{k_{3}}\left(\frac{1}{\epsilon^{1 / 3} \delta^{1 / 3}}-\frac{1}{\rho}\right)$, and between these two sections, $\dot{r}_{3}>\epsilon^{2 / 3} k_{3}$.
We now fix $\beta$ small enough so as to satisfy Lemma 4.3.

### 4.6 Dynamics in $\mathcal{K}_{2}$

In the chart $\mathcal{K}_{2}$, the desingularized equations in the new variables are given by

$$
\begin{align*}
x_{2}^{\prime} & =-y_{2}+x_{2}^{2}+\mathcal{O}\left(r_{2}\right)  \tag{4.42}\\
y_{2}^{\prime} & =-1+\mathcal{O}\left(r_{2}\right) \\
r_{2}^{\prime} & =0,
\end{align*}
$$

where $^{\prime}=\frac{d}{d t_{2}}=\frac{1}{r_{2}} \frac{d}{d t}$ denotes differentiation with respect to a rescaled time variable $t_{1}=r_{2} t$. For $r_{2}=0$, this reduces to the Riccati equation

$$
\begin{align*}
x_{2}^{\prime} & =-y_{2}+x_{2}^{2}  \tag{4.43}\\
y_{2}^{\prime} & =-1
\end{align*}
$$

whose solutions can be expressed in terms of special functions. We quote the relevant results:
Proposition 4.3 ([19, §II.9]). The system (4.43) has the following properties:
(i) There exists a special solution $\gamma_{2,0}(t)=\left(x_{2,0}(t), y_{2,0}(t)\right)$ which can be represented as a graph $y_{2,0}(t)=$ $s_{2,0}\left(x_{2,0}(t)\right)$ for an invertible function $s_{2,0}$ which satisfies $s_{2,0}(x)>\left(x^{2}+\frac{1}{2 x}\right)$ for $x<0$ and $s_{2,0}(x)<x^{2}$ for all $x$. In addition,

$$
s_{2,0}(x)=-\Omega_{0}+1 / x+\mathcal{O}\left(1 / x^{3}\right) \quad \text { as } \quad x \rightarrow \infty
$$

where $\Omega_{0}$ is the smallest positive zero of

$$
J_{-1 / 3}\left(2 z^{3 / 2} / 3\right)+J_{1 / 3}\left(2 z^{3 / 2} / 3\right)
$$

where $J_{-1 / 3}, J_{1 / 3}$ are Bessel functions of the first kind.
(ii) The special solution $\gamma_{2,0}(t)=\left(x_{2,0}(t), y_{2,0}(t)\right)$ satisfies $x_{2,0}^{\prime}(t), x_{2,0}^{\prime \prime}(t)>0$ for all $t$ and $x_{2,0}(t) \rightarrow \pm \infty$ as $t \rightarrow \pm \infty$.

We now fix $\delta$ small enough to satisfy the results of Sections 4.4 and 4.5 as well as taking $2 \Omega_{0} \delta^{2 / 3}<\beta$, where $\beta$ is the small constant fixed at the end of $\S 4.5$. The lemma below follows from a regular perturbation argument.

Lemma 4.4. The special solution $\gamma_{2,0}$ has the following properties:
(i) Let $\tau_{1}, \tau_{2}$ be the times at which $y_{2,0}\left(\tau_{1}\right)=\delta^{-2 / 3}$ and $x_{2,0}\left(\tau_{2}\right)=\delta^{-1 / 3}$. Then there exists $k_{2}$ such that $x_{2,0}^{\prime}(t)>3 k_{2}$ for $t \in\left[\tau_{1}, \tau_{2}\right]$.
(ii) There exists $r_{2}^{*}>0$ such that for any $(c, a) \in I_{c} \times I_{a}$ and any $0<r_{2}<r_{2}^{*}$, the special solution $\gamma_{2,0}$ persists as a solution $\gamma_{2, r_{2}}(t ; c, a)=\left(x_{2, r_{2}}(t ; c, a), y_{2, r_{2}}(t ; c, a), r_{2}\right)$ of (4.42), and solution similarly can be represented as a graph $y=s_{2, r_{2}}(x ; c, a)$ for an invertible function $s_{2, r_{2}}(x ; c, a)$ which is $\mathcal{C}^{1}-\mathcal{O}\left(r_{2}\right)$ close to $s_{2,0}(x)$ on the interval $x \in\left[s_{2, r_{2}}^{-1}\left(\delta^{-2 / 3} ; c, a\right), \delta^{-1 / 3}\right]$. Furthermore, we have $x_{2, r_{2}}^{\prime}(t ; c, a)>2 k_{2}$ for $x \in$ $\left[s_{2, r_{2}}^{-1}\left(\delta^{-2 / 3} ; c, a\right), \delta^{-1 / 3}\right]$.

Remark 4.3. We note that the set

$$
\begin{equation*}
M_{2}^{+}(c, a):=\left\{\left(x_{2, r_{2}}, s_{2, r_{2}}\left(x_{2, r_{2}} ; c, a\right), r_{2}\right): x_{2, r_{2}} \in\left[s_{2, r_{2}}^{-1}\left(\delta^{-2 / 3} ; c, a\right), \delta^{-1 / 3}\right], 0<r_{2}<r_{2}^{*}\right\} \tag{4.44}
\end{equation*}
$$

is in fact a piece of the manifold $M^{+}(c, a)$ in the $\mathcal{K}_{2}$ coordinates.
In the $\mathcal{K}_{2}$ coordinates, we have that $\kappa_{12}\left(\Sigma_{1}^{o u t}\right)$ is contained in the set

$$
\begin{equation*}
\Sigma_{2}^{i n}=\left\{\left(x_{2}, y_{2}, r_{2}\right): 0 \leq\left|x_{2}-s_{2, r_{2}}^{-1}\left(\delta^{-2 / 3} ; c, a\right)\right|<\sigma \rho^{2} \delta^{2 / 3}, y_{2}=\delta^{-2 / 3}, 0<r_{2} \leq \rho \delta^{1 / 3}\right\} \tag{4.45}
\end{equation*}
$$

We also define the exit set

$$
\begin{equation*}
\Sigma_{2}^{o u t}=\left\{\left(x_{2}, y_{2}, r_{2}\right): x_{2}=\delta^{-1 / 3}, 0<r_{2} \leq \rho \delta^{1 / 3}\right\} \tag{4.46}
\end{equation*}
$$

The set up is shown in Figure 14. We have the following


Figure 14: Shown is the set up in the chart $\mathcal{K}_{2}$.

Lemma 4.5. For any $(c, a) \in I_{c} \times I_{a}$ and any sufficiently small $\sigma, \rho$, any solution $\gamma_{2}(t)=\left(x_{2}(t), y_{2}(t), r_{2}\right)$ satisfying $\gamma(0) \in \Sigma_{2}^{i n}$ will reach $\Sigma_{2}^{o u t}$ and between these two sections, this solution satisfies $x_{2}^{\prime}(t)>k_{2}$ and $\left|y_{2}(t)-s_{2, r_{2}}\left(x_{2}(t) ; c, a\right)\right| \leq \Omega_{0}$.

Proof. For sufficiently small $\rho<\delta^{-1 / 3} r_{2}^{*}$, we can appeal to Lemma 4.4 (ii), so that for any $r_{2}<\rho \delta^{1 / 3}$, the special solution $\gamma_{2, r_{2}}$ does in fact reach $\Sigma_{2}^{o u t}$ with $x_{2, r_{2}}^{\prime}(t)>2 k_{2}$ between $\Sigma_{2}^{i n}$ and $\Sigma_{2}^{o u t}$. We can also ensure that $y_{2}^{\prime}>-1 / 2$.
Now consider any solution $\gamma_{2}(t)=\left(x_{2}(t), y_{2}(t), r_{2}\right)$ with $\gamma(0) \in \Sigma_{2}^{i n}$. By taking $\sigma$ small, we can control how close $\gamma_{2}$ and $\gamma_{2, r_{2}}$ are in $\Sigma_{2}^{i n}$. Thus we can ensure that $\gamma_{2}$ reaches $\Sigma_{2}^{\text {out }}$ and $x_{2}^{\prime}(t)>k_{2}$ between $\Sigma_{2}^{i n}$ and $\Sigma_{2}^{o u t}$.
By shrinking $\sigma$ if necessary, it is also possible to control the difference $\left|y_{2}(t)-s_{2, r_{2}}\left(x_{2}(t) ; c, a\right)\right|$.

### 4.7 Proof of Proposition 4.1

The following argument holds for any $\rho, \sigma$ small enough to satisfy the analysis in Sections 4.4, 4.5, and 4.6 (the parameters $\beta$ and $\delta$ were already fixed in Sections 4.5 and 4.6 , respectively).

To prove (i), we follow the section $\Sigma_{i}^{+}$, utilizing the results of the analysis in the previous sections. We consider a solution $\gamma(t)=(x(t), y(t), \epsilon, c, a)$ which starts in $\Sigma_{i}^{+}$. As outlined in $\S 4.4$, in the $\mathcal{K}_{1}$ coordinates, $\Sigma_{i}^{+}$is given by the section $\Sigma_{1}^{i n}$. The section $\Sigma_{1}^{i n}$ is carried to $\Sigma_{1}^{o u t}$ by the flow and between these two sections, using (4.36) we can also compute

$$
\begin{align*}
\frac{d x}{d t} & =\frac{d r_{1}}{d t} x_{1}+r_{1} \frac{d x_{1}}{d t}  \tag{4.47}\\
& =r_{1}\left(\frac{d r_{1}}{d t_{1}} x_{1}+r_{1} \frac{d x_{1}}{d t_{1}}\right) \\
& >k_{1} \rho r_{1}^{2} \epsilon_{1} \\
& >k_{1} \epsilon
\end{align*}
$$

As noted in $\S 4.6, \kappa_{12}\left(\Sigma_{1}^{o u t}\right) \subseteq \Sigma_{2}^{i n}$. Between the two sections $\Sigma_{2}^{i n}$ and $\Sigma_{2}^{o u t}$, Lemma 4.5 gives

$$
\begin{align*}
\frac{d x}{d t} & =r_{2} \frac{d x_{2}}{d t}  \tag{4.48}\\
& =r_{2}^{2} \frac{d x_{2}}{d t_{2}} \\
& >k_{2} r_{2}^{2} \\
& >k_{2} \epsilon^{2 / 3}
\end{align*}
$$

and in addition, by the choice of $2 \Omega_{0} \delta^{2 / 3}<\beta$ in $\S 4.6$, we have that $\kappa_{23}\left(\Sigma_{2}^{o u t}\right)$ is contained in the set $\Sigma_{3}^{i n}$. In chart $\mathcal{K}_{3}$, Lemma 4.3 implies that $\dot{x}(t)>k_{3} \epsilon^{2 / 3}$ between $\Sigma_{3}^{i n}$ and $\Sigma_{3}^{o u t}$. Taking $\tilde{k}<\min \left\{k_{i}: i=1,2,3\right\}$ gives $\dot{x}(t)>\tilde{k} \epsilon$ between $\Sigma^{i n}$ and $\Sigma^{o u t}$, which completes the proof of (i).
It remains to prove the estimates (ii) and (iii) for the function $s_{\epsilon}(x ; c, a)$. In the chart $\mathcal{K}_{1}, S_{0}^{+}(c, a)$ is given by the graph $x_{1}=h_{+}\left(r_{1}, 0, c, a\right)=-1+\mathcal{O}\left(r_{1}\right)$, and for small positive $\epsilon$, between the sections $\Sigma_{1}^{i n}$ and $\Sigma_{1}^{o u t}, S_{\epsilon}^{+}(c, a)$ lies on the manifold defined by the graph

$$
\begin{equation*}
x_{1}=h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)=h_{+}\left(r_{1}, 0, c, a\right)+\mathcal{O}\left(\epsilon_{1}\right) \tag{4.49}
\end{equation*}
$$

We now compute $\frac{d s_{0}}{d x}(x ; c, a)$ and $\frac{d s_{\epsilon}}{d x}(x ; c, a)$ as

$$
\begin{align*}
\frac{d s_{0}}{d x}(x ; c, a) & =\frac{d y / d r_{1}}{d x / d r_{1}}=\frac{2 r_{1}}{h_{+}\left(r_{1}, 0, c, a\right)+r_{1} \partial_{r_{1}} h_{+}\left(r_{1}, 0, c, a\right)}  \tag{4.50}\\
& =r_{1}\left(-2+\mathcal{O}\left(r_{1}\right)\right) \\
\frac{d s_{\epsilon}}{d x}(x ; c, a) & =\frac{d y / d r_{1}}{d x / d r_{1}}=\frac{2 r_{1}}{h_{+}\left(r_{1}, 0, c, a\right)+r_{1} \partial_{r_{1}} h_{+}\left(r_{1}, 0, c, a\right)+\mathcal{O}\left(\epsilon_{1}\right)} \\
& =\frac{d s_{0}}{d x}(x ; c, a)+\mathcal{O}\left(r_{1} \epsilon_{1}\right)
\end{align*}
$$

Between $\Sigma_{1}^{i n}$ and $\Sigma_{1}^{o u t}$, we have that $r_{1} \geq(\epsilon / \delta)^{1 / 3}$. This implies that between $\Sigma_{1}^{i n}$ and $\Sigma_{1}^{o u t}$, we have

$$
\begin{equation*}
\frac{d s_{\epsilon}}{d x}(x ; c, a)-\frac{d s_{0}}{d x}(x ; c, a)=\mathcal{O}\left(\epsilon^{1 / 3}\right) \tag{4.51}
\end{equation*}
$$

To estimate $\left|s_{\epsilon}(x ; c, a)-s_{0}(x ; c, a)\right|$, we write

$$
\begin{equation*}
s_{0}(x ; c, a)=s_{\epsilon}(x ; c, a)+\int_{0}^{1} \frac{d s_{0}}{d x}(x+t(\bar{x}-x) ; c, a) \cdot(x-\bar{x}) d t \tag{4.52}
\end{equation*}
$$

where $\bar{x}=s_{0}^{-1}\left(s_{\epsilon}(x ; c, a) ; c, a\right)$. By (4.50), we have

$$
\begin{align*}
\frac{d s_{0}}{d x}(x ; c, a) & =r_{1}\left(-2+\mathcal{O}\left(r_{1}\right)\right)  \tag{4.53}\\
& =\frac{r_{1} x_{1}\left(-2+\mathcal{O}\left(r_{1}\right)\right)}{h_{+}\left(r_{1}, \epsilon_{1}, c, a\right)}  \tag{4.54}\\
& =\mathcal{O}(x) \tag{4.55}
\end{align*}
$$

by (4.32). Therefore

$$
\begin{equation*}
s_{0}(x ; c, a)=s_{\epsilon}(x ; c, a)+\mathcal{O}\left(x(x-\bar{x}),(x-\bar{x})^{2}\right) \tag{4.56}
\end{equation*}
$$

By (4.49), we have that $(x-\bar{x})=\mathcal{O}\left(r_{1} \epsilon_{1}\right)$ which gives

$$
\begin{align*}
\left|s_{\epsilon}(x ; c, a)-s_{0}(x ; c, a)\right| & =\mathcal{O}\left(r_{1}^{2} \epsilon_{1}\right)  \tag{4.57}\\
& =\mathcal{O}\left(\epsilon^{2 / 3}\right)
\end{align*}
$$

where again we used the fact that between $\Sigma_{1}^{i n}$ and $\Sigma_{1}^{o u t}$, we have that $r_{1} \geq(\epsilon / \delta)^{1 / 3}$.
In the chart $\mathcal{K}_{2}$, the function $s_{\epsilon}(x ; c, a)$ is given by $-r_{2}^{2} s_{2, r_{2}}\left(x r_{2}^{-1} ; c, a\right)$. By Lemma 4.4 (ii), we have that

$$
\begin{equation*}
s_{\epsilon}(x ; c, a)=-r_{2}^{2} s_{2, r_{2}}\left(x r_{2}^{-1} ; c, a\right)=\mathcal{O}\left(\epsilon^{2 / 3}\right) \tag{4.58}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d s_{\epsilon}}{d x}(x ; c, a)=-r_{2} \frac{d s_{2, r_{2}}}{d x}\left(x r_{2}^{-1} ; c, a\right)=\mathcal{O}\left(\epsilon^{1 / 3}\right) \tag{4.59}
\end{equation*}
$$

between $\Sigma_{2}^{i n}$ and $\Sigma_{2}^{o u t}$. Since $s_{0}(x ; c, a)=\mathcal{O}\left(x^{2}\right)$ near $x=0$, we have that in this region in the chart $\mathcal{K}_{2}, s_{0}(x ; c, a)$ satisfies $s_{0}(x ; c, a)=\mathcal{O}\left(\epsilon^{2 / 3}\right)$ and $\frac{d s_{0}}{d x}(x ; c, a)=\mathcal{O}\left(\epsilon^{1 / 3}\right)$.

Once the trajectory exits the chart $\mathcal{K}_{2}$ via $\Sigma_{2}^{o u t}$, we are in a region of positive $x$ where $s_{0}(x ; c, a)=\frac{d s_{0}}{d x}(x ; c, a)=0$, and we can determine the dynamics in the chart $\mathcal{K}_{3}$. From above, we know that in the chart $\mathcal{K}_{3}$, the $y$-coordinate changes by no more than $\mathcal{O}\left(\epsilon^{2 / 3}\right)$ so that $s_{\epsilon}(x ; c, a)=\mathcal{O}\left(\epsilon^{2 / 3}\right)$. Also, in the chart $\mathcal{K}_{3}$, we have that $\dot{x}>k_{3} \epsilon^{2 / 3}$ which gives

$$
\begin{align*}
\frac{d s_{\epsilon}}{d x}(x ; c, a) & =\frac{\dot{y}}{\dot{x}}  \tag{4.60}\\
& \leq \frac{g_{M} \epsilon}{k_{3} \epsilon^{2 / 3}} \\
& =\mathcal{O}\left(\epsilon^{1 / 3}\right)
\end{align*}
$$

This completes the proof of (ii) and (iii).

### 4.8 Proof of Corollary 4.1

Except for the estimates on $T, y_{f}$, the result follows directly from the statement of Proposition 4.1 and the implicit function theorem. To obtain the estimates on the time of flight $T$, we write

$$
\begin{equation*}
T\left(\epsilon, x_{i}, x_{f}, c, a\right)=\int_{\tilde{x}_{\epsilon}(c, a)+x_{i}}^{x_{f}} \frac{1}{\dot{x}} d x \tag{4.61}
\end{equation*}
$$

By Proposition 4.1 (i), we have that $T=\mathcal{O}\left(\epsilon^{-1}\right)$. Since the vector field is smooth, we can differentiate (4.61) and using Proposition 4.1 (i), we obtain the required bounds on the derivatives of $T$ with respect to $x_{i}, x_{f}, c, a$.

To obtain $y_{f}=\mathcal{O}\left(x_{i}\right)$, we look at the evolution of $\tilde{y}=y-s_{\epsilon}(x ; c, a)$. We first note that since the graph $y=s_{\epsilon}(x ; c, a)$ defines a solution to (4.14), we can plug in this solution to get

$$
\begin{equation*}
\frac{d}{d t} s_{\epsilon}(x ; c, a)=\epsilon g\left(x, s_{\epsilon}(x ; c, a), \epsilon, c, a\right) \tag{4.62}
\end{equation*}
$$

Plugging $y=\tilde{y}+s_{\epsilon}(x ; c, a)$ into (4.14) and using (4.62) gives

$$
\begin{align*}
|\dot{\tilde{y}}| & =\epsilon\left|g\left(x, \tilde{y}+s_{\epsilon}(x ; c, a), \epsilon, c, a\right)-g\left(x, s_{\epsilon}(x ; c, a), \epsilon, c, a\right)\right|  \tag{4.63}\\
& =\left|\int_{0}^{1} \epsilon g_{y}\left(x, s \tilde{y}+s_{\epsilon}(x ; c, a), \epsilon, c, a\right) \tilde{y} d s\right| \\
& \leq H_{1} \epsilon|\tilde{y}|
\end{align*}
$$

for some constant $H_{1}$ since the function $g$ is smooth. Therefore $\tilde{y}$ can grow with rate at most $\mathcal{O}(\epsilon)$, and we can deduce that

$$
\begin{equation*}
\left|y_{f}\right|=|\tilde{y}(T)| \leq|\tilde{y}(0)| e^{H_{1} \epsilon T} \tag{4.64}
\end{equation*}
$$

which, by using the bound on $T$ above, we can reduce to

$$
\begin{equation*}
\left|y_{f}\right| \leq H_{2}|\tilde{y}(0)| \tag{4.65}
\end{equation*}
$$

for some constant $H_{2}$. To determine $\tilde{y}(0)$, we write

$$
\begin{align*}
|\tilde{y}(0)| & =\left|s_{\epsilon}\left(x_{\epsilon}(c, a)+x_{i} ; c, a\right)-s_{\epsilon}\left(x_{\epsilon}(c, a) ; c, a\right)\right|  \tag{4.66}\\
& =\left|\int_{0}^{1} \frac{d s_{\epsilon}}{d x}\left(x_{\epsilon}(c, a)+s x_{i} ; c, a\right) \cdot x_{i} d s\right| \\
& \leq \frac{g_{M}}{\tilde{k}}\left|x_{i}\right|
\end{align*}
$$

where we used (4.19).

To obtain the bound on $D y_{f}$, we integrate (4.63) to obtain

$$
\begin{equation*}
y_{f}=\tilde{y}(T)=\int_{0}^{T\left(\epsilon, x_{i}, x_{f}, c, a\right)} \int_{0}^{1} \epsilon g_{y}\left(x, s \tilde{y}+s_{\epsilon}(x ; c, a), \epsilon, c, a\right) \tilde{y}(t) d s d t . \tag{4.67}
\end{equation*}
$$

Using the fact that the function $g$ is smooth and the estimates on $T$ and $D T$ above, we obtain the desired estimate for the first derivative of $y_{f}$ with respect to $x_{i}, x_{f}, c, a$.
The bound on $z_{i}$ comes directly from the equations, but to ensure that $z_{i}$ and its derivatives are exponentially small in $1 / \epsilon$, it is necessary to find a lower bound for the time of flight $T$. We now write

$$
\begin{align*}
T\left(\epsilon, x_{i}, x_{f}, c, a\right) & =\int_{-\rho^{2}}^{y_{f}} \frac{1}{\dot{y}} d y  \tag{4.68}\\
& \geq \frac{1}{g_{M} \epsilon}\left(y_{f}+\rho^{2}\right)
\end{align*}
$$

We note that by Proposition 4.1 (ii) and the analysis above using the fact that $\left|x_{i}\right| \leq \sigma \epsilon$, we have $y_{f}=$ $s_{0}\left(x_{f} ; c, a\right)+\mathcal{O}\left(\epsilon^{2 / 3}\right)$. Since $x_{f} \in\left[-\bar{x}_{f}, \bar{x}_{f}\right]$, we have $s_{0}\left(x_{f} ; c, a\right) \geq s_{0}\left(-\bar{x}_{f} ; c, a\right)$. So we can deduce the existence of $\tau_{0}$ such that for all sufficiently small $\bar{x}_{f}$ and for all sufficiently small $\epsilon$,

$$
\begin{equation*}
T\left(\epsilon, x_{i}, x_{f}, c, a\right) \geq \frac{\tau_{0}}{\epsilon} \tag{4.69}
\end{equation*}
$$

## 5 Tying together the Exchange Lemma and fold analysis

### 5.1 Set-up and transversality

To find connections between the strong unstable manifold $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ of the origin and the stable manifold $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ of the left segment of the slow manifold, we will need two transversality results. The first describes transversality of the manifolds $\mathcal{W}_{0}^{u}(0 ; c, a)$ and $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ with respect to varying the wave speed parameter $c$.

Proposition 5.1. There exists $\epsilon_{0}>0$ and $\mu>0$ such that for each $a \in I_{a}$ and $\epsilon \in\left[0, \epsilon_{0}\right]$, the manifold $\bigcup_{c \in I} \mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ intersects $\bigcup_{c \in I} \mathcal{W}_{\epsilon}^{s, r}(c, a)$ transversely in uvwc-space with the intersection occurring at $c=\tilde{c}(a, \epsilon)$ $c \in I_{c} \quad{ }_{c \in I_{c}}$
for a smooth function $\tilde{c}: I_{a} \times\left[0, \epsilon_{0}\right] \rightarrow I_{c}$ where $\tilde{c}(a, \epsilon)=c^{*}(a)-\mu \epsilon+\mathcal{O}(\epsilon(|a|+\epsilon))$.
Proof. We aim to show that the manifold defined by $\bigcup_{c \in I_{c}} \mathcal{W}_{0}^{u}(0 ; c, a)$ intersects $\bigcup_{c \in I_{c}} \mathcal{W}^{s}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ transversely in $u v w c$-space at $c=c^{*}(a)$ and that this transverse intersection persists for $\epsilon \in\left[0, \epsilon_{0}\right]$. To do this, we note that for each $a \in I_{a}$, there is an intersection of these manifolds occurring along the Nagumo front $\phi_{f}$ for $c=c^{*}(a), \epsilon=0$ in the plane $w=0$. It suffices to show that the intersection at $(c, a, \epsilon)=\left(c^{*}(0), 0,0\right)$ is transverse with respect to varying the wave speed $c$, so that for all sufficiently small $a \in I_{a}$ and $\epsilon>0$, we can solve for an intersection at $c=c(a, \epsilon)$. This amounts to a Melnikov computation along the Nagumo front $\phi_{f}$.
For each $a \in I_{a}$, we consider the planar system (2.4)

$$
\begin{aligned}
\dot{u} & =v \\
\dot{v} & =c v-f(u)
\end{aligned}
$$

obtained by considering (2.1) with $w=\epsilon=0$. As stated above, for $(c, a)=\left(c^{*}(0), 0\right)$, this system possesses a heteroclinic connection $\phi_{f}(t)=\left(u_{f}(t), v_{f}(t)\right)$ (the Nagumo front) between the critical points $(u, v)=(0,0)=p_{0}$ and $(u, v)=(1,0)=p_{1}$ that lies in the intersection of $\mathcal{W}^{u}\left(p_{0}\right)$ and $\mathcal{W}^{s}\left(p_{1}\right)$. We now compute the distance
between $\mathcal{W}^{u}\left(p_{0}\right)$ and $\mathcal{W}^{s}\left(p_{1}\right)$ to first order in $c-c^{*}(0)$. We consider the adjoint equation of the linearization of (2.4) about the Nagumo front $\phi_{f}$ given by

$$
\dot{\psi}=\left(\begin{array}{cc}
0 & \frac{d f}{d u}\left(u_{f}(t)\right)  \tag{5.1}\\
-1 & -c^{*}(0)
\end{array}\right) \psi
$$

Let $\psi_{f}$ be a nonzero bounded solution of (5.1), and let $F_{0}$ denote the right hand side of (2.4). Then

$$
\begin{equation*}
M_{f}^{c}=\int_{-\infty}^{\infty} D_{c} F_{0}\left(\phi_{f}(t)\right) \cdot \psi_{f}(t) d t \tag{5.2}
\end{equation*}
$$

measures the distance between $\mathcal{W}^{u}\left(p_{0}\right)$ and $\mathcal{W}^{s}\left(p_{1}\right)$ to first order in $c-c^{*}(0)$. Thus it remains to show that $M_{f}^{c}$ is nonzero. Up to multiplication by a constant, we have that $\psi_{f}(t)=e^{-c^{*}(0) t}\left(-\dot{v}_{f}(t), \dot{u}_{f}(t)\right)=$ $e^{-c^{*}(0) t}\left(-\dot{v}_{f}(t), v_{f}(t)\right)$ which gives

$$
\begin{equation*}
M_{f}^{c}=\int_{-\infty}^{\infty} e^{-c^{*}(0) t} v_{f}(t)^{2} d t>0 \tag{5.3}
\end{equation*}
$$

as required.
Similarly, we may also compute the distance between $\mathcal{W}^{u}\left(p_{0}\right)$ and $\mathcal{W}^{s}\left(p_{1}\right)$ to first order in $a$ as

$$
\begin{align*}
M_{f}^{a} & =\int_{-\infty}^{\infty} D_{a} F_{0}\left(\phi_{f}(t)\right) \cdot \psi_{f}(t) d t  \tag{5.4}\\
& =\int_{-\infty}^{\infty} e^{-c^{*}(0) t} v_{f}(t) u_{f}(t)\left(1-u_{f}(t)\right) d t
\end{align*}
$$

By (2.9), for $a=0$ the Nagumo front satisfies the relation $v_{f}(t)=\frac{1}{\sqrt{2}} u_{f}(t)\left(1-u_{f}(t)\right)$. Hence

$$
\begin{equation*}
M_{f}^{a}=\sqrt{2} M_{f}^{c} \tag{5.5}
\end{equation*}
$$

To understand how the intersection of $\mathcal{W}_{0}^{u}\left(0 ; c^{*}(0), 0\right)$ and $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{r}\left(c^{*}(0), 0\right)\right)$ breaks as we vary $\epsilon$, we now consider the full three-dimensional system (2.1)

$$
\begin{aligned}
\dot{u} & =v \\
\dot{v} & =c v-f(u)+w \\
\dot{w} & =\epsilon(u-\gamma w) .
\end{aligned}
$$

Note that the function $\left(u_{f}(t), v_{f}(t), 0\right)$ obtained by appending $w=0$ to the Nagumo front $\phi_{f}(t)$ is a solution to this system for $\epsilon=a=0$ and $c=c^{*}(0)$. We consider the adjoint equation of the linearization of (2.1) about this solution given by

$$
\dot{\Psi}=\left(\begin{array}{ccc}
0 & \frac{d f}{d u}\left(u_{f}(t)\right) & 0  \tag{5.6}\\
-1 & -c^{*}(0) & 0 \\
0 & -1 & 0
\end{array}\right) \Psi .
$$

The space of solutions to (5.6) that grow at most algebraically is two-dimensional and spanned by

$$
\begin{equation*}
\Psi_{1}=\left(-e^{-c^{*}(0) t} \dot{v}_{f}(t), e^{-c^{*}(0) t} \dot{u}_{f}(t), \int_{0}^{t}-e^{-c^{*}(0) s} v_{f}(s) d s\right) \tag{5.7}
\end{equation*}
$$

and $\Psi_{2}=(0,0,1)$. The function

$$
\begin{equation*}
\Psi=\left(-e^{-c^{*}(0) t} \dot{v}_{f}(t), e^{-c^{*}(0) t} \dot{u}_{f}(t), \int_{t}^{\infty} e^{-c^{*}(0) s} v_{f}(s) d s\right) \tag{5.8}
\end{equation*}
$$

is the unique such solution to (5.6) (up to multiplication by a constant) satisfying $\Psi(t) \rightarrow 0$ as $t \rightarrow \infty$. Let $F_{1}$ denote the right hand side of (2.1); then by Melnikov theory, we can describe the distance between $\mathcal{W}_{\epsilon}^{u}(c, a)$ and $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ to first order in $\epsilon$ by the integral:

$$
\begin{align*}
M_{f}^{\epsilon} & =\int_{-\infty}^{\infty} D_{\epsilon} F_{1}\left(u_{f}(t), v_{f}(t), 0\right) \cdot \Psi(t) d t  \tag{5.9}\\
& =\int_{-\infty}^{\infty}\left(\int_{t}^{\infty} e^{-c^{*}(0) s} v_{f}(s) d s\right) u_{f}(t) d t>0
\end{align*}
$$

The distance function $d(c, a, \epsilon)$ which defines the separation between $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ and $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ can now be expanded as

$$
\begin{equation*}
d(c, a, \epsilon)=M_{f}^{c}\left(c-c^{*}(0)+\sqrt{2} a\right)+M_{f}^{\epsilon} \epsilon+\mathcal{O}\left(\left(\left|c-c^{*}(0)\right|+a+\epsilon\right)^{2}\right) \tag{5.10}
\end{equation*}
$$

To find an intersection between $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ and $\mathcal{W}_{\epsilon}^{s, r}(c, a)$, we now solve the equation $d(c, a, \epsilon)=0$ for $c$ and obtain $c=\tilde{c}(a, \epsilon)=c^{*}(a)-\mu \epsilon+\mathcal{O}(\epsilon(|a|+\epsilon))$ where $\mu:=M_{f}^{\epsilon} / M_{f}^{c}>0$ due to (5.3) and (5.9), and we used the fact that $\sqrt{2} a=c^{*}(0)-c^{*}(a)$. The lack of $\mathcal{O}\left(a^{2}\right)$ terms in the expression for $\tilde{c}(a, \epsilon)$ is due to the fact that for $\epsilon=0$ the intersection occurs at $c=c^{*}(a)$.

The second result needed is transversality of $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ along the back for $a=0$. The problem here is that $\mathcal{M}_{0}^{r}(c, a)$ is not actually normally hyperbolic at the fold and therefore Fenichel theory does not ensure smooth persistence of the manifold $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ in this region for $\epsilon>0$ : we will have to appeal to results from $\S 4$ to obtain the necessary transversality.
To start with, in the neighborhood $O_{f}$, in the center manifold near the fold point, we extend the right branch $\mathcal{M}_{0}^{r}(c, a)$ of the critical manifold by concatenating it with the fast unstable fiber leaving the fold point (see the description of the function $s_{0}(x ; c, a)$ in $\left.\S 4.3\right)$ and call this new manifold $\mathcal{M}_{0}^{r,+}(c, a)$. It now makes sense to define $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r,+}(c, a)\right)$ as the union of the strong unstable fibers of this singular trajectory. The advantage is now that Proposition 4.1 shows that $\mathcal{M}_{0}^{r,+}(c, a)$ persists as a trajectory $\mathcal{M}_{\epsilon}^{r,+}(c, a)$ which is $\mathcal{C}^{1}-\mathcal{O}\left(\epsilon^{1 / 3}\right)$ close to $\mathcal{M}_{0}^{r,+}(c, a)$. We can then define $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ to be the union of the strong unstable fibers of this perturbed solution. We are ready to state the following result.

Proposition 5.2. For each $(c, a) \in I_{c} \times I_{a}, \mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ intersects $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r,+}(c, a)\right)$ transversely in uvwcspace along the Nagumo back $\phi_{b}$, and this transverse intersection persists for $\epsilon \in\left[0, \epsilon_{0}\right]$. Furthermore, for each $(c, a, \epsilon) \in I_{c} \times I_{a} \times\left[0, \epsilon_{0}\right]$, the manifold $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ intersects $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ transversely.

Proof. We note that past the fold point, $\mathcal{M}_{0}^{r,+}(c, a)$ lies in a plane of constant $w$ since in this region $\mathcal{M}_{0}^{r,+}(c, a)$ is described by the fast $\epsilon=0$ flow. Thus we proceed as in the proof of Proposition 5.1, though now we show transversality of the manifolds $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}(c, a)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}(c, a)\right)$ with respect to $w$, which is a parameter for the fast $\epsilon=0$ flow.

It suffices to prove transversality at $(\epsilon, a)=(0,0)$. By the $\mathcal{C}^{1}$ dependence of the manifolds with respect to $a$, this transversality persists for $a \in I_{a}$. The fact that this transversality persists for small $\epsilon>0$ follows from the $\mathcal{C}^{1}-\mathcal{O}\left(\epsilon^{1 / 3}\right)$ closeness of $\mathcal{M}_{\epsilon}^{r,+}(c, a)$ and $\mathcal{M}_{0}^{r,+}(c, a)$. This implies that $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r,+}(c, a)\right)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ are also $\mathcal{C}^{1}-\mathcal{O}\left(\epsilon^{1 / 3}\right)$ close.

To continue, we consider the planar system

$$
\begin{align*}
\dot{u} & =v  \tag{5.11}\\
\dot{v} & =c v-f(u)+w
\end{align*}
$$

obtained by considering (2.1) with $a=\epsilon=0$. For $c=c^{*}(0)$ and $w=w^{*}(0)$, this system possesses a heteroclinic connection $\phi_{b}(t)=\left(u_{b}(t), v_{b}(t)\right)$ (the Nagumo back) between the critical points $(u, v)=\left(u_{1}, 0\right)=q_{1}$ and
$(u, v)=\left(u_{0}, 0\right)=q_{0}$ where $u_{0}$ and $u_{1}$ are the smallest and largest zeros of $w^{*}(0)-f(u)$, respectively. That is, there exists an intersection of $\mathcal{W}^{u}\left(q_{1}\right)$ and $\mathcal{W}^{s}\left(q_{0}\right)$ defined by the connection $\phi_{b}$.
Thus the manifolds $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}\left(c^{*}(0), 0\right)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}\left(c^{*}(0), 0\right)\right)$ intersect in the full system along the Nagumo back $\phi_{b}$. Since $\mathcal{M}_{0}^{r,+}\left(c^{*}(0), 0\right)$ lies in the plane $w=w^{*}(0)$ past the fold point (and thus so do its fast fibers since the fast flow is confined to $w=$ const planes), we have that $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}\left(c^{*}(0), 0\right)\right)$ is tangent to the plane $w=w^{*}(0)$ along $\phi_{b}$. In (5.11), from regular perturbation theory, the stable manifold of the leftmost equilibrium (given by the trajectory $\phi_{b}$ at $\left.w=w^{*}(0)\right)$ breaks smoothly in $w$ and thus $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}\left(c^{*}(0), 0\right)\right)$ is transverse to planes $w=$ const; in particular this gives the necessary transversality of $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}\left(c^{*}(0), 0\right)\right)$ and $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r}\left(c^{*}(0), 0\right)\right)$.
We therefore obtain the desired transversality of $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ for all $(c, a, \epsilon) \in I_{c} \times I_{a} \times\left[0, \epsilon_{0}\right]$.

### 5.2 Exchange Lemma

In this section we use the Exchange Lemma of [21] to track the manifold $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ near the right branch $\mathcal{M}_{\epsilon}^{r}(c, a)$ of the slow manifold up to a fixed neighborhood of the fold point. The analysis of $\S 4$ defines a fixed neighborhood $O_{f}$ of the fold point in uvw-coordinates for $(c, a) \in I_{c} \times I_{a}$ in which the flow is well understood. The neighborhood $O_{f}$ corresponds to the neighborhood $U_{f}$ in $x y z c a$-coordinates in which the section $\Sigma_{i}^{+}$defines points along trajectories satisfying the desired estimates.

We may assume that the manifold $\mathcal{M}_{\epsilon}^{r}(c, a)$ extends into this neighborhood past the section $\Sigma_{i}^{+}$but ends before the fold (in $U_{f}$ note that $\mathcal{M}_{\epsilon}^{r}(c, a)$, where defined, coincides with $S_{\epsilon}^{+}(c, a)$ up to errors exponentially small in $1 / \epsilon$ due to the non-uniqueness of the center manifold in $\S 4)$. Here $\mathcal{M}_{\epsilon}^{r}(c, a)$ is normally hyperbolic, and thus there exists a $C^{r+1}$ Fenichel normal form for the equations in a neighborhood of $\mathcal{M}_{\epsilon}^{r}(c, a)$ :

$$
\begin{align*}
X^{\prime} & =-A(X, Y, Z, c, a, \epsilon) X  \tag{5.12}\\
Y^{\prime} & =B(X, Y, Z, c, a, \epsilon) Y \\
Z^{\prime} & =\epsilon(1+E(X, Y, Z, c, a, \epsilon) X Y) \\
c^{\prime} & =0 \\
a^{\prime} & =0
\end{align*}
$$

where the functions $A$ and $B$ are positive and bounded away from 0 uniformly in all variables. These equations are valid in a neighborhood $U_{e}$ of $\mathcal{M}_{\epsilon}^{r}(c, a), c \in I_{c}, a \in I_{a}$ which we assume to be given by $X, Y \in(-\Delta, \Delta)$ for some small $\Delta>0$ and $(Z, c, a) \in V=\left(-\Delta, Z_{0}+\Delta\right) \times I_{c} \times I_{a}$ for appropriate $Z_{0}>\Delta$. In $U_{e}$, for each $c, a, \epsilon$, the manifold $\mathcal{M}_{\epsilon}^{r}(c, a)$ is given by $X=Y=0$. Similarly the manifolds $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ and $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ are given by $X=0$ and $Y=0$ respectively. We denote the change of coordinates from $(X, Y, Z, c, a)$ to the $(u, v, w, c, a)$ coordinates by $\Phi_{e}: U_{e} \rightarrow O_{e}$ where $O_{e}$ is the corresponding neighborhood of $\mathcal{M}_{\epsilon}^{r}(c, a)$ in $(u, v, w)$-coordinates for $(c, a) \in I_{c} \times I_{a}$. Since $O_{e}$ is by construction a neighborhood of a normally hyperbolic segment of $\mathcal{M}_{\epsilon}^{r}(c, a)$ which extends into $O_{f}$, there is an overlap of the neighborhoods $O_{e}$ and $O_{f}$ where the fold analysis is valid. We now comment on the constants $\Delta, Z_{0}$ : since $\mathcal{M}_{\epsilon}^{r}(c, a)$ extends past the section $\Sigma_{i}^{+}$in the neighborhood $U_{f}$, for $\Delta$ sufficiently small, we can think of $Z_{0}$ as being the height in the $U_{e}$ coordinates at which $\mathcal{M}_{\epsilon}^{r}(c, a)$ hits $\Sigma_{i}^{+}$for $(c, a, \epsilon)=\left(c^{*}(0), 0,0\right)$; see $\S 5.3$ for details.
We note that due to the non-uniqueness of the center manifold in $\S 4$, the coordinate descriptions of the manifolds $\mathcal{M}_{\epsilon}^{r}(c, a), \mathcal{W}_{\epsilon}^{u, r}(c, a)$, and $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ in the two neighborhoods $O_{e}$ and $O_{f}$ are only equal up to errors exponentially small in $1 / \epsilon$. Since these errors are taken into account in the analysis below, for simplicity we will use the same notation for these manifolds in the different coordinate systems.
For each $\epsilon \in\left[0, \epsilon_{0}\right]$ we define the two-dimensional incoming manifold

$$
\begin{equation*}
N_{\epsilon}^{i n}=\left(\bigcup_{c \in I_{c}, a \in I_{a}} \mathcal{W}_{\epsilon}^{u}(0 ; c, a)\right) \cap\{X=\Delta\} \tag{5.13}
\end{equation*}
$$

which, under the flow of (5.12) becomes a manifold $N_{\epsilon}^{*}$ of dimension three. Define

$$
\begin{equation*}
\mathcal{A}=\left\{(Y, Z, a): Y \in(-\Delta, \Delta), Z \in\left(Z_{0}-\Delta, Z_{0}+\Delta\right), a \in I_{a}\right\} \tag{5.14}
\end{equation*}
$$

The necessary transversality of the incoming manifold $N_{\epsilon}^{i n}$ with $\{Y=0\}$ is given by Proposition 5.1. The generalized Exchange Lemma now gives the following

Theorem 5.1 ([21, Theorem 3.1]). There exist functions $\tilde{X}, \tilde{W}: \mathcal{A} \times\left[0, \epsilon_{0}\right] \rightarrow \mathbb{R}$ which satisfy
(i) For $\epsilon>0$, the set $\{(X, Y, Z, a, c):(Y, Z, a) \in \mathcal{A}, X=\tilde{X}(Y, Z, a, \epsilon), c=\tilde{c}(a, \epsilon)+\tilde{W}(Y, Z, a, \epsilon)\}$ is contained in $N_{\epsilon}^{*}$.
(ii) $\tilde{X}(Y, Z, a, 0)=0, \tilde{W}(Y, Z, a, 0)=0, \tilde{W}(0, Z, a, \epsilon)=0$
(iii) There exists $q>0$ such that $\left|D_{j} \tilde{X}\right|,\left|D_{j} \tilde{W}\right|=\mathcal{O}\left(e^{-q / \epsilon}\right)$ for any $0 \leq j \leq r$.

We comment on the interpretation of Theorem 5.1. For each choice of $a$, height $Z$ and unstable component $Y$ lying in $\mathcal{A}$, provided the offset $c-\tilde{c}(a, \epsilon)$ is adjusted by the quantity $\tilde{W}(Y, Z, a, \epsilon)$, the theorem guarantees a solution which starts in $N_{\epsilon}^{i n}$ which hits the point $(X, Y, Z, a, c)$ where $X=\tilde{X}(Y, Z, a, c)$. In (ii), the property $\tilde{W}(0, Z, a, \epsilon)=0$ refers to the fact that for $c=\tilde{c}(a, \epsilon)$, the manifold $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ in fact lies in the stable foliation $Y=0$, which was proved in Proposition 5.1. The final properties $\tilde{X}(Y, Z, a, 0)=0, \tilde{W}(Y, Z, a, 0)=0$, and property (iii) state that the functions $\tilde{X}, \tilde{W} \rightarrow 0$ uniformly in the limit $\epsilon \rightarrow 0$, and that this convergence is in fact exponential in derivatives up to order $r$.

### 5.3 Set-up in $U_{e}$

We will use Theorem 5.1 to describe the flow up to a neighborhood of the fold point, then we will use the results of $\S 4$. We first place a section $\Sigma^{i n}$ in the neighborhood $U_{f}$ of the fold point which we define by

$$
\begin{equation*}
\Sigma^{i n}=\left\{(x, y, z, c, a, \epsilon) \in U_{f}: y=-\rho^{2},\left|x-\tilde{x}_{0}\left(c^{*}(0), 0\right)\right| \leq \Delta^{\prime}, z \in\left[-\Delta^{\prime}, \Delta^{\prime}\right],(c, a, \epsilon) \in I\right\} \tag{5.15}
\end{equation*}
$$

for some small choice of $\Delta^{\prime}$ where $I=I_{c} \times I_{a} \times\left[0, \epsilon_{0}\right]$. As described above, there is an of overlap of the regions described by $U_{f}$ and the neighborhood $U_{e}$ where the Fenichel normal form is valid. We denote the change of coordinates between these neighborhoods by $\Phi_{\text {ef }}: \Phi_{e}^{-1}\left(O_{e} \cap O_{f}\right) \subseteq U_{e} \rightarrow U_{f}$ where $\Phi_{\text {ef }}=\Phi_{f}^{-1} \circ \Phi_{e}$. From the construction the section $\Phi_{\text {ef }}^{-1}\left(\Sigma^{i n}\right)$ will be given by a section in $X Y Z$-space transverse to the sets $X=$ const and $Y=$ const. We can therefore represent $\Phi_{\text {ef }}^{-1}\left(\Sigma^{i n}\right)$ in $X Y Z$-space as $\Phi_{\text {ef }}^{-1}\left(\Sigma^{i n}\right)=\{(X, Y, Z, c, a, \epsilon): Z=$ $\psi(X, Y, c, a, \epsilon)\}$ for some smooth function

$$
\begin{equation*}
\psi:[-\Delta, \Delta] \times[-\Delta, \Delta] \times I \rightarrow\left[-\Delta+Z_{0}, \Delta+Z_{0}\right] \tag{5.16}
\end{equation*}
$$

where we assume that $Z_{0}$ has been chosen so that $\psi\left(0,0, c^{*}(0), 0,0\right)=Z_{0}$. It is important to note that since $\mathcal{M}_{\epsilon}^{r}(c, a)$ and $S_{\epsilon}^{+}(c, a)$ are equal up to exponentially small errors in the $U_{f}$ coordinates, $\Phi_{\text {ef }}(0,0, Z, c, a, \epsilon)$ maps onto $S_{\epsilon}^{+}(c, a)$ up to errors exponentially small in $1 / \epsilon$. Figure 15 shows the setup as well as the passage of a trajectory according to the Exchange Lemma. Figure 16 shows the continuation of this trajectory past the fold. The idea is to show that for each $a \in I_{a}$ and each $\epsilon \in\left(0, \epsilon_{0}\right)$, we can find $c$ such that this solution connects $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ to $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ as shown.

### 5.4 Entering $U_{f}$ via the Exchange lemma

We now use the Exchange Lemma to solve for solutions which cross $\Phi_{\text {ef }}^{-1}\left(\Sigma^{i n}\right)$. For each $Y \in[-\Delta, \Delta], a \in I_{a}$, and $\epsilon \in\left(0, \epsilon_{0}\right]$, we can find a solution which reaches the point

$$
\begin{equation*}
(X, Y, \psi(X, Y, c, a, \epsilon), c, a) \in \Phi_{\mathrm{ef}}^{-1}\left(\Sigma^{i n}\right) \tag{5.17}
\end{equation*}
$$



Figure 15: Shown is the set up of the Exchange Lemma (Theorem 5.1).


Figure 16: Shown is the flow near the fold point.
provided we can solve

$$
\begin{align*}
X & =\tilde{X}(Y, \psi(X, Y, c, a, \epsilon), a, \epsilon)  \tag{5.18}\\
c & =\tilde{c}(a, \epsilon)+\tilde{W}(Y, \psi(X, Y, c, a, \epsilon), a, \epsilon)
\end{align*}
$$

in terms of $(Y, a, \epsilon)$ where $\tilde{X}, \tilde{W}$ are the functions from Theorem 5.1. Using the fact that $\psi$ is smooth and that $\tilde{X}, \tilde{W}$ and their derivatives are $\mathcal{O}\left(e^{-q / \epsilon}\right)$, we can solve by the implicit function theorem for $(X, c-\tilde{c}(a, \epsilon))$ near $(0,0)$ in terms of the variables $(Y, a, \epsilon)$ to obtain

$$
\begin{align*}
X & =X^{*}(Y, a, \epsilon)  \tag{5.19}\\
c & =\tilde{c}(a, \epsilon)+W^{*}(Y, a, \epsilon)
\end{align*}
$$

where the smooth functions $X^{*}, W^{*}$ and their derivatives are $\mathcal{O}\left(e^{-q / \epsilon}\right)$, where we may need to take $q$ smaller. To sum up, we have just shown the following:

Proposition 5.3. For each $Y \in[-\Delta, \Delta]$, $a \in I_{a}$, and $\epsilon \in\left(0, \epsilon_{0}\right]$, we can find a solution which reaches the point

$$
\begin{equation*}
(X, Y, \psi(X, Y, c, a, \epsilon), c, a) \in \Phi_{e f}^{-1}\left(\Sigma^{i n}\right) \tag{5.20}
\end{equation*}
$$

where $X=X^{*}(Y, a, \epsilon)$ and $c=\tilde{c}(a, \epsilon)+W^{*}(Y, a, \epsilon)$. The functions $X^{*}$ and $W^{*}$ and their derivatives are $\mathcal{O}\left(e^{-q / \epsilon}\right)$.

### 5.5 Connecting to $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ : analysis in $U_{f}$

What we conclude from Proposition 5.3 is that for any sufficiently small choice of $(Y, a, \epsilon)$ we can find a solution which enters a neighborhood of the fold at a distance $Y$ from $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ along the unstable fibers provided $c$ is adjusted from $\tilde{c}(a, \epsilon)$ by $\mathcal{O}\left(e^{-q / \epsilon}\right)$. In addition the distance from $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ is $\mathcal{O}\left(e^{-q / \epsilon}\right)$. By applying the smooth transition map $\Phi_{\text {ef }}$, it is convenient to rewrite Proposition 5.3 in the $U_{f}$ coordinates as

Proposition 5.4. For each $z \in\left[-\Delta^{\prime}, \Delta^{\prime}\right], a \in I_{a}$, and $\epsilon \in\left(0, \epsilon_{0}\right]$, we can find a solution which reaches the point

$$
\begin{equation*}
\left(x,-\rho^{2}, z, c, a\right) \in \Sigma^{i n} \tag{5.21}
\end{equation*}
$$

where $x=\tilde{x}_{\epsilon}(c, a)+x^{*}(z, a, \epsilon)$ and $c=\tilde{c}(a, \epsilon)+w^{*}(z, a, \epsilon)$. The functions $x^{*}, w^{*}$ and their derivatives are $\mathcal{O}\left(e^{-q / \epsilon}\right)$.

Remark 5.1. Though the manifolds $\mathcal{W}_{\epsilon}^{s, r}(c, a)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ are not unique, the errors we incur by transforming to the $U_{f}$ coordinates are exponentially small in $1 / \epsilon$ and can be absorbed in the functions $x^{*}, w^{*}$ without changing the result.

We will use this result along with the center manifold analysis of $\S 4$ to find such a solution for each $(a, \epsilon)$ which connects to $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$. We first determine the location of $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ in the neighborhood $U_{f}$. From Proposition 5.2, we know that $\mathcal{W}^{s}\left(\mathcal{M}_{0}^{\ell}\left(c^{*}(0), 0\right)\right)$ intersects $\mathcal{W}^{u}\left(\mathcal{M}_{0}^{r,+}\left(c^{*}(0), 0\right)\right)$ transversely for $\epsilon=0$ along the Nagumo back $\phi_{b}$, and this intersection persists for $(c, a, \epsilon) \in I_{c} \times I_{a} \times\left(0, \epsilon_{0}\right)$. This means that $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ will transversely intersect the manifold $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ which is composed of the union of the unstable fibers of the continuation of the slow manifold $\mathcal{M}_{\epsilon}^{r,+}(c, a)$ found in $\S 4$. We therefore place an exit section $\Sigma^{o u t}$ defined by

$$
\begin{equation*}
\Sigma^{o u t}=\left\{(x, y, z, c, a, \epsilon) \in U_{f}: z=\Delta^{\prime}\right\} \tag{5.22}
\end{equation*}
$$

For $(c, a, \epsilon) \in I_{c} \times I_{a} \times\left(0, \epsilon_{0}\right)$, the intersection of $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ and $\mathcal{W}_{\epsilon}^{u, r}(c, a)$ occurs at a point

$$
\begin{equation*}
(x, y, z, c, a, \epsilon)=\left(x_{\ell}(c, a, \epsilon), s_{\epsilon}\left(x_{\ell}(c, a, \epsilon) ; c, a\right), \Delta^{\prime}, c, a, \epsilon\right) \in \Sigma^{o u t} \tag{5.23}
\end{equation*}
$$

and thus we may expand $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ in $\Sigma^{o u t}$ as

$$
\begin{equation*}
\left(x, y-s_{\epsilon}(x ; c, a)\right)=\left(x_{\ell}(c, a, \epsilon)+\mathcal{O}(\tilde{y}, \epsilon), \tilde{y}\right), \quad \tilde{y} \in\left[-\Delta_{y}, \Delta_{y}\right] \tag{5.24}
\end{equation*}
$$

for some small $\Delta_{y}$. Now using Corollary 4.1, we aim to find a solution to match with $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ at $z=\Delta^{\prime}$. From Corollary 4.1 for each $\left(\epsilon, c, a, x_{i}, x_{f}\right)$, we get a solution $\phi\left(t ; \epsilon, x_{i}, x_{f}, c, a\right)$ and time of flight $T\left(\epsilon, x_{i}, x_{f}, c, a\right)$ satisfying

$$
\begin{align*}
\phi\left(0 ; \epsilon, x_{i}, x_{f}, c, a\right) & =\left(\tilde{x}_{\epsilon}(c, a)+x_{i},-\rho^{2}, z_{i}, \epsilon\right)  \tag{5.25}\\
\phi\left(T ; \epsilon, x_{i}, x_{f}, c, a\right) & =\left(x_{f}, s_{\epsilon}\left(x_{f} ; c, a\right)-y_{f}, \Delta^{\prime}, \epsilon\right)
\end{align*}
$$

Thus finding a connection between $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ and $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ for a given $(a, \epsilon)$ amounts to solving the following system of equations

$$
\begin{align*}
x_{i} & =x^{*}(z, a, \epsilon)  \tag{5.26}\\
c & =\tilde{c}(a, \epsilon)+w^{*}(z, a, \epsilon) \\
x_{f} & =x_{\ell}(c, a, \epsilon)+\mathcal{O}(\tilde{y}, \epsilon) \\
y_{f}\left(x_{i}, x_{f}, \epsilon, a, c\right) & =\tilde{y} \\
z_{i}\left(\Delta^{\prime}, x_{i}, x_{f}, \epsilon, a, c\right) & =z
\end{align*}
$$

for all variables in terms of $(a, \epsilon)$.
We start by substituting $x_{i}=x^{*}(z, a, \epsilon)$ into the equation for $\tilde{y}$. Using the fact that $x^{*}(z, a, \epsilon)=\mathcal{O}\left(e^{-q / \epsilon}\right)$ and the estimates $y_{f}=\mathcal{O}\left(x_{i}\right)$ and $D y_{f}=\mathcal{O}\left(x_{i} / \epsilon\right)$ from Corollary 4.1, we can solve for $\tilde{y}=\tilde{y}(z, a, \epsilon)$ by the implicit function theorem where $\tilde{y}, D \tilde{y}=\mathcal{O}\left(e^{-q / \epsilon}\right)$, where $q$ may need to be taken smaller. We now substitute everything into the equation for $z$. Using the estimates on $z_{i}$ from Corollary 4.1 and the estimates on $\tilde{y}$ above, we can then solve for $z=z(a, \epsilon)$ (and subsequently all other variables) by the implicit function theorem.
In particular, we note that the wave speed $c$ is given by

$$
\begin{align*}
c(a, \epsilon) & =\tilde{c}(a, \epsilon)+\mathcal{O}\left(e^{-q / \epsilon}\right)  \tag{5.27}\\
& =c^{*}(a)-\mu \epsilon+\mathcal{O}(\epsilon(|a|+\epsilon))
\end{align*}
$$

where $\mu>0$ is the constant from Proposition 5.1 , and we have absorbed the exponentially small terms in the $\mathcal{O}\left(\epsilon^{2}\right)$ term.

## 6 Convergence of $M_{\epsilon}^{l}(c, a)$ to the equilibrium

The analysis of Sections 4 and 5 shows that for each $a \in I_{a}$ and for any sufficiently small $\epsilon$, there exists a wave speed $c$ such that the manifold $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ intersects $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$. Upon entering a neighborhood of the origin, this trajectory will be exponentially close to the perturbed slow manifold $\mathcal{M}_{\epsilon}^{\ell}(c, a)$. It remains to show that $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and as well as nearby trajectories on $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ in fact converge to the equilibrium at the origin. This result is not immediate, as for $a=0$, the origin is on the lower left knee where $\mathcal{M}_{0}^{\ell}(c, a)$ is not normally hyperbolic. In this section, a center manifold analysis of the origin produces conditions on $(a, \epsilon)$ which ensure this result.

### 6.1 Preparation of equations

To study the stability properties of the equilibrium at $(u, v, w)=(0,0,0)$ of (2.1) for small $\epsilon$, $a$, we append equations for $a$ and $\epsilon$ to (2.1) and obtain

$$
\begin{align*}
\dot{u} & =v  \tag{6.1}\\
\dot{v} & =c v-f(u)+w \\
\dot{w} & =\epsilon(u-\gamma w) \\
\dot{a} & =0 \\
\dot{\epsilon} & =0 .
\end{align*}
$$

For $a=\epsilon=0$, the origin coincides with the lower left knee on the critical manifold. System (6.1) has the family of equilibria $(u, 0, f(u), a, 0)$ where $u$ varies near 0 . We are interested in the lower left knee of $w=f(u)$ as a function of $a$. For $(c, a) \in I_{c} \times I_{a}$, the knee is given by the family $\left(u^{\dagger}(a), 0, w^{\dagger}(a), a, 0\right)$ where

$$
\begin{align*}
u^{\dagger}(a) & =\frac{1}{3}\left(a+1-\sqrt{a^{2}-a+1}\right)  \tag{6.2}\\
w^{\dagger}(a) & =f\left(u^{\dagger}(a)\right)
\end{align*}
$$

The linearization of (6.1) about the knee at $(a, \epsilon)=0$ is given by

$$
J=\left(\begin{array}{lllll}
0 & 1 & 0 & 0 & 0  \tag{6.3}\\
0 & c & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

There is one positive eigenvalue $\lambda=c$ with eigenvector ( $1, c, 0,0,0$ ) and a quadruple zero eigenvalue. By making the coordinate transformation

$$
\begin{align*}
& z_{1}=u-u^{\dagger}-\frac{v}{c}-\frac{w-w^{\dagger}}{c^{2}}  \tag{6.4}\\
& z_{2}=-\frac{w-w^{\dagger}}{c} \\
& z_{3}=\frac{v}{c}+\frac{w-w^{\dagger}}{c^{2}}
\end{align*}
$$

we arrive at the system

$$
\begin{align*}
\dot{z}_{1} & =z_{2}+\frac{1}{c}\left(\sqrt{a^{2}-a+1}\right)\left(z_{1}+z_{3}\right)^{2}-\frac{1}{c}\left(z_{1}+z_{3}\right)^{3}-\frac{\epsilon}{c^{2}}\left(z_{1}+z_{3}+c \gamma z_{2}+u^{\dagger}-\gamma w^{\dagger}\right)  \tag{6.5}\\
\dot{z}_{2} & =-\frac{\epsilon}{c}\left(z_{1}+z_{3}+c \gamma z_{2}+u^{\dagger}-\gamma w^{\dagger}\right) \\
\dot{z_{3}} & =c z_{3}-\frac{1}{c}\left(\sqrt{a^{2}-a+1}\right)\left(z_{1}+z_{3}\right)^{2}+\frac{1}{c}\left(z_{1}+z_{3}\right)^{3}+\frac{\epsilon}{c^{2}}\left(z_{1}+z_{3}+c \gamma z_{2}+u^{\dagger}-\gamma w^{\dagger}\right) \\
\dot{a} & =0 \\
\dot{\epsilon} & =0
\end{align*}
$$

which, for $\epsilon=0$, is in Jordan normal form for the three dynamic variables $\left(z_{1}, z_{2}, z_{3}\right)$. To understand the dynamics near the fold point, we separate the nonhyperbolic dynamics which occur on a four-dimensional center manifold. In a small neighborhood of the fold point, this manifold can be represented as a graph

$$
\begin{align*}
z_{3} & =F\left(z_{1}, z_{2}, \epsilon\right)  \tag{6.6}\\
& =\beta_{0} z_{1}+\beta_{1} z_{2}+\beta_{2} z_{1}^{2}+\mathcal{O}\left[\epsilon, z_{1} z_{2}, z_{2}^{2}, z_{1}^{3}\right]
\end{align*}
$$

We can directly compute the coefficients $\beta_{i}$, and we find that

$$
\begin{equation*}
\beta_{0}=\beta_{1}=0, \quad \beta_{2}=\frac{1}{c^{2}}\left(\sqrt{a^{2}-a+1}\right)=\frac{1}{c^{2}}+\mathcal{O}(a) \tag{6.7}
\end{equation*}
$$

We now make the following change of coordinates

$$
\begin{align*}
x & =\frac{1}{c^{1 / 2}}\left(\sqrt{a^{2}-a+1}\right) z_{1}  \tag{6.8}\\
y & =-\left(\sqrt{a^{2}-a+1}\right) z_{2} \\
\alpha & =\frac{a}{2 c^{1 / 2}}
\end{align*}
$$

and rescale time by $c^{1 / 2}$ which gives the flow on the center manifold in the coordinates $(x, y, \alpha, \epsilon)$ as

$$
\begin{align*}
\dot{x} & =-y(1+\mathcal{O}(x, y, \alpha, \epsilon))+x^{2}(1+\mathcal{O}(x, y, \alpha, \epsilon))+\epsilon \mathcal{O}(x, y, \alpha, \epsilon)  \tag{6.9}\\
\dot{y} & =\epsilon[x(1+\mathcal{O}(x, y, \alpha, \epsilon))+\alpha(1+\mathcal{O}(x, y, \alpha, \epsilon))+\mathcal{O}(y)] \\
\dot{\alpha} & =0 \\
\dot{\epsilon} & =0
\end{align*}
$$

Making one further coordinate transformation in the variable $z_{3}$ to straighten out the unstable fibers, we arrive at the full system

$$
\begin{align*}
\dot{x} & =-y+x^{2}+\mathcal{O}\left(\epsilon, x y, y^{2}, x^{3}\right)  \tag{6.10}\\
\dot{y} & =\epsilon[x(1+\mathcal{O}(x, y, \alpha, \epsilon))+\alpha(1+\mathcal{O}(x, y, \alpha, \epsilon))+\mathcal{O}(y)] \\
\dot{z} & =z(c+\mathcal{O}(x, y, z, \epsilon)) \\
\dot{\alpha} & =0 \\
\dot{\epsilon} & =0
\end{align*}
$$

We note that the $(x, y)$ coordinates are in the canonical form for a canard point (compare [17]), that is,

$$
\begin{align*}
\dot{x} & =-y h_{1}(x, y, \alpha, \epsilon, c)+x^{2} h_{2}(x, y, \alpha, \epsilon, c)+\epsilon h_{3}(x, y, \alpha, \epsilon, c)  \tag{6.11}\\
\dot{y} & =\epsilon\left(x h_{4}(x, y, \alpha, \epsilon, c)+\alpha h_{5}(x, y, \alpha, \epsilon, c)+y h_{6}(x, y, \alpha, \epsilon, c)\right) \\
\dot{z} & =z(c+\mathcal{O}(x, y, z, \epsilon)) \\
\dot{\alpha} & =0 \\
\dot{\epsilon} & =0
\end{align*}
$$

where we have

$$
\begin{align*}
h_{3}(x, y, \alpha, \epsilon, c) & =\mathcal{O}(x, y, \alpha, \epsilon)  \tag{6.12}\\
h_{j}(x, y, \alpha, \epsilon, c) & =1+\mathcal{O}(x, y, \alpha, \epsilon), \quad j=1,2,4,5
\end{align*}
$$

We have now separated the hyperbolic dynamics (given by the $z$-coordinate) from the nonhyperbolic dynamics which are isolated on a four-dimensional center manifold parameterized by the variables $(x, y, \epsilon, \alpha)$ on which the origin is a canard point in the sense of [17]. Geometrically, in a singularly perturbed system a canard point is characterized by a folded critical manifold with one attracting and one repelling branch and a singular "canard" trajectory traveling down the attracting branch and continuing up the repelling branch (see Figure 17). Such points are associated with "canard explosion" phenomena in which small scale oscillations near the equilibrium undergo a rapid transformation in an exponentially small region in parameter space and emerge as large relaxation cycles [18]. We note that Figure 23 in $\S 7$ provides a visualization of what such a canard explosion looks like, though in this case the solutions depicted are homoclinic pulse solutions rather than periodic orbits; we refer to $\S 7$ for a more detailed discussion.

### 6.2 Tracking $M_{\epsilon}^{l}(c, a)$ close to the canard point - blowup and rescaling

From Fenichel theory, we know that away from the canard point, the left branch $M_{0}^{l}(c, a)$ of the critical manifold perturbs to a slow manifold $M_{\epsilon}^{l}(c, a)$ for small $\epsilon>0$ (see Figure 17). This slow manifold is unique up to errors exponentially small in $1 / \epsilon$; as the preceding analysis is valid for any such choice of $M_{\epsilon}^{l}(c, a)$, we may now fix a choice of $M_{\epsilon}^{l}(c, a)$ which lies in the center manifold $z=0$. In addition, there is a stable equilibrium $p_{0}=p_{0}(\alpha)$ for the slow flow on $M_{0}^{l}(c, a)$ for $\alpha>0$. The goal of this section is to show that under suitable conditions on $\epsilon$ and


Figure 17: Shown is the flow near the canard point for $\epsilon=\alpha=0$. Away from the canard point, the left branch $M_{0}^{l}(c, a)$ of the critical manifold perturbs smoothly to a slow manifold $M_{\epsilon}^{l}(c, a)$ for small $\epsilon>0$.
$\alpha, p_{0}$ persists as a stable equilibrium $p_{\epsilon}(\alpha)$, and the perturbed slow manifold $M_{\epsilon}^{l}(c, a)$ and nearby trajectories converge to $p_{\epsilon}$.
To do this, the idea will be to track a section $\Delta_{\text {in }}(\rho, \sigma)=\left\{(x, y):|x+\rho| \leq \sigma \rho, y=\rho^{2}\right\}$ (see Figure 17) for some small $\rho, \sigma>0$ and show that all trajectories crossing this section converge to the equilibrium. We have the following

Proposition 6.1. Consider the section $\Delta_{i n}(\rho, \sigma)$ for the system (6.11) in the center manifold $z=0$. For each $K>0$, there exists $\tilde{\rho}, \tilde{\sigma}, \tilde{\epsilon}, \tilde{\alpha}$ such that for $(\rho, \sigma, \epsilon, \alpha) \in D$ and $c \in I_{c}$, there is a stable equilibrium for (6.11), where $D$ is given by

$$
\begin{equation*}
D=\left\{(\rho, \sigma, \epsilon, \alpha): \rho \in(0, \tilde{\rho}), \sigma \in(0, \tilde{\sigma}), 0<\epsilon<\rho^{2} \tilde{\epsilon}, 0<\alpha<\rho \tilde{\alpha}, 0<\epsilon<K \alpha^{2}\right\} \tag{6.13}
\end{equation*}
$$

Furthermore, under these conditions, all trajectories passing through $\Delta_{i n}(\rho, \sigma)$ converge to the equilibrium.

From this we have the following
Corollary 6.1. For each $K>0$, there exists a choice of the parameters $\rho, \sigma, a_{0}, \epsilon_{0}$ such that for all $(c, a, \epsilon) \in$ $I_{c} \times\left(0, a_{0}\right) \times\left(0, \epsilon_{0}\right)$ satisfying $\epsilon<K a^{2}$, the manifold $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ crosses the section $\Delta^{i n}(\rho, \sigma)$ and thus converges to the equilibrium.

Remark 6.1. The aim of Corollary 6.1 is to prove convergence of the tails of the pulses constructed in Sections 4 and 5. Thus far, we have found an intersection of $\mathcal{W}_{\epsilon}^{u}(0 ; c, a)$ with $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$; this trajectory will therefore be exponentially attracted to the perturbed slow manifold $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ upon entering a neighborhood of the origin. The manifold $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$, however, is only unique up to errors exponentially small in $1 / \epsilon$, though the justification for the intersection holds for any such choice of $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$. Therefore, we may now fix $\mathcal{W}_{\epsilon}^{s, \ell}(c, a)$ to be the manifold formed by evolving the section $\Delta^{i n}(\rho, \sigma)$ in backwards time in the center manifold $z=0$.

We now fix an arbitrary $K>0$. The section $\Delta_{\text {in }}(\rho, \sigma)$ will be tracked using blowup methods as in [17]. Restricting to the center manifold $z=0$, we proceed as in $\S 4$, though now the blow up transformation is given by

$$
\begin{equation*}
x=\bar{r} \bar{x}, \quad y=\bar{r}^{2} \bar{y}, \quad \alpha=\bar{r} \bar{\alpha}, \quad \epsilon=\bar{r}^{2} \bar{\epsilon}, \tag{6.14}
\end{equation*}
$$

defined on the manifold $B_{c}=S^{2} \times\left[0, \bar{r}_{0}\right] \times\left[-\bar{\alpha}_{0}, \bar{\alpha}_{0}\right]$ for sufficiently small $\bar{r}_{0}, \bar{\alpha}_{0}$ with $(\bar{x}, \bar{y}, \bar{\epsilon}) \in S^{2}$. There are three relevant coordinate charts which will be needed for the analysis of the flow on the manifold $B_{c}$. Keeping the same notation as in [17] and [18], the first is the chart $\mathcal{K}_{1}$ which uses the coordinates

$$
\begin{equation*}
x=r_{1} x_{1}, \quad y=r_{1}^{2}, \quad \alpha=r_{1} \alpha_{1}, \quad \epsilon=r_{1}^{2} \epsilon_{1} \tag{6.15}
\end{equation*}
$$

the second chart $\mathcal{K}_{2}$ uses the coordinates

$$
\begin{equation*}
x=r_{2} x_{2}, \quad y=r_{2}^{2} y_{2}, \quad \alpha=r_{2} \alpha_{2}, \quad \epsilon=r_{2}^{2} \tag{6.16}
\end{equation*}
$$

and the third chart $\mathcal{K}_{4}$ uses the coordinates

$$
\begin{equation*}
x=r_{4} x_{4}, \quad y=r_{4}^{2} y_{4}, \quad \alpha=r_{4}, \quad \epsilon=r_{4}^{2} \epsilon_{4} . \tag{6.17}
\end{equation*}
$$

With these three sets of coordinates, a short calculation gives the following
Lemma 6.1. The transition map $\kappa_{12}: K_{1} \rightarrow K_{2}$ between the coordinates in $\mathcal{K}_{1}$ and $\mathcal{K}_{2}$ is given by

$$
\begin{equation*}
x_{2}=\frac{x_{1}}{\epsilon_{1}^{1 / 2}}, \quad y_{2}=\frac{1}{\epsilon_{1}}, \quad \alpha_{2}=\frac{\alpha_{1}}{\epsilon_{1}^{1 / 2}}, \quad r_{2}=r_{1} \epsilon_{1}^{1 / 2}, \quad \text { for } \epsilon_{1}>0 \tag{6.18}
\end{equation*}
$$

and the transition map $\kappa_{14}: K_{1} \rightarrow K_{4}$ between the coordinates in $\mathcal{K}_{1}$ and $\mathcal{K}_{4}$ is given by

$$
\begin{equation*}
x_{4}=\frac{x_{1}}{\alpha_{1}}, \quad y_{4}=\frac{1}{\alpha_{1}^{2}}, \quad \epsilon_{4}=\frac{\epsilon_{1}}{\alpha_{1}^{2}}, \quad r_{4}=r_{1} \alpha_{1}, \quad \text { for } \alpha_{1}>0 . \tag{6.19}
\end{equation*}
$$

### 6.3 Dynamics in $\mathcal{K}_{1}$

Here we outline the relevant dynamics in $\mathcal{K}_{1}$ as described in [17]. After desingularizing the equations in the new variables, we arrive at the following system

$$
\begin{align*}
x_{1}^{\prime} & =-1+x_{1}^{2}-\frac{1}{2} \epsilon_{1} x_{1} F\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}, c\right)+\mathcal{O}\left(r_{1}\right)  \tag{6.20}\\
r_{1}^{\prime} & =\frac{1}{2} r_{1} \epsilon_{1} F\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}, c\right) \\
\epsilon_{1}^{\prime} & =-\epsilon_{1}^{2} F\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}, c\right) \\
\alpha_{1}^{\prime} & =-\frac{1}{2} \alpha_{1} \epsilon_{1} F\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}, c\right),
\end{align*}
$$

where

$$
\begin{equation*}
F\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}, c\right)=x_{1}+\alpha_{1}+\mathcal{O}\left(r_{1}\right) \tag{6.21}
\end{equation*}
$$

Here we collect a few results from [17]. The hyperplanes $r_{1}=0, \epsilon_{1}=0, \alpha_{1}=0$ are all invariant. Their intersection is the invariant line $l_{1}=\left\{\left(x_{1}, 0,0,0\right): x_{1} \in \mathbb{R}\right\}$, and the dynamics on $l_{1}$ evolve according to $x_{1}^{\prime}=-1+x_{1}^{2}$. There are two equilibria $p_{a}=(-1,0,0,0)$ and $p_{r}=(1,0,0,0)$. The equilibrium we are interested in, $p_{a}$ has eigenvalue -2 for the flow along $l_{1}$. There is a normally hyperbolic curve of equilibria $S_{0,1}^{+}(c)$ emanating from $p_{a}$ which exactly corresponds to the manifold $\mathcal{M}_{0}^{\ell}(c, 0)$ in the original coordinates. Restricting attention to the set

$$
\begin{equation*}
D_{1}=\left\{\left(x_{1}, r_{1}, \epsilon_{1}\right):-2<x_{1}<2,0 \leq r_{1} \leq \rho, 0 \leq \epsilon_{1} \leq \tilde{\epsilon},-\tilde{\alpha} \leq \alpha_{1} \leq \tilde{\alpha}\right\} \tag{6.22}
\end{equation*}
$$

we have the following result which will be useful in obtaining an expression for $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ :
Proposition 6.2 ([17, Proposition 3.4]). Consider the system (6.20). For any $c \in I_{c}$ and all sufficiently small $\rho, \tilde{\epsilon}, \tilde{\alpha}>0$, there exists a three-dimensional attracting center manifold $M_{1}^{+}(c)$ at $p_{a}$ which contains the line of equilibria $S_{0,1}^{+}(c)$. In $D_{1}, M_{1}^{+}(c)$ is given as a graph $x_{1}=h_{+}\left(r_{1}, \epsilon_{1}, \alpha_{1}, c\right)=-1+\mathcal{O}\left(r_{1}, \epsilon_{1}, \alpha_{1}\right)$.

We now consider the following section

$$
\Sigma_{1}^{\text {in }}:=\left\{\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}\right):\left|1+x_{1}\right|<\sigma, 0<\epsilon_{1} \leq K \alpha_{1}^{2}, 0<\alpha_{1} \leq \tilde{\alpha}, r_{1}=\rho\right\}
$$

where $\sigma, \tilde{\alpha}, \rho$ will be chosen appropriately. It is clear that in the chart $\mathcal{K}_{1}, \Delta_{\text {in }}(\rho, \sigma)$ is contained in the section $\Sigma_{1}^{\mathrm{in}}$ for $\epsilon$ and $\alpha$ in the desired range, hence the goal will be to track the evolution of $\Sigma_{1}^{\mathrm{in}}$. To accomplish this, we consider two subsets of $\Sigma_{1}^{\text {in }}$ defined by

$$
\begin{aligned}
& \Sigma_{14}^{\mathrm{in}}:=\left\{\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}\right):\left|1+x_{1}\right|<\sigma, 0<\epsilon_{1} \leq 2 \delta \alpha_{1}^{2}, 0<\alpha_{1} \leq \tilde{\alpha}, r_{1}=\rho\right\} \\
& \Sigma_{12}^{\mathrm{in}}:=\left\{\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}\right):\left|1+x_{1}\right|<\sigma, \delta \alpha_{1}^{2}<\epsilon_{1} \leq K \alpha_{1}^{2}, 0<\epsilon_{1} \leq \tilde{\epsilon}, r_{1}=\rho\right\}
\end{aligned}
$$

where $\delta$ and $\tilde{\epsilon} \ll \delta$ will be chosen appropriately small. The evolution of $\Sigma_{14}^{\mathrm{in}}$ and $\Sigma_{12}^{\mathrm{in}}$ will be governed by the dynamics of the charts $\mathcal{K}_{4}$ and $\mathcal{K}_{2}$, respectively.

We now consider the two exit sections defined by

$$
\begin{aligned}
\Sigma_{14}^{\text {out }} & :=\left\{\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}\right):\left|1+x_{1}\right|<\sigma, 0<\epsilon_{1} \leq 2 \delta \tilde{\alpha}^{2}, 0 \leq r_{1} \leq \rho, \alpha_{1}=\tilde{\alpha}\right\} \\
\Sigma_{12}^{\text {out }} & :=\left\{\left(x_{1}, r_{1}, \epsilon_{1}, \alpha_{1}\right):\left|1+x_{1}\right|<\sigma, \frac{\tilde{\epsilon}}{K}<\alpha_{1}^{2} \leq \frac{\tilde{\epsilon}}{\delta}, 0 \leq r_{1} \leq \rho, \epsilon_{1}=\tilde{\epsilon}\right\}
\end{aligned}
$$

The following lemma describes the flow in the chart $\mathcal{K}_{1}$ for (6.20).
Lemma 6.2. There exists $\tilde{\alpha}>0$ and $k_{1}^{*}>0$ such that the following hold for all $c \in I_{c}$ :
(i) For any $\rho, \sigma, \delta<k_{1}^{*}$, the flow maps $\Sigma_{14}^{i n}$ into $\Sigma_{14}^{o u t}$.
(ii) Fix $\delta<k_{1}^{*}$. There exists $\tilde{\epsilon}_{1}^{*}<\delta$ such that for any $\tilde{\epsilon}<\tilde{\epsilon}_{1}^{*}$ and any $\rho, \sigma<k_{1}^{*}$, the flow maps $\Sigma_{12}^{i n}$ into $\Sigma_{12}^{o u t}$.

Thus once $\tilde{\alpha}>0$ and $k_{1}^{*}>0$ are fixed as in the above lemma, it is possible to define the transition map $\Pi_{14}: \Sigma_{14}^{\text {in }} \rightarrow \Sigma_{14}^{\text {out }}$ for any $\rho, \sigma, \delta<k_{1}^{*}$. Once $\delta<k_{1}^{*}$ is fixed, we may then also define the transition map $\Pi_{12}: \Sigma_{12}^{\mathrm{in}} \rightarrow \Sigma_{12}^{\text {out }}$. Hence we first determine the evolution of $\Pi_{14}\left(\Sigma_{14}^{\mathrm{in}}\right) \subseteq \Sigma_{14}^{\text {out }}$ in the chart $\mathcal{K}_{4}$ in order to choose $\delta$ appropriately. Then it will be possible to consider the evolution of $\Pi_{12}\left(\Sigma_{12}^{\mathrm{in}}\right) \subseteq \Sigma_{12}^{\text {out }}$ in the chart $\mathcal{K}_{2}$.

### 6.4 Dynamics in $\mathcal{K}_{4}$

Fix $\tilde{\alpha}$ as in Lemma 6.2. We desingularize the equations in the new variables and arrive at the following system

$$
\begin{align*}
x_{4}^{\prime} & =-y_{4}+x_{4}^{2}+\mathcal{O}\left(r_{4}\right)  \tag{6.23}\\
y_{4}^{\prime} & =\epsilon_{4}\left(1+x_{4}+\mathcal{O}\left(r_{4}\right)\right) \\
r_{4}^{\prime} & =0 \\
\epsilon_{4}^{\prime} & =0
\end{align*}
$$

We think of this system as a singularly perturbed system with two slow variables $y_{4}$ and $r_{4}$, one fast variable $x_{4}$, and singular perturbation parameter $\epsilon_{4}$.
It is possible to define a critical manifold $S_{0}\left(r_{4}\right)$ in each fixed $r_{4}$ slice for $r_{4} \in\left[0, r_{4}^{*}\right]$ for some small $r_{4}^{*}$. At $r_{4}=0$ this critical manifold can be taken as any segment of the curve $y_{4}=x_{4}^{2}$ for $x_{4}$ in any negative compact interval bounded away from 0 , say for $x_{4} \in\left[-x_{4}^{\ell},-x_{4}^{r}\right]$ where we can take $x_{4}^{\ell}>2 / \tilde{\alpha}$ and $0<x_{4}^{r}<1 / 2$. For each fixed $r_{4} \in\left[0, r_{4}^{*}\right]$, there is a similar critical manifold for the same range of $x_{4}$. Define $M_{0}$ to be the union of the curves $S_{0}\left(r_{4}\right)$ over $r_{4} \in\left[0, r_{4}^{*}\right]$. Then $M_{0}$ is a compact 2-dimensional critical manifold for $\epsilon_{4}=0$ for the full 3-dimensional system. In addition, provided $r_{4}^{*}$ is sufficiently small, for each fixed $r_{4}$ the slow flow on $S_{0}\left(r_{4}\right)$ has a stable equilibrium $p_{0}\left(r_{4}\right)$ with $p_{0}(0)=(-1,1)$. Figure 18 shows the setup for $\epsilon_{4}=0$.
In addition $M_{0}$ has a stable manifold $\mathcal{W}^{s}\left(M_{0}\right)$ consisting of the planes $y_{4}=$ const. In particular, we consider the subset of $\mathcal{W}^{s}\left(M_{0}\right)$ defined by

$$
\begin{equation*}
\mathcal{W}_{0}=\left\{\left(x_{4}, y_{4}, \epsilon_{4}, r_{4}\right): x_{4} \in\left[-x_{4}^{\ell},-x_{4}^{r}\right], y_{4} \in\left[\left(x_{4}^{r}\right)^{2},\left(x_{4}^{\ell}\right)^{2}\right], \epsilon_{4}=0, r_{4} \in\left[0, r_{4}^{*}\right]\right\} \tag{6.24}
\end{equation*}
$$

It follows from Fenichel theory that the critical manifold $M_{0}$ and its stable manifold $\mathcal{W}^{s}\left(M_{0}\right)$ perturb smoothly for small $\epsilon_{4}>0$ to invariant manifolds $M_{\epsilon_{4}}$ and $\mathcal{W}^{s}\left(M_{\epsilon_{4}}\right)$. Further, the equilibria $p_{0}\left(r_{4}\right)$ persist as stable equilibria $p_{\epsilon_{4}}\left(r_{4}\right)$, and in each fixed $r_{4}$ slice, all orbits lying on $\mathcal{W}^{s}\left(M_{\epsilon_{4}}\right)$ converge to $p_{\epsilon_{4}}\left(r_{4}\right)$. The dynamics for $\epsilon_{4}>0$ are shown in Figure 19. In particular, there exists $\tilde{\epsilon}_{4}^{*}>0$ such that for $0<\epsilon_{4}<\tilde{\epsilon}_{4}^{*}$, the set $\mathcal{W}_{0}$ perturbs to a set $\mathcal{W}_{\epsilon_{4}}$, all points of which converge to $p_{\epsilon_{4}}\left(r_{4}\right)$.


Figure 18: Shown is the critical manifold $M_{0}$ in chart $\mathcal{K}_{4}$ for $\epsilon_{4}=0$. Also shown is the $\epsilon_{4}=0$ curve of equilibria $p_{0}\left(r_{4}\right)$ for the slow flow on $M_{0}$. The section $\Sigma_{4}^{\mathrm{in}}$ as in the proof of Proposition 6.1 is also shown.


Figure 19: Shown is the perturbed slow manifold $M_{\epsilon_{4}}$ in chart $\mathcal{K}_{4}$ for $\epsilon_{4}>0$. All trajectories on $\mathcal{W}^{s}\left(M_{\epsilon_{4}}\right)$ converge to $p_{\epsilon_{4}}\left(r_{4}\right)$.

Using the transition map $\kappa_{14}(6.19)$, we have in chart $\mathcal{K}_{4}$ that $\kappa_{14}\left(\Sigma_{14}^{\text {out }}\right)$ is contained in the set

$$
\begin{equation*}
\Sigma_{4}^{\text {in }}=\left\{\left(x_{4}, y_{4}, \epsilon_{4}, r_{4}\right):\left|\frac{1}{\tilde{\alpha}}+x_{4}\right|<\frac{\sigma}{\tilde{\alpha}}, 0 \leq \epsilon_{4} \leq 2 \delta, 0<r_{4} \leq \rho \tilde{\alpha}, y_{4}=\frac{1}{\tilde{\alpha}^{2}}\right\}, \tag{6.25}
\end{equation*}
$$

which is shown in Figure 18 for $\epsilon_{4}=0$.
We can now prove the following
Lemma 6.3. There exists $k_{4}^{*}$ such that for any $\rho, \delta<k_{4}^{*}$ and any $\sigma<1 / 2$, there exists a curve of equilibria $p_{\epsilon_{4}}\left(r_{4}\right)$ such that all trajectories crossing the section $\Sigma_{4}^{i n}$ converge to $p_{\epsilon_{4}}\left(r_{4}\right)$.

Proof. For any $\rho<r_{4}^{*} / \tilde{\alpha}$ and any $\sigma<1 / 2$, the set $\Sigma_{4}^{\mathrm{in}} \cap\left\{\epsilon_{4}=0\right\}$ lies in $\mathcal{W}_{0}$. Thus by taking $\delta<\tilde{\epsilon}_{4}^{*} / 2$, we have that all trajectories passing through $\Sigma_{4}^{\mathrm{in}}$ converge to the unique equilibrium on the slow manifold $M_{\epsilon_{4}}\left(r_{4}\right)$ for each $r_{4} \in(0, \rho \tilde{\alpha})$. Thus taking $k_{4}^{*}<\min \left(\frac{r_{4}^{*}}{\tilde{\alpha}}, \frac{\tilde{\epsilon}_{4}^{*}}{2}\right)$ proves the result.

### 6.5 Dynamics in $\mathcal{K}_{2}$

We now fix $\delta<\min \left(k_{1}^{*}, k_{4}^{*}\right)$ and desingularize the equations in the $\mathcal{K}_{2}$ coordinates to arrive at the following system

$$
\begin{align*}
x_{2}^{\prime} & =-y_{2}+x_{2}^{2}+\mathcal{O}\left(r_{2}\right)  \tag{6.26}\\
y_{2}^{\prime} & =x_{2}+\alpha_{2}+\mathcal{O}\left(r_{2}\right) \\
r_{2}^{\prime} & =0 \\
\alpha_{2}^{\prime} & =0
\end{align*}
$$

Making the change of variables $\tilde{x}_{2}=x_{2}+\alpha_{2}$ and $\tilde{y}_{2}=y_{2}-\alpha_{2}^{2}$, we arrive at the system

$$
\begin{align*}
\tilde{x}_{2}^{\prime} & =-\tilde{y}_{2}+\tilde{x}_{2}^{2}-2 \tilde{x}_{2} \alpha_{2}+\mathcal{O}\left(r_{2}\right)  \tag{6.27}\\
\tilde{y}_{2}^{\prime} & =\tilde{x}_{2}+\mathcal{O}\left(r_{2}\right) \\
r_{2}^{\prime} & =0 \\
\alpha_{2}^{\prime} & =0
\end{align*}
$$

For $r_{2}=\alpha_{2}=0$, the system is integrable with constant of motion

$$
\begin{equation*}
H\left(\tilde{x}_{2}, \tilde{y}_{2}\right)=\frac{1}{2} e^{-2 \tilde{y}_{2}}\left(\tilde{y}_{2}-\tilde{x}_{2}^{2}+\frac{1}{2}\right) . \tag{6.28}
\end{equation*}
$$

The function $H$ has a continuous family of closed level curves

$$
\begin{equation*}
\Gamma^{h}=\left\{\left(\tilde{x}_{2}, \tilde{y}_{2}\right): H\left(\tilde{x}_{2}, \tilde{y}_{2}\right)=h\right\}, \quad h \in(0,1 / 4) \tag{6.29}
\end{equation*}
$$

contained in the interior of the parabola $\tilde{y}_{2}=\tilde{x}_{2}^{2}-1 / 2$, which is the level curve for $h=0$ (see Figure 20).
For $r_{2}=0$, we have that

$$
\begin{equation*}
\frac{d H}{d t}=2 e^{-2 \tilde{y}_{2}} \alpha_{2} \tilde{x}_{2}^{2} \tag{6.30}
\end{equation*}
$$

so that for positive $\alpha_{2}$, all trajectories in the interior of the parabola $\tilde{y}_{2}=\tilde{x}_{2}^{2}-1 / 2$ converge to the unique equilibrium $\left(\tilde{x}_{2}, \tilde{y}_{2}\right)=(0,0)$ corresponding to the maximum value $h=1 / 4$. For sufficiently small $r_{2}$, this equilibrium persists, and we denote it by $p_{2}\left(r_{2}\right)$ with $p_{2}(0)=(0,0)$.


Figure 20: Shown are the dynamics in the chart $\mathcal{K}_{2}$ as well as the section $\Sigma_{2}^{\mathrm{in}}$ for the cases of $\alpha_{2}=0$ and $\alpha_{2}>0$. For $\alpha_{2}=0$, the orbits are given by the level curves $\Gamma^{h}$ for the function $H\left(\tilde{x}_{2}, \tilde{y}_{2}\right)$.

Using the transition map $\kappa_{12}(6.18)$, we have in chart $\mathcal{K}_{2}$ that $\kappa_{12}\left(\Sigma_{12}^{\text {out }}\right)$ is contained in the set

$$
\begin{equation*}
\Sigma_{2}^{\operatorname{in}}=\left\{\left(x_{2}, y_{2}, \alpha_{2}, r_{2}\right):\left|\frac{1}{\tilde{\epsilon}^{1 / 2}}+x_{2}\right|<\frac{\sigma}{\tilde{\epsilon}^{1 / 2}}, \frac{1}{K^{1 / 2}} \leq \alpha_{2} \leq \frac{1}{\delta^{1 / 2}}, 0<r_{2} \leq \rho^{2} \tilde{\epsilon}, y_{2}=\frac{1}{\tilde{\epsilon}}\right\} \tag{6.31}
\end{equation*}
$$

which in the coordinates $\left(\tilde{x}_{2}, \tilde{y}_{2}\right)$ is the set

$$
\begin{equation*}
\Sigma_{2}^{\text {in }}=\left\{\left(\tilde{x}_{2}, \tilde{y}_{2}, \alpha_{2}, r_{2}\right):\left|\frac{1}{\tilde{\epsilon}^{1 / 2}}+\tilde{x}_{2}-\alpha_{2}\right|<\frac{\sigma}{\tilde{\epsilon}^{1 / 2}}, \frac{1}{K^{1 / 2}} \leq \alpha_{2} \leq \frac{1}{\delta^{1 / 2}}, 0 \leq r_{2} \leq \rho^{2} \tilde{\epsilon}, \tilde{y}_{2}=\frac{1}{\tilde{\epsilon}}-\alpha_{2}^{2}\right\} \tag{6.32}
\end{equation*}
$$

We assume that $\tilde{\epsilon}<1 / K$ so that $\Sigma_{2}^{\text {in }}$ lies in a region of positive $\tilde{y}_{2}$. We also define the set

$$
\begin{equation*}
\Sigma_{2,0}^{\mathrm{in}}=\Sigma_{2}^{\mathrm{in}} \cap\left\{r_{2}=0\right\} \tag{6.33}
\end{equation*}
$$

We can now prove the following
Lemma 6.4. There exists $\tilde{\epsilon}_{2}^{*}>0$ such that the following holds. For each $\tilde{\epsilon}<\tilde{\epsilon}_{2}^{*}$, there exists $k_{2}^{*}>0$ such that for $\rho<k_{2}^{*}$ and $\sigma<1 / 2$, all trajectories crossing the section $\Sigma_{2}^{i n}$ converge to the equilibrium $p_{2}\left(r_{2}\right)$ of (6.26).

Proof. It suffices to show that for $\sigma$ small enough, all trajectories crossing $\Sigma_{2,0}^{\mathrm{in}}$ eventually enter the interior of the parabola $\tilde{y}_{2}=\tilde{x}_{2}^{2}-1 / 2$ when $r_{2}=0$ for any $\alpha_{2} \in\left[1 / K^{1 / 2}, 1 / \delta^{1 / 2}\right]$. By a regular perturbation argument, this also holds for small $r_{2}>0$. Thus by taking $\rho$ sufficiently small we can ensure all points in $\Sigma_{2}^{\mathrm{in}}$ converge to $p_{2}\left(r_{2}\right)$.
For $r_{2}=0$ and $\alpha_{2} \in\left[1 / K^{1 / 2}, 1 / \delta^{1 / 2}\right]$, we consider the flow for points in the set $\Sigma_{2,0}^{\operatorname{in}} \cap\left\{\tilde{x}_{2}^{2}>\tilde{y}_{2}+1 / 2\right\}$ (as the other points already lie in the interior of the parabola $\left.\tilde{y}_{2}=\tilde{x}_{2}^{2}-1 / 2\right)$. Note that all such points satisfy $\tilde{x}_{2}<-1 / \sqrt{2}$. In this region for any $\alpha_{2} \in\left[1 / K^{1 / 2}, 1 / \delta^{1 / 2}\right]$, we have

$$
\begin{equation*}
\tilde{x}_{2}^{\prime}>\frac{1}{2}+\left(\frac{1}{2 K}\right)^{1 / 2} \tag{6.34}
\end{equation*}
$$

so that any orbit starting in $\Sigma_{2}^{\text {in }}$ either reaches $\tilde{x}_{2}=-1 / \sqrt{2}$ in finite time or enters the interior of the parabola $\tilde{y}_{2}=\tilde{x}_{2}^{2}-1 / 2$. The idea will be to show that all orbits starting in $\Sigma_{2,0}^{\mathrm{in}}$ enter the interior of the parabola before reaching $x_{0}=-1 / \sqrt{2}$. For an orbit starting in $\Sigma_{2}^{\mathrm{in}} \cap\left\{\tilde{x}_{2}^{2}>\tilde{y}_{2}+1 / 2\right\}$ at $t=0$ with $\tilde{x}_{2}(0)=x_{0}$, which reaches $\tilde{x}_{2}=-1 / \sqrt{2}$ at time $t=t_{0}$, the condition that this orbit has crossed $\tilde{y}_{2}=\tilde{x}_{2}^{2}-1 / 2$ is satisfied if $\tilde{y}_{2}\left(t_{0}\right)>0$. We
have

$$
\begin{align*}
\tilde{y}_{2}\left(t_{0}\right) & =\tilde{y}_{2}(0)+\int_{0}^{t_{0}} \tilde{y}_{2}^{\prime}(t) d t  \tag{6.35}\\
& =\tilde{y}_{2}(0)+\int_{0}^{t_{0}} \tilde{x}_{2}(t) d t \\
& =\tilde{y}_{2}(0)+\int_{0}^{t_{0}} \tilde{x}_{2}(t) \frac{-\tilde{y}_{2}(t)+\tilde{x}_{2}^{2}(t)-2 \alpha_{2} \tilde{x}_{2}(t)}{-\tilde{y}_{2}(t)+\tilde{x}_{2}^{2}(t)-2 \alpha_{2} \tilde{x}_{2}(t)} d t \\
& >\tilde{y}_{2}(0)+\int_{0}^{t_{0}}-\frac{\left(-\tilde{y}_{2}(t)+\tilde{x}_{2}^{2}(t)-2 \alpha_{2} \tilde{x}_{2}(t)\right)}{2 \alpha_{2}} d t \\
& =\tilde{y}_{2}(0)+\int_{x_{0}}^{-1 / \sqrt{2}}-\frac{1}{2 \alpha_{2}} d \tilde{x}_{2} \\
& =\frac{1}{\tilde{\epsilon}}-\alpha_{2}^{2}+\frac{1}{2 \alpha_{2}}\left(x_{0}+\frac{1}{\sqrt{2}}\right)
\end{align*}
$$

Thus the condition is satisfied if

$$
\begin{equation*}
\frac{1}{2 \alpha_{2}}\left(x_{0}+\frac{1}{\sqrt{2}}\right)+\frac{1}{\tilde{\epsilon}}-\alpha_{2}^{2}>0 \tag{6.36}
\end{equation*}
$$

for any initial condition $x_{0}$ of a trajectory in $\Sigma_{2}^{\text {in }} \cap\left\{\tilde{x}_{2}^{2}>\tilde{y}_{2}+1 / 2\right\}$ and any $\alpha_{2} \in\left[1 / K^{1 / 2}, 1 / \delta^{1 / 2}\right]$. In particular, this holds for any $\sigma<1 / 2$ and any $\tilde{\epsilon}<\min \left(\delta^{3}, \frac{1}{2 \sqrt{K}}\right)$. Therefore we set $\tilde{\epsilon}_{2}^{*}=\min \left(\delta^{3}, \frac{1}{2 \sqrt{K}}\right)$.
Now fix any $\tilde{\epsilon}<\tilde{\epsilon}_{2}^{*}$. Then all points in $\Sigma_{2,0}^{\mathrm{in}}$ converge to the equilibrium for $r_{2}=0$, and there exists $r_{2}^{*}$ such that this continues to be true for $0<r_{2}<r_{2}^{*}$ and any $\sigma<1 / 2$. Thus for any $\rho<\left(r_{2}^{*} / \tilde{\epsilon}\right)^{1 / 2}$, all points in $\Sigma_{2}^{\text {in }}$ converge to the equilibrium. So we set $k_{2}^{*}=\left(r_{2}^{*} / \tilde{\epsilon}\right)^{1 / 2}$.

Proof of Proposition 6.1. To prove the main result, we just need to choose constants appropriately and identify $\Delta_{\mathrm{in}}(\rho, \sigma)$ in the chart $\mathcal{K}_{1}$. We fix $\tilde{\alpha}$ and $k_{1}^{*}$ as in Lemma 6.2. Then for $\rho, \sigma, \delta<k_{1}^{*}$ we have that $\Pi_{14}\left(\Sigma_{14}^{\text {in }}\right) \subseteq \Sigma_{14}^{\text {out }}$, and thus we may apply Lemma 6.3 from $\S 6.4$. Therefore for any $\rho, \delta<\min \left(k_{1}^{*}, k_{4}^{*}\right)$ and any $\sigma<\min \left(k_{1}^{*}, 1 / 2\right)$, all points in $\Sigma_{14}^{\mathrm{in}}$ converge to the equilibrium.
We now fix $\delta<\min \left(k_{1}^{*}, k_{4}^{*}\right)$. By Lemma 6.2 (ii) for any $\tilde{\epsilon}<\tilde{\epsilon}_{1}^{*}$ and any $\rho, \sigma<k_{1}^{*}$, we have that $\Pi_{12}\left(\Sigma_{12}^{\text {in }}\right) \subseteq \Sigma_{12}^{\text {out }}$ and we may apply Lemma 6.4 of $\S 6.5$. We fix $\tilde{\epsilon}<\min \left(\tilde{\epsilon}_{1}^{*}, \tilde{\epsilon}_{2}^{*}\right)$. Then Lemma 6.4 gives $k_{2}^{*}$ such that for any $\rho<\min \left(k_{1}^{*}, k_{2}^{*}\right)$ and any $\sigma<\min \left(k_{1}^{*}, 1 / 2\right)$, all points in $\Sigma_{12}^{\text {in }}$ converge to the equilibrium.

Taking $\tilde{\rho}<\min \left(k_{1}^{*}, k_{2}^{*}, k_{4}^{*}\right)$ and $\tilde{\sigma}<\min \left(k_{1}^{*}, 1 / 2\right)$, we have the following. For each $\rho<\tilde{\rho}$ and $\sigma<\tilde{\sigma}$, the union of $\Delta_{\text {in }}(\rho, \sigma)$ over $\alpha \in(0, \rho \tilde{\alpha}), \epsilon \in\left(0, \rho^{2} \tilde{\epsilon}\right)$ and $0<\epsilon \leq K \alpha^{2}$ is contained in the union of the sections $\Sigma_{12}^{\mathrm{in}} \cup \Sigma_{14}^{\mathrm{in}}$, and we can apply Lemmas 6.3 and 6.4 as just described.
With these choices of $\tilde{\alpha}, \tilde{\epsilon}, \tilde{\rho}, \tilde{\sigma}$, the result holds on all of $D$.

Proof of Corollary 6.1. Fix $K>0$. Proposition 6.1 then gives $\tilde{\rho}, \tilde{\sigma}, \tilde{\epsilon}, \tilde{\alpha}$ such that for all $(\rho, \sigma, \epsilon, \alpha) \in D$, any trajectory crossing $\Delta^{i n}(\rho, \sigma)$ converges to the equilibrium. We therefore need to show that the parameters can be chosen in such a way as to continue to satisfy Proposition 6.1 with $\mathcal{M}_{\epsilon}^{\ell}(c, a) \operatorname{crossing} \Delta^{i n}(\rho, \sigma)$. Using Proposition 6.2, we can obtain an expression for $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ at $y=r_{1}^{2}=\rho^{2}$ :

$$
\begin{align*}
x=\rho x_{1} & =-\rho+\mathcal{O}\left(\rho \alpha_{1}, \rho \epsilon_{1}, \rho^{2}\right)  \tag{6.37}\\
& =-\rho+\mathcal{O}\left(\alpha, \frac{\epsilon}{\rho}, \rho^{2}\right)
\end{align*}
$$

For $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ to hit $\Delta^{i n}(\rho, \sigma)$, we need $|x+\rho| \leq \sigma \rho$. Provided $\alpha<\rho^{2}$ and $\epsilon<\rho^{3}$, we have that $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ reaches $y=\rho^{2}$ at

$$
\begin{equation*}
x=-\rho+\mathcal{O}\left(\rho^{2}\right) \tag{6.38}
\end{equation*}
$$



Figure 21: Plotted are the homoclinic C-curve and banana obtained by continuing the pulse solution in the parameters $(a, c)$ for $\epsilon=0.021$. The red square and green circle refer to the locations of the oscillatory pulse and double pulse of Figure 22, respectively.

So fix any $\sigma<\tilde{\sigma}$. Then for any sufficiently small $\rho$, we can ensure that $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ hits $\Delta^{i n}(\rho, \sigma)$. Fix such a value of $\rho$. Now take $\epsilon_{0}=\min \left(\rho^{3}, \rho^{2} \tilde{\epsilon}\right)$ and choose $a_{0}$ so that $a_{0} c^{-1 / 2}<\min \left(\rho^{2}, \rho \tilde{\alpha}\right)$ for all $c \in I_{c}$. Then the result follows from Proposition 6.1.

## 7 Numerical analysis

In this section we present a numerical analysis, performed using the continuation software AUTO, which describes a possible termination mechanism for the branch of pulses found analytically above. Throughout this section, we fix $\gamma=1 / 2$.

### 7.1 Homoclinic C-curve and stability

Starting with a monotone 1-pulse, we begin by fixing $\epsilon=0.021$ and continuing in the parameters $(c, a)$ and obtain the homoclinic "C-curve" (Figure 21a) which connects the branch of fast monotone pulses with the branch of slow pulses (see the schematic diagram in Figure 7). When continuing along the upper branch towards the left corner of the diagram, due to the Belyakov transition occurring at the origin, the tail of the pulse solution changes from monotone to oscillatory. The branch eventually turns around sharply as the pulse undergoes a transition from a single to a double pulse. The continuation then follows the C-curve in reverse, and similar sharp turn occurs when the lower branch appears to terminate in the lower left corner of the diagram, during which the double pulse transitions back to a single pulse. To better visualize this curve of solutions, in Figure 21 b , the $L^{2}$-norm of the solutions is plotted against the parameter $a$. This gives the homoclinic "banana" as described in [3].

One question that arises is that of the stability of the pulses of Theorem 1.1. We choose an oscillatory pulse and a double pulse near the end of the homoclinic C-curve and compute their spectra in Matlab. The pulses and their spectra are shown in Figure 22. It is known ( $[13,23]$ ) that the classical fast pulse solution is stable, and one expects that after the onset of oscillations in the tail, the pulse remains stable. In addition, we expect that as the homoclinic C-curve turns around and the oscillatory pulse transitions into a double pulse, stability is lost so that the double pulse is unstable. This is supported by the numerical computations for the pulses shown in Figure 22. Due to translation invariance, they each possess an eigenvalue at zero, but the double pulse has an additional positive real eigenvalue to the right of the essential spectrum arising from the interaction of the Nagumo front $\phi_{f}$ and back $\phi_{b}$ (the eigenfunction for the unstable eigenvalue is plotted along with the double


Figure 22: Plotted are examples of an oscillatory pulse and a double pulse along the homoclinic C-curve along with their spectra. The colored shapes refer to their location along the homoclinic C-curve and banana of Figure 21.
pulse in Figure 22). The oscillatory pulse on the other hand has the remainder of its spectrum confined to the left half plane.

We expect that stability of the oscillatory pulse should follow from the analysis used to construct the pulse and following techniques as in $[11,12]$, where existence and stability of pulses for the discrete FitzHugh-Nagumo equations were obtained. As with the classical pulse, the difficulty comes from tracking the two eigenvalues near the origin arising from the translation invariance of the Nagumo front and back. For the pulse, one eigenvalue remains at the origin due to translation invariance and we expect that the second eigenvalue, representing the interaction of front and back, moves into the left half plane. Again, complications arise due to the loss of normal hyperbolically at the fold, but it should be possible to treat this with analysis similar to that used to construct the pulse.

In previous works where stability was obtained for pulses passing near non-hyperbolic fold points (e.g. in $[1,4]$ ), only one eigenvalue was contributed near the origin due to the front. However, as mentioned previously in §2.2, the singular 'back' solution in this case leaves the fold along a center manifold rather than the strong unstable manifold and thus does not contribute an eigenvalue ([22]). We reiterate that for the pulses constructed in the case of FitzHugh-Nagumo, the singular back solution does in fact leave the fold along the strong unstable manifold which accounts for the additional interaction eigenvalue.

### 7.2 Single-to-double pulse transition

Numerical explorations of the FitzHugh-Nagumo system have resulted in possible explanations for the termination of the branch of pulses in the upper left corner of the C-curve and the structure of the homoclinic banana ([3, 7, 8]). The major contributing factor to this behavior is the singular Hopf bifurcation occurring at the origin. As the Hopf bifurcation is subcritical in the region in question, the onset of small unstable periodic solutions nearby block the convergence of the homoclinic to the equilibrium. However, the exact nature of the sharp turn in the C-curve, and in particular the relation to the transition between the single and double pulse, is not well understood. Guided by the analysis to construct the pulse in the previous sections and the investigation of the canard point at the origin, we propose a geometric mechanism for the transition from the single to double pulse, and we use the numerical continuation to visualize this transition. Figure 23 shows a zoom of the upper left part of the banana for a lower value of $\epsilon$ as well as six different pulses along the curve plotted together.

When viewing this progression, it becomes clear how the second pulse must be added. Starting from the oscillatory pulse, after passing near the equilibrium, the tail follows the completely unstable middle branch of the slow manifold for some amount of time before jumping off and returning to $\mathcal{M}_{\epsilon}^{\ell}(c, a)$ and then converging to the equilibrium. Eventually the pulse follows the entire middle branch up to the fold point before jumping back to $\mathcal{M}_{\epsilon}^{\ell}(c, a)$. In Figure 24a, we see this progression fills out a surface when many such pulses are plotted together. As the transition continues, the pulse instead jumps from the middle branch to $\mathcal{M}_{\epsilon}^{r}(c, a)$ which it follows until reaching the fold point, then jumps back to $\mathcal{M}_{\epsilon}^{\ell}(c, a)$, culminating in a double pulse. Figure 24 b shows a surface filled out by this part of the sequence with many pulses plotted together. The entire progression of the tail from small oscillations to a full additional pulse resembles a classical canard explosion (see Figure 23b).

After extending the results of the previous sections, it should be possible to describe this entire sequence analytically using the same geometric framework used to construct the pulse with oscillatory tail. As with the proof of Theorem 1.1, up to understanding the flow near the fold points, each pulse along the transition can be constructed using classical geometric singular perturbation theory and the Exchange Lemma. Near the fold points, additional blow up charts will be required to fully construct the pulses.
For $(c, a, \epsilon)=(1 / \sqrt{2}, 0,0)$, away from the fold points, the center branch of the critical manifold (which we now denote as $\mathcal{M}_{0}^{c}$ ) is completely unstable and exhibits heteroclitic connections to both the left branch $\mathcal{M}_{0}^{\ell}$ and the right branch $\mathcal{M}_{0}^{r}$ which we denote by $\phi_{\ell}$ and $\phi_{r}$, respectively. We therefore expect that the transition pulses can be constructed as perturbations of two new types of singular double pulses for $(c, a, \epsilon)=(1 / \sqrt{2}, 0,0)$ : "left" double pulses and "right" double pulses. The left/right descriptor refers to whether the double pulse involves a jump from $\mathcal{M}_{0}^{c}$ to the left branch $\mathcal{M}_{0}^{\ell}$ or a jump to the right branch $\mathcal{M}_{0}^{r}$. These two types of singular pulses are shown in Figure 25.

As $\mathcal{M}_{0}^{c}$ is normally hyperbolic away from the folds, standard geometric singular perturbation theory and the Exchange Lemma can be used to track the pulses along any of the three branches of the critical manifolds and along the heteroclinic connections away from the fold points. As before, the difficulty comes in tracking the solutions near the folds, where additional blow up charts will be required in order to fully understand the transition. At the lower fold, one must describe how solutions close to $M_{\epsilon}^{\ell}$ pass near the equilibrium (see Figure 26) then follow the middle branch for some amount of time before jumping off towards either $M_{\epsilon}^{r}$ or $M_{\epsilon}^{\ell}$. Eventually these trajectories will return to converge to the equilibrium along $M_{\epsilon}^{\ell}$, and an issue then arises in determining how to distinguish such trajectories from those that follow the middle branch as they are all exponentially close to $M_{\epsilon}^{\ell}$. Additional charts will also be required in the blow up analysis at the upper right fold, especially at the interface between the left and right double pulses where the pulse follows $\mathcal{M}_{0}^{c}$ into a neighborhood of the fold before jumping off along the Nagumo back (see Figure 27).

(c) Six pulses along the banana in the full $u v w$-space showing the transition from a single to a double pulse.

Figure 23: Transition from single to double pulse in the top left of the homoclinic banana for $\epsilon=0.0036$. The solutions labelled $1,2,3$ are left pulses, and those labelled $4,5,6$ are right pulses.


Figure 24: Transition from single to double pulse in the top left of the homoclinic banana for $\epsilon=0.0036$.

(a) Singular left double pulse follows the sequence: $\phi_{f}, \mathcal{M}_{0}^{r}$, $\phi_{b}, \mathcal{M}_{0}^{\ell}, \mathcal{M}_{0}^{c} \phi_{\ell}, \mathcal{M}_{0}^{\ell}$.

(b) Singular right double pulse follows the sequence: $\phi_{f}$, $\mathcal{M}_{0}^{r}, \phi_{b}, \mathcal{M}_{0}^{\ell}, \mathcal{M}_{0}^{c} \phi_{r}, \mathcal{M}_{0}^{r}, \phi_{b}, \mathcal{M}_{0}^{\ell}$.

Figure 25: Singular $\epsilon=0$ double pulses for $(c, a)=(1 / \sqrt{2}, 0)$.


Figure 26: Shown is a schematic for the expected flow near the canard point for pulses along the single-to-double pulse transition.


Figure 27: Shown is a schematic for the expected flow near the upper fold point for left and right pulses along the single-to-double pulse transition.

## 8 Discussion

In this paper we showed that the FitzHugh-Nagumo equations

$$
\begin{align*}
u_{t} & =u_{x x}+f(u)-w  \tag{8.1}\\
w_{t} & =\delta(u-\gamma w)
\end{align*}
$$

admit a traveling pulse solution $(u, w)(x, t)=(u, w)(x+c t)$ with wave speed $c=c(a, \epsilon)$ for $0<a<1 / 2$ and sufficiently small $\epsilon>0$. We showed that the region of existence near $a=0$ encompasses a Belyakov transition occurring at the equilibrium $(u, v, w)=(0,0,0)$ where two real stable eigenvalues split as a complex conjugate pair and thus describes the onset of small scale oscillations in the tails of the pulses. This result extends the classical existence result for traveling pulses in FitzHugh-Nagumo.
We employed many of the same techniques used in the classical existence proof in the context of geometric singular perturbation theory. Fenichel's theorems and the Exchange Lemma were used to construct the pulse up to understanding the flow near two non-hyperbolic fold points of the critical manifold. To understand the flow near the folds, we used blow up techniques to extend results from [17] and obtain estimates on the flow in small neighborhoods of these points.

We further investigated the oscillatory pulses through a numerical analysis. Firstly, we provided numerical evidence for the stability of the pulses. In addition, we contextualized the above existence result in the study of FitzHugh-Nagumo as a "CU-system", and we described a geometric mechanism for how the onset of oscillations leads to the addition of a full second pulse. This transition also explains the termination of the branch of pulses constructed above as we approach the region in parameter space containing canard solutions arising from the singular Hopf bifurcation occurring at the origin. We believe it is possible to describe this transition analytically, and we outlined a strategy which employs the same techniques used in the above existence proof.

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