Stability of traveling pulses with oscillatory tails in the FitzHugh-Nagumo system

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Abstract

The FitzHugh-Nagumo equations are known to admit fast traveling pulses that have monotone tails and arise as the concatenation of Nagumo fronts and backs in an appropriate singular limit, where a parameter ε goes to zero. These pulses are known to be nonlinearly stable with respect to the underlying PDE. Recently, the existence of fast pulses with oscillatory tails was proved for the FitzHugh-Nagumo equations. In this paper, we prove that the fast pulses with oscillatory tails are also nonlinearly stable. Similar to the case of monotone tails, stability is decided by the location of a nontrivial eigenvalue near the origin of the PDE linearization about the traveling pulse. We prove that this real eigenvalue is always negative. However, the expression that governs the sign of this eigenvalue for oscillatory pulses differs from that for monotone pulses, and we show indeed that the nontrivial eigenvalue in the monotone case scales with ε, while the relevant scaling in the oscillatory case is $\varepsilon^{2/3}$.

1 Introduction

The FitzHugh-Nagumo system

$$\begin{aligned} u_t &= u_{xx} + u(u-a)(1-u) - w, \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \tag{1.1}$$

with $\gamma > 0$, $0 < a < \frac{1}{2}$ and $0 < \varepsilon \ll 1$ serves as a simple model for the propagation of nerve impulses in axons [10, 27]. The FitzHugh-Nagumo system is also a paradigm for singularly perturbed partial differential equations: many of its features and solutions have been studied in great detail over the past decades. Nerve impulses correspond to traveling waves that propagate with constant speed without changing their profile, and the FitzHugh-Nagumo system indeed supports many different localized traveling waves, or pulses. Slow pulses have wave speeds close to zero and arise as regular perturbations from the limit $\varepsilon \to 0$. Fast pulses, on the other hand, have speeds that are bounded away from zero as $\varepsilon \to 0$: their profiles do not arise as a regular perturbation from the $\varepsilon = 0$ limit. Both slow and fast pulses have monotone tails as $x \to \pm\infty$. Numerical simulations of (1.1) reveal that it admits traveling pulses with exponentially decaying oscillatory tails: this observation is interesting as it opens up the possibility of constructing multi-pulses, which consist of several well-separated copies of the original pulses that are glued together and propagate without changes of speed and profile. Recently, the existence of oscillatory pulses was shown in [3] in the region where $0 < a, \varepsilon \ll 1$. The oscillations in the tails were shown to arise along with a canard mechanism [22] in a local center manifold of the equilibrium; such a mechanism is associated with the onset of periodic canard orbits and relaxation oscillations, for instance in the van der Pol equation [23]. Pulses with oscillatory tails have been found previously [13] in the FitzHugh–Nagumo system; however the methods used were topological and provide insufficient information to deduce the existence of multipulses or determine stability.

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The emphasis of this paper is to investigate the stability of the traveling pulses with oscillatory tails that were found in [3]. It is known [11] that the slow pulses are unstable as traveling-wave solutions to (1.1). In contrast, it was proved independently by Jones [17] and Yanagida [33] that the fast pulses are stable for each fixed $0 < a < \frac{1}{2}$ provided $\varepsilon > 0$ is sufficiently small. The idea behind the stability proofs published in [17, 33] is as follows: first, (1.1) is linearized about a fast pulse, and the eigenvalue problem associated with the resulting linear operator is then analysed to see whether it has any eigenvalues with positive real part. By counting the zeros of the Evans-function, it was shown in [17, 33] that there are at most two eigenvalues near or to the right of the imaginary axis: one of these eigenvalues stays at the origin due to translational invariance of the family of pulses (obtained by shifting the profile in space). The key was then to show that the second critical eigenvalue has a negative sign. In [17, 33], this was established using a parity argument by proving that the derivative of the Evans function at 0 is strictly positive, which, in turn, follows from geometric properties of the pulse profile in the limit $\varepsilon \to 0$. We mention that these results were extended in [6] to the long-wavelength spatially-periodic wave trains that accompany the fast pulses in the FitzHugh-Nagumo equation.

In this paper, we prove that the pulses with oscillatory tails are also stable. In particular, we show that their stability is again determined by the location of two eigenvalues near the origin, and we prove that the nonzero critical eigenvalue has always negative real part. While the result is the expected one, the stability criterion that ensures negativity of the critical eigenvalue is actually very different from the criterion for monotone pulses. Furthermore, the nonzero eigenvalue scales differently in the monotone and oscillatory regimes: we show that the critical eigenvalue is of order ε for monotone pulses, while there are oscillatory pulses for which the eigenvalue scales with $\varepsilon^{2/3}$ as $\varepsilon \to 0$.

In contrast to [17, 33], our proof is not based on Evans functions but relies instead on Lin's method [16, 25, 30] to construct potential eigenfunctions of the linearization for each potential eigenvalue λ near and to the right of the imaginary axis. We show that we can construct a piecewise continuous eigenfunction with exactly two jumps for each choice of λ: finding proper eigenvalues then reduces to finding values of λ for which the two jumps vanish. One advantage of our method over the Evans-function analysis in [17, 33] is that we obtain, as outlined above, the leading-order expressions for the eigenvalues near the origin. In addition, our approach allows us to derive the leading-order asymptotics of the associated eigenfunctions and the adjoint eigenfunctions, which helps us understand the dynamics of the pulse profile under perturbations: our results imply that small perturbations centered around the back affect only the back but do not affect the position of the front, and therefore of the pulse.

While we restrict ourselves to the FitzHugh-Nagumo system, our approach via Lin's method applies more generally to stability problems of pulses in singularly perturbed reaction-diffusion systems: in particular, the method can be applied to singular pulse profiles that are constructed by concatenating fast jumps with parts of the slow manifolds. Provided the slow manifolds have a consistent splitting of fast transverse stable and unstable fibers, our method reduces the PDE eigenvalue problem to a matrix eigenvalue problem whose dimension is equal to the number of fast jumps.

Finally, we comment on the presence of the second critical eigenvalue that determines stability. The fast traveling pulses are constructed by gluing pieces of the nullcline $w = u(u - a)(1 - u)$ together with traveling fronts and backs of the FitzHugh-Nagumo system with $\varepsilon = 0$. These pulses will develop oscillatory tails when $a \approx 0$: this coincides with the region where the traveling fronts and backs jump off from the maxima and minima of the nullcline $w = u(u-a)(1-u)$ (we refer to Figure 2 below for an illustration). Depending on exactly how the back jumps off the maximum of the nullcline, the nontrivial second eigenvalue is either present or not: in previous work [1, 14] the stability of similar types of traveling pulses is considered, but the critical eigenvalue is not present and the pulses are therefore automatically stable. We comment in more detail in section 9 on the differences between [1, 14] and the present work.

This paper is organized as follows. The next section is devoted to an overview of our main results, including their precise statements which are contained in Theorems 2.2 and 2.4. We then give an overview of the known existence results in section 3. In section 4, we collect and prove pointwise estimates of the pulses in the limit $\varepsilon \to 0$ that will be crucial in our stability analysis, which will be carried out in section 5 for the essential spectrum and in section 6 for the point spectrum of the linearization about the pulses: these results are then collected in section 7 to prove Theorems 2.2 and 2.4 and conclude stability. We illustrate our results with numerical simulations in section 8, and end with a discussion of our results and the underlying method in section 9.

2 Overview of main results

We consider the FitzHugh-Nagumo system

$$\begin{aligned} u_t &= u_{xx} + f(u) - w, \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \tag{2.1}$$

where $f(u) = f(u;a) = u(u-a)(1-u)$, $0 < a < \frac{1}{2}$ and $0 < \varepsilon \ll 1$. Moreover, we take $0 < \gamma < 4$ such that (2.1) has a single equilibrium rest state $(u,w) = (0,0)$. Using geometric singular perturbation theory [9] and the Exchange Lemma [18] one can construct traveling-pulse solutions to (2.1):

Theorem 2.1 ([3, 19])**.** *There exists $K^* > 0$ such that for each $\kappa > 0$ and $K > K^*$ the following holds. There exists $\varepsilon_0 > 0$ such that for each $(a,\varepsilon) \in [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$ system* (2.1) *admits a traveling-pulse solution $\hat{\phi}_{a,\varepsilon}(x,t) := \tilde{\phi}_{a,\varepsilon}(x + \check{c}t)$ with wave speed $\check{c} = \check{c}(a,\varepsilon)$ approximated (a-uniformly) by*

$$\check{c} = \sqrt{2}\left(\tfrac{1}{2} - a\right) + O(\varepsilon).$$

Furthermore, if we have in addition $\varepsilon > K^* a^2$, then the tail of the pulse is oscillatory.

Figure 1 depicts a schematic bifurcation diagram of the region of existence of pulses guaranteed by Theorem 2.1. This theorem encompasses two different existence results: the well known classical existence result [19] in the region where $0 < \varepsilon \ll a < \frac{1}{2}$, and the extension [3] to the regime $0 < a, \varepsilon \ll 1$, where the onset of oscillations in the tails of the pulses is observed. In the following, we refer to these two regimes as the hyperbolic and nonhyperbolic regimes, respectively, due to the use of (non)-hyperbolic geometric singular perturbation theory in the respective existence proofs.

In the co-moving frame $\xi = x + \check{c}t$, the solution $\tilde{\phi}_{a,\varepsilon}(\xi) = (u_{a,\varepsilon}(\xi), w_{a,\varepsilon}(\xi))$ is a stationary solution to

$$\begin{aligned} u_t &= u_{\xi\xi} - \check{c}u_\xi + f(u) - w, \\ w_t &= -\check{c}w_\xi + \varepsilon(u - \gamma w). \end{aligned} \tag{2.2}$$

We are interested in the stability of the traveling pulse $\hat{\phi}_{a,\varepsilon}(x,t)$ as solution to (2.1) or equivalently the stability of $\tilde{\phi}_{a,\varepsilon}(\xi)$ as solution to (2.2). Linearizing (2.2) about $\tilde{\phi}_{a,\varepsilon}(\xi)$ yields a linear differential operator $\mathcal{L}_{a,\varepsilon}$ on $C_{ub}(\mathbb{R},\mathbb{R}^2)$ given by

$$\mathcal{L}_{a,\varepsilon}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u_{\xi\xi} - \check{c}u_\xi + f'(u_{a,\varepsilon}(\xi))u - w \\ -\check{c}w_\xi + \varepsilon(u - \gamma w) \end{pmatrix}.$$

The stability of the pulse is determined by the spectrum of $\mathcal{L}_{a,\varepsilon}$, i.e. the values $\lambda \in \mathbb{C}$ for which the operator $\mathcal{L}_{a,\varepsilon} - \lambda$ is not boundedly invertible. The associated eigenvalue problem $\mathcal{L}_{a,\varepsilon}\psi = \lambda\psi$ can be written as the ODE

$$\psi_\xi = A_0(\xi,\lambda)\psi, \quad A_0(\xi,\lambda) = A_0(\xi,\lambda;a,\varepsilon) := \begin{pmatrix} 0 & 1 & 0 \\ \lambda - f'(u_{a,\varepsilon}(\xi)) & \check{c} & 1 \\ \dfrac{\varepsilon}{\check{c}} & 0 & -\dfrac{\lambda + \varepsilon\gamma}{\check{c}} \end{pmatrix}. \tag{2.3}$$

Invertibility of $\mathcal{L}_{a,\varepsilon} - \lambda$ can fail in two ways [31]: either the asymptotic matrix

$$\hat{A}_0(\lambda) = \hat{A}_0(\lambda;a,\varepsilon) := \begin{pmatrix} 0 & 1 & 0 \\ \lambda + a & \check{c} & 1 \\ \dfrac{\varepsilon}{\check{c}} & 0 & -\dfrac{\lambda + \varepsilon\gamma}{\check{c}} \end{pmatrix},$$

of system (2.3) is nonhyperbolic (λ is in the essential spectrum), or there exists a nontrivial exponentially localized solution to (2.3) (λ is in the point spectrum). In the latter case we call λ an eigenvalue of $\mathcal{L}_{a,\varepsilon}$ or of (2.3). The spaces of exponentially localized solutions to $(\mathcal{L}_{a,\varepsilon} - \lambda)\psi = 0$ or to (2.3) are referred to as eigenspaces and its nontrivial elements are called eigenfunctions. This brings us to our main result.

Theorem 2.2. *There exists $b_0, \varepsilon_0 > 0$ such that the following holds. In the setting of Theorem 2.1, let $\tilde{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to* (2.2) *for $0 < \varepsilon < \varepsilon_0$ with associated linear operator $\mathcal{L}_{a,\varepsilon}$. The spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in*

$$\{0\} \cup \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq -\varepsilon b_0\}.$$

More precisely, the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half plane $\mathrm{Re}(\lambda) \leq -\varepsilon\gamma$. The point spectrum of $\mathcal{L}_{a,\varepsilon}$ to the right hand side of the essential spectrum consists of the simple translational eigenvalue $\lambda_0 = 0$ and at most one other real eigenvalue $\lambda_1 = \lambda_1(a, \varepsilon) < 0$.

Theorem 2.2 will be proved in section 7. Combining Theorem 2.2 with [7] and [8, Theorem 2] yields nonlinear stability of the traveling pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$.

Theorem 2.3. *In the setting of Theorem 2.2, the traveling pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$ is nonlinearly stable in the following sense. There exists $d > 0$ such that, if $\phi(\xi, t)$ is a solution to* (2.2) *satisfying $\|\phi(\xi, 0) - \tilde{\phi}_{a,\varepsilon}(\xi)\| \leq d$, then there exists $\xi_0 \in \mathbb{R}$ such that $\|\phi(\xi + \xi_0, t) - \tilde{\phi}_{a,\varepsilon}(\xi)\| \to 0$ as $t \to \infty$.*

In specific cases we have more information about the critical eigenvalue λ_1 of $\mathcal{L}_{a,\varepsilon}$. In the hyperbolic regime, where a is bounded below by an ε-independent constant $a_0 > 0$, the nontrivial eigenvalue λ_1 can be approximated explicitly to leading order $O(\varepsilon)$. In the nonhyperbolic regime we have $0 < a, \varepsilon \ll 1$; if we restrict ourselves to a wedge $K_0 a^3 < \varepsilon < K a^2$, then the second eigenvalue λ_1 can be approximated to leading order $O(\varepsilon^{2/3})$ by an a-independent expression in terms of Bessel functions. Thus, regarding the potential other eigenvalue λ_1 we have the following result.

Theorem 2.4. *In the setting of Theorem 2.2, we have the following:*

(i) *(Hyperbolic regime) For each $a_0 > 0$ there exists $\varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in [a_0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ the potential eigenvalue $\lambda_1 < 0$ of $\mathcal{L}_{a,\varepsilon}$ is approximated (a-uniformly) by*

$$\lambda_1 = -M_1\varepsilon + O\left(|\varepsilon \log \varepsilon|^2\right),$$

where $M_1 = M_1(a) > 0$ can be determined explicitly; see (7.1). *If the condition $M_1 < \gamma + a^{-1}$ is satisfied, then λ_1 is contained in the point spectrum of $\mathcal{L}_{a,\varepsilon}$ and lies to the right hand side of the essential spectrum.*

(ii) *(Non-hyperbolic regime) There exists $\varepsilon_0 > 0$ and $K_0, k_0 > 1$ such that, if $(a, \varepsilon) \in (0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfies $K_0 a^3 < \varepsilon$, then the eigenvalue $\lambda_1 < 0$ of $\mathcal{L}_{a,\varepsilon}$ lies to the right hand side of the essential spectrum and satisfies*

$$\varepsilon^{2/3}/k_0 < \lambda_1 < k_0\varepsilon^{2/3}.$$

In particular, if $(a, \varepsilon) \in (0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfies $K_0 a^3 < \varepsilon^{1+\alpha}$ for some $\alpha > 0$, then λ_1 is approximated (a- and α-uniformly) by

$$\lambda_1 = -\frac{(18 - 4\gamma)^{2/3}\, \zeta_0}{3}\varepsilon^{2/3} + O\left(\varepsilon^{(2+\alpha)/3}\right), \tag{2.4}$$

where $\zeta_0 \in \mathbb{R}$ is the smallest positive solution to the equation

$$J_{-2/3}\left(\tfrac{2}{3}\zeta^{3/2}\right) = J_{2/3}\left(\tfrac{2}{3}\zeta^{3/2}\right),$$

where J_r denote Bessel functions of the first kind.

The regions in (c, a, ε)-parameter space considered in Theorems 2.1 and 2.4 are shown in Figure 1. We emphasize that Theorem 2.4 (ii) covers the regime $\varepsilon > K^* a^2$ of oscillatory tails. Theorem 2.4 will be proved in section 7.

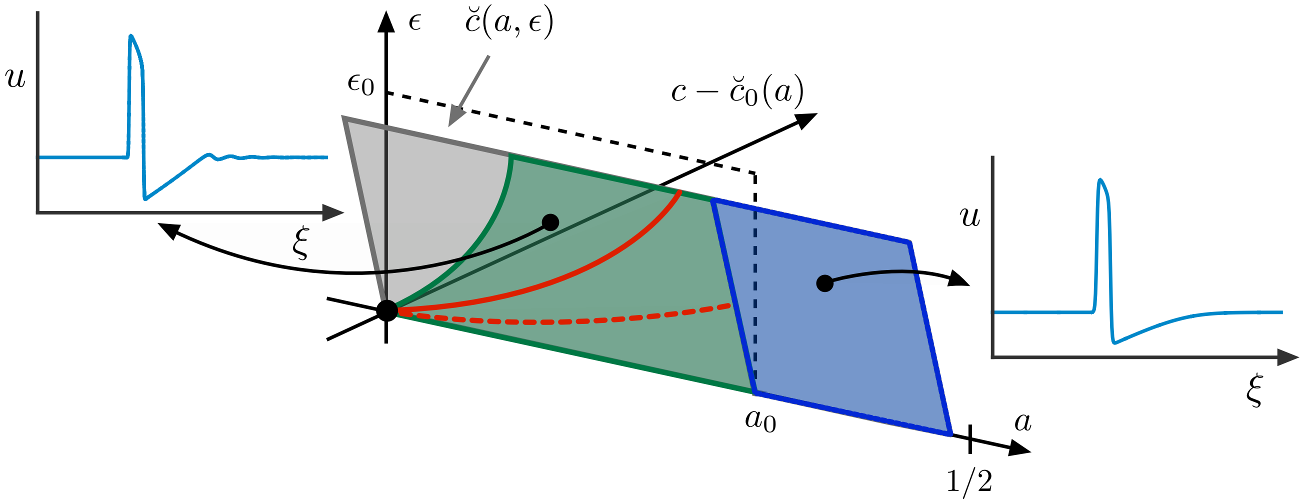


Figure 1: Shown is a schematic bifurcation diagram of the regions in (c, a, ε)-parameter space considered in Theorems 2.1 and 2.4. The green surface denotes the region of existence of pulses in the nonhyperbolic regime, and the blue surface represents the hyperbolic regime. The solid red curve $\varepsilon = K^* a^2$ represents the transition from monotone to oscillatory behavior in the tails of the pulses. The dashed red curve denotes $\varepsilon = K_0 a^3$; the region above this curve gives the parameter values for which the results of Theorem 2.4 (ii) are valid.

3 Overview of existence results

The stability analysis in this paper relies crucially on how the underlying pulse solutions are approximated by the singular limit structure. In particular, we need detailed pointwise estimates of the pulse solutions whose stability we are interested in. We will prove these estimates in Section 4; however, the proof will require an understanding of aspects of the existence construction in [3]. In this section, we therefore provide an overview of the existence analysis in [3], omitting technical details as much as possible. Then, in Section 4, we collect the technical results from [3] necessary for the pointwise approximation result.

The traveling-pulse solutions in Theorem 2.1 arise from a concatenation of solutions to a series of reduced systems in the singular limit $\varepsilon \to 0$. Their construction can be understood best in the setting of the traveling-wave ODE

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= cv - f(u) + w, \\ w_\xi &= \frac{\varepsilon}{c}(u - \gamma w), \end{aligned} \tag{3.1}$$

which is obtained from (2.1) by substituting the Ansatz $(u, w)(x, t) = (u, w)(x + ct)$ for wave speed $c > 0$ and putting $\xi = x + ct$. We consider a pulse solution $\tilde{\phi}_{a,\varepsilon}(\xi) = (u_{a,\varepsilon}(\xi), w_{a,\varepsilon}(\xi))$ as in Theorem 2.1. Equivalently, $\phi_{a,\varepsilon}(\xi) = (u_{a,\varepsilon}(\xi), u'_{a,\varepsilon}(\xi), w_{a,\varepsilon}(\xi))$ is a solution to (3.1) homoclinic to $(u, v, w) = (0, 0, 0)$ with wave speed $c = \breve{c}(a, \varepsilon)$.

The singular limit $\phi_{a,0}$ of $\phi_{a,\varepsilon}$ can be understood via the fast/slow decomposition of the traveling-wave ODE (3.1). We begin this section with defining the singular limit $\phi_{a,0}$, followed by an outline of the construction of the pulse solution $\phi_{a,\varepsilon}$ from the singular limit $\phi_{a,\varepsilon}$ in both the hyperbolic and nonhyperbolic regimes.

3.1 Singular limit

We separately consider (3.1), which we call the fast system, and the system below obtained by rescaling $\hat{\xi} = \varepsilon\xi$, which we call the slow system

$$\begin{aligned} \varepsilon u_{\hat{\xi}} &= v, \\ \varepsilon v_{\hat{\xi}} &= cv - f(u) + w, \\ w_{\hat{\xi}} &= \frac{1}{c}(u - \gamma w). \end{aligned} \tag{3.2}$$

Note that (3.1) and (3.2) are equivalent for any $\varepsilon > 0$. Taking the singular limit $\varepsilon \to 0$ in each of (3.1) and (3.2) results in simpler lower dimensional systems from which enough information can be obtained to determine the behavior in the full system for $0 < \varepsilon \ll 1$. We first set $\varepsilon = 0$ in (3.1) and obtain the layer problem

$$\begin{aligned} u_{\xi} &= v, \\ v_{\xi} &= cv - f(u) + w, \\ w_{\xi} &= 0, \end{aligned} \tag{3.3}$$

so that w becomes a parameter for the flow, and the manifold

$$\mathcal{M}_0 := \{(u, v, w) \in \mathbb{R}^3 : v = 0,\ w = f(u)\},$$

defines a set of equilibria. Considering this layer problem in the plane $w = 0$ and for $c = \breve{c}_0(a) = \sqrt{2}(\frac{1}{2} - a)$, we obtain the Nagumo system

$$\begin{aligned} u_{\xi} &= v, \\ v_{\xi} &= \breve{c}_0 v - f(u). \end{aligned} \tag{3.4}$$

For each $0 \leq a \leq 1/2$, this system possesses a heteroclinic front solution $\phi_{\mathrm{f}}(\xi) = (u_{\mathrm{f}}(\xi), v_{\mathrm{f}}(\xi))$ which connects the equilibria $p_{\mathrm{f}}^0 = (0, 0)$ and $p_{\mathrm{f}}^1 = (1, 0)$. In (3.3) this manifests as a connection in the plane $w = 0$ between the left and right branches of $\mathcal{M}_0$, when the wave speed c equals $\breve{c}_0$. In addition, there exists a heteroclinic solution $\phi_{\mathrm{b}}(\xi) = (u_{\mathrm{b}}(\xi), v_{\mathrm{b}}(\xi))$ (the Nagumo back) to the system

$$\begin{aligned} u_{\xi} &= v, \\ v_{\xi} &= \breve{c}_0 v - f(u) + w_{\mathrm{b}}^1, \end{aligned} \tag{3.5}$$

which connects the equilibria $p_{\mathrm{b}}^1 = (u_{\mathrm{b}}^1, 0)$ and $p_{\mathrm{b}}^0 = (u_{\mathrm{b}}^0, 0)$, where $u_{\mathrm{b}}^0 = \frac{1}{3}(2a - 1)$ and $u_{\mathrm{b}}^1 = \frac{2}{3}(1 + a)$ satisfy $f(u_{\mathrm{b}}^0) = f(u_{\mathrm{b}}^1) = w_{\mathrm{b}}^1$. Thus, for the same wave speed $c = \breve{c}_0$ there exists a connection between the left and right branches of $\mathcal{M}_0$ in system (3.3) in the plane $w = w_{\mathrm{b}}^1$.

Remark 3.1. The front $\phi_{\mathrm{f}}(\xi)$ can be determined explicitly by substituting the Ansatz $v = bu(u - 1)$, $b \in \mathbb{R}$ in the Nagumo equations (3.4). Subsequently, the back $\phi_{\mathrm{b}}(\xi)$ is established by using the symmetry of $f(u)$ about its inflection point. We obtain

$$\phi_{\mathrm{f}}(\xi) = \begin{pmatrix} u_\diamond(\xi + \xi_{\mathrm{f},0}) \\ u_\diamond'(\xi + \xi_{\mathrm{f},0}) \end{pmatrix}, \quad \phi_{\mathrm{b}}(\xi) = \begin{pmatrix} \frac{2}{3}(1 + a) - u_\diamond(\xi + \xi_{\mathrm{b},0}) \\ -u_\diamond'(\xi + \xi_{\mathrm{b},0}) \end{pmatrix}, \quad \text{with } u_\diamond(\xi) := \frac{1}{e^{-\frac{1}{2}\sqrt{2}\xi} + 1}, \tag{3.6}$$

where $\xi_{\mathrm{b},0}, \xi_{\mathrm{f},0} \in \mathbb{R}$ depends on the initial translation. We emphasize that we do not use the explicit expressions in (3.6) to prove our main stability result Theorem 2.2. However, they are useful to evaluate the leading order expressions for the second eigenvalue close to 0; see Theorem 2.4. Here we make use of the explicit formulas above with $\xi_{\mathrm{b},0}, \xi_{\mathrm{f},0} = 0$, but we could have made any choice of initial translate.

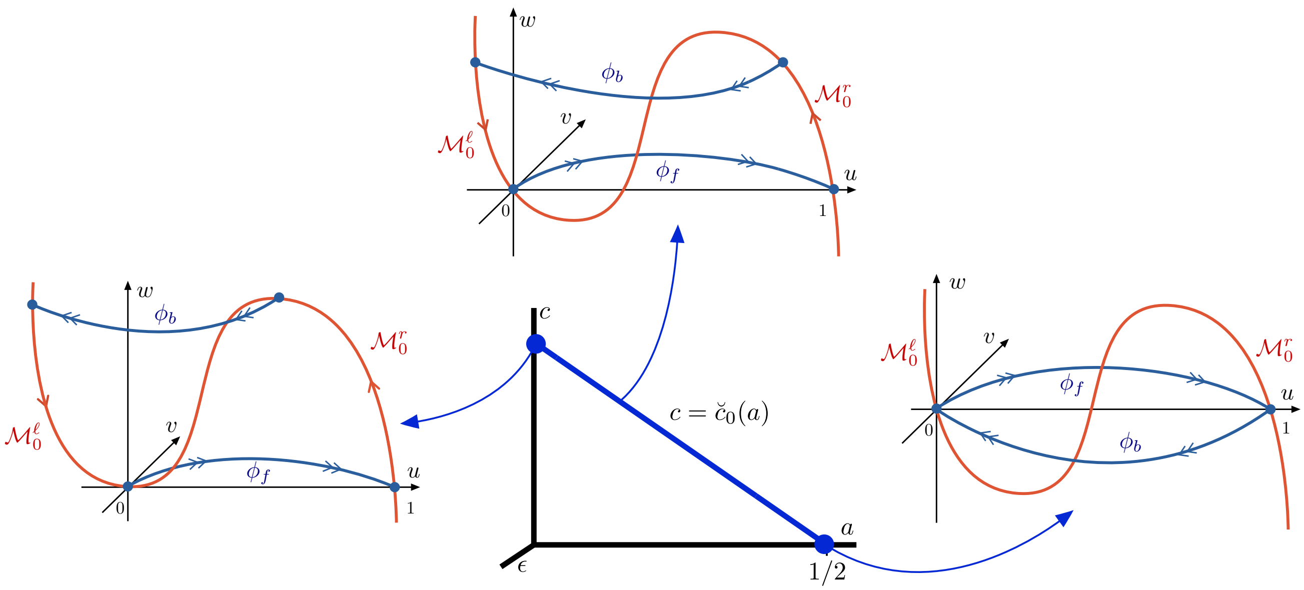


Figure 2: Shown is the singular pulse for $\varepsilon = 0$ in the nonhyperbolic regime (left), the hyperbolic regime (center), and the heteroclinic loop case [21] (right).

We note that for any $0 < a < 1/2$ the heteroclitic orbits ϕ_f and ϕ_b connect equilibria which lie on normally hyperbolic segments of the right and left branches of $\mathcal{M}_0$ given by

$$\mathcal{M}_0^r := \{(u, 0, f(u)) : u \in [u_\mathrm{b}^1, 1]\}, \quad \mathcal{M}_0^\ell := \{(u, 0, f(u)) : u \in [u_\mathrm{b}^0, 0]\}, \tag{3.7}$$

respectively. However, for $a = 0$, ϕ_f and ϕ_b leave precisely at the fold points on the critical manifold where normal hyperbolicity is lost (see Figure 2). This determines the distinction in the singular structure between the hyperbolic and nonhyperbolic cases. Furthermore, we note that for $a = 1/2$, ϕ_f and ϕ_b form a heteroclinic loop, but we do not consider this case in this paper; see [21].

We now set $\varepsilon = 0$ in (3.2) and obtain the reduced problem

$$\begin{aligned} 0 &= v, \\ 0 &= cv - f(u) + w, \\ w_{\hat{\xi}} &= \frac{1}{c}(u - \gamma w), \end{aligned}$$

where the flow is now restricted to the set $\mathcal{M}_0$ and the dynamics are determined by the equation for w. Putting together the information from the layer problem and reduced problem, there is for $c = \breve{c}_0$ a singular homoclinic orbit $\phi_{a,0}$ obtained by following ϕ_f, then up $\mathcal{M}_0^r$, back across ϕ_b, then down $\mathcal{M}_0^\ell$; see Figure 2. Thus, we define $\phi_{a,0}$ as the singular concatenation

$$\phi_{a,0} := \{(\phi_\mathrm{f}(\xi), 0) : \xi \in \mathbb{R}\} \cup \left\{(\phi_\mathrm{b}(\xi), w_\mathrm{b}^1) : \xi \in \mathbb{R}\right\} \cup \mathcal{M}_0^r \cup \mathcal{M}_0^\ell, \tag{3.8}$$

where $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ are defined in (3.7). Note that $\phi_{a,0}$ exists purely as a formal object as the two subsystems are not equivalent to (3.1) for $\varepsilon = 0$.

3.2 Existence analysis

Theorem 2.1 combines the classical existence result for fast pulses as well as an extension to the regime of pulses with oscillatory tails proved in [3]. We begin by introducing the classical existence result and its proof in the context of geometric singular perturbation theory and then proceed by describing how to overcome the difficulties encountered in the case $0 < a, \varepsilon \ll 1$. We refer to these cases as the hyperbolic and nonhyperbolic regimes, respectively.

3.2.1 Hyperbolic regime

The classical result is stated as follows

Theorem 3.2. *For each $0 < a < 1/2$, there exists $\varepsilon_0 = \varepsilon_0(a) > 0$ such that for $0 < \varepsilon < \varepsilon_0$ system* (2.1) *admits a traveling-pulse solution with wave speed $\check{c} = \check{c}(a, \varepsilon)$ satisfying*

$$\check{c}(a, \varepsilon) = \sqrt{2}\left(\frac{1}{2} - a\right) + O(\varepsilon).$$

The above result is well known and has been obtained using a variety of methods including classical singular perturbation theory [12] and the Conley index [2]. We describe a proof of this result similar to that in [19], using geometric singular perturbation theory [9] and the Exchange Lemma [18].

It is possible to construct a pulse for $\varepsilon > 0$ as a perturbation of the singular structure $\phi_{a,0}$ given by (3.8) as follows. By Fenichel theory the segments $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ persist for $\varepsilon > 0$ as locally invariant manifolds $\mathcal{M}_\varepsilon^r$ and $\mathcal{M}_\varepsilon^\ell$. In addition, the manifolds $\mathcal{W}^s(\mathcal{M}_0^r)$ and $\mathcal{W}^u(\mathcal{M}_0^r)$ defined as the union of the stable and unstable fibers, respectively, of $\mathcal{M}_0^r$ persist as locally invariant manifolds $\mathcal{W}_\varepsilon^{s,r}$ and $\mathcal{W}_\varepsilon^{u,r}$. Similarly the stable and unstable foliations of $\mathcal{M}_0^\ell$ persist as locally invariant manifolds $\mathcal{W}_\varepsilon^{s,\ell}$ and $\mathcal{W}_\varepsilon^{u,\ell}$. By Fenichel fibering the manifold $\mathcal{W}_\varepsilon^{s,\ell}$ coincides with $\mathcal{W}_\varepsilon^s(0)$, the stable manifold of the origin. The origin also has a one-dimensional unstable manifold $\mathcal{W}_0^u(0)$ which persists for $\varepsilon > 0$ as $\mathcal{W}_\varepsilon^u(0)$. By tracking $\mathcal{W}_\varepsilon^u(0)$ forwards and $\mathcal{W}_\varepsilon^s(0)$ backwards, it is possible to find an intersection provided that $c \approx \check{c}_0$ is chosen appropriately. The Exchange Lemma is needed to track these manifolds in a neighborhood of the right branch $\mathcal{M}_\varepsilon^r$, where the flow spends time of order ε^{-1}. There exists for any $r \in \mathbb{Z}_{>0}$ an ε-independent open neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$ and a C^r-change of coordinates $\Psi_\varepsilon \colon \mathcal{U}_E \to \mathbb{R}^3$, depending C^r-smoothly on ε, in which the flow is given by the Fenichel normal form [9, 18]

$$\begin{aligned} U' &= -\Lambda(U, V, W; c, a, \varepsilon)U, \\ V' &= \Gamma(U, V, W; c, a, \varepsilon)V, \\ W' &= \varepsilon(1 + H(U, V, W; c, a, \varepsilon)UV), \end{aligned} \tag{3.9}$$

where the functions Λ, Γ and H are C^r, and Λ and Γ are bounded below away from zero. In the local coordinates $\mathcal{M}_\varepsilon^r$ is given by $U = V = 0$, and $\mathcal{W}_\varepsilon^{u,r}$ and $\mathcal{W}_\varepsilon^{s,r}$ are given by $U = 0$ and $V = 0$, respectively. We assume that the Fenichel neighborhood contains a box

$$\Psi_\varepsilon(\mathcal{U}_E) \supseteq \{(U, V, W) : U, V \in [-\Delta, \Delta], W \in [-\Delta, W^* + \Delta]\}, \tag{3.10}$$

for $W^* > 0$ and some small $0 < \Delta \ll W^*$, both independent of ε. The Exchange Lemma [18] then states that for sufficiently small $\Delta > 0$ and $\varepsilon > 0$, any sufficiently large T, and any $|W_0| < \Delta$, there exists a solution $(U(\xi), V(\xi), W(\xi))$ to (3.9) that lies in $\Psi_\varepsilon(\mathcal{U}_E)$ for $\xi \in [0, T]$ and satisfies $U(0) = \Delta$, $W(0) = W_0$, and $V(T) = \Delta$ and the norms $|U(T)|$, $|V(0)|$,and $|W(T) - W_0 - \varepsilon W^*|$ are of order e^{-qT} for some $q > 0$, independent of ε.

We now track $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^s(0)$ up to the neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$ and determine how they behave at $U = \Delta$ and $V = \Delta$. This gives a system of equations in c, T, ε which can solved for $c = \check{c}(a, \varepsilon) = \check{c}_0(a) + O(\varepsilon)$ to connect $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^s(0)$ via a solution given by the Exchange lemma, completing the construction of the pulse of Theorem 3.2. The full pulse solution $\phi_{a,\varepsilon}$ is shown in Figure 4.

3.2.2 Nonhyperbolic regime

We now move on to the case $0 < a, \varepsilon \ll 1$. For certain values of the parameters a, ε, the tails of the pulses develop small oscillations near the equilibrium. The onset of the oscillations in the tail of the pulse is due to a transition occurring in the linearization of (3.1) about the origin in which the two stable real eigenvalues collide and emerge as a complex conjugate pair as a decreases for fixed ε. If a pulse/homoclinic orbit is present when eigenvalues change in this fashion, then this situation is referred to as a Belyakov transition [15, §5.1.4]. In [3], it was shown that for sufficiently small $a, \varepsilon > 0$ this transition occurs when

$$\varepsilon = \frac{a^2}{4} + O\left(a^3\right), \tag{3.11}$$

and the following result capturing the existence of pulses on either side of this transition was proved.

Theorem 3.3. *[3, Theorem 1.1] There exists $K^*, \mu > 0$ such that the following holds. For each $K > K^*$, there exists $a_0, \varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, system (2.1) admits a traveling-pulse solution with wave speed $\check{c} = \check{c}(a, \varepsilon)$ given by*

$$\check{c}(a, \varepsilon) = \sqrt{2}\left(\tfrac{1}{2} - a\right) - \mu\varepsilon + O(\varepsilon(a + \varepsilon)).$$

Furthermore, for $\varepsilon > K^* a^2$, the tail of the pulse is oscillatory.

Remark 3.4. In fact, by the identity (3.11), the constant $K^* > 0$ in Theorem 3.3 can be any value larger than $1/4$.

The difficulties in the proof of Theorem 3.3 arise from the fact that the pulses are constructed as perturbations from the highly singular limit in which $a = \varepsilon = 0$ (see Figure 2). In this limit, the origin sits at the lower left fold on the critical manifold $\mathcal{M}_0$, and the Nagumo front and back solutions $\phi_{\mathrm{f,b}}$ leave $\mathcal{M}_0^\ell$ and $\mathcal{M}_0^r$ precisely at the folds where these manifolds are no longer normally hyperbolic. Near such points, standard Fenichel theory and the Exchange Lemma break down, and geometric blow-up techniques are used to track the flow in these regions.

However, away from the folds, standard geometric singular perturbation theory applies, and many of the arguments from the classical case carry over. Outside of neighborhoods of the two fold points, the manifolds $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ persist for $\varepsilon > 0$ as locally invariant manifolds $\mathcal{M}_\varepsilon^r$ and $\mathcal{M}_\varepsilon^\ell$ as do their (un)stable foliations $\mathcal{W}_\varepsilon^{s,\ell}, \mathcal{W}_\varepsilon^{u,\ell}, \mathcal{W}_\varepsilon^{s,r}, \mathcal{W}_\varepsilon^{u,r}$. The origin has a strong unstable manifold $\mathcal{W}_\varepsilon^u(0)$ which persists for $\varepsilon > 0$ and can be tracked along $\mathcal{M}_\varepsilon^r$ through the neighborhood $\mathcal{U}_E$ given in (3.10) via the Exchange Lemma into a neighborhood $\mathcal{U}_F$ of the upper right fold point. The stable foliation $\mathcal{W}_\varepsilon^{s,\ell}$ of the left branch can be tracked backwards from a neighborhood of the equilibrium to a neighborhood of the upper right fold point. Constructing the pulse solution then amounts to the following two technical difficulties. First, one must find an intersection of $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$ near the upper right fold point. Second, since the exponentially attracting properties of the manifold $\mathcal{W}_\varepsilon^{s,\ell}$ are only defined along a normally hyperbolic segment of $\mathcal{M}_\varepsilon^\ell$, the flow can only be tracked up to a neighborhood of the equilibrium at the origin. Hence additional arguments are required to justify that the tails of the pulses in fact converge to the equilibrium upon entering this neighborhood. Overcoming these difficulties is therefore reduced to local analyses near the two fold points.

We begin with the upper right fold point; by the Exchange Lemma the manifold $\mathcal{W}_\varepsilon^u(0)$ is exponentially close to $\mathcal{M}_\varepsilon^r$ upon entering an a- and ε-independent neighborhood $\mathcal{U}_F$ of the fold point. The goal is therefore to track $\mathcal{M}_\varepsilon^r$ and nearby trajectories in this neighborhood. The fold point is given by the fixed point $(u^*, 0, w^*)$ of the layer problem (3.3) where

$$u^* = \tfrac{1}{3}\left(a + 1 + \sqrt{a^2 - a + 1}\right),$$

and $w^* = f(u^*)$. The linearization of (3.3) about this fixed point has one positive real eigenvalue $c > 0$ and a double zero eigenvalue, since $f'(u^*) = 0$. In a neighborhood of the fold point there exists a local change of coordinates that brings system (3.1) into the canonical form for a fold point, as studied in [22] using blow-up analysis. By tracking solutions near the fold, it is possible to find a solution which connects $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$. This leads to the following result from [3, §5.5].

Proposition 3.5. *There exists $\mu, a_0, \varepsilon_0 > 0$ such that the following holds. For each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$, there exists $c = \check{c}(a, \varepsilon)$ satisfying*

$$\check{c}(a, \varepsilon) = \sqrt{2}\left(\tfrac{1}{2} - a\right) - \mu\varepsilon + O(\varepsilon(a + \varepsilon)),$$

such that in system (3.1) *the manifolds $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$ intersect.*

After finding an intersection between $\mathcal{W}^u_\varepsilon(0)$ and $\mathcal{W}^{s,\ell}_\varepsilon$, it remains to show that solutions on the manifold $\mathcal{W}^{s,\ell}_\varepsilon$ converge to the equilibrium. As previously stated, using standard geometric singular perturbation theory arguments, it is possible to track $\mathcal{W}^{s,\ell}_\varepsilon$ into a neighborhood of the origin, but more work is required to show that the tail of the pulse in fact converges to the equilibrium after entering this neighborhood. We have the following result which follows from the analysis in [3, §6].

Proposition 3.6. *For each $K > 0$ and each sufficiently small $\sigma_0 > 0$, there exists $a_0, \varepsilon_0, d_0 > 0$ such that the following holds. For each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, the equilibrium $(u, v, w) = (0, 0, 0)$ in system (3.1) is stable with two-dimensional stable manifold $\mathcal{W}^s_\varepsilon(0)$. Furthermore, any solution on $\mathcal{W}^{s,\ell}_\varepsilon$ which enters the ball $B(0, \sigma_0)$ at a distance $\leq d_0$ from $\mathcal{M}^\ell_\varepsilon$ lies in the stable manifold $\mathcal{W}^s_\varepsilon(0)$ and remains in $B(0, \sigma_0)$ until converging to the equilibrium.*

Theorem 3.3 then follows from Propositions 3.5 and 3.6.

3.2.3 Main existence result

Combining Theorems 3.2 and 3.3, we obtain Theorem 2.1, repeated here for convenience, which encompasses both the hyperbolic and nonhyperbolic regimes.

Theorem 2.1. *There exists $K^* > 0$ such that for each $\kappa > 0$ and $K > K^*$ the following holds. There exists $\varepsilon_0 > 0$ such that for each $(a, \varepsilon) \in [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$ system (2.1) admits a traveling-pulse solution $\hat{\phi}_{a,\varepsilon}(x, t) := \tilde{\phi}_{a,\varepsilon}(x + \check{c}t)$ with wave speed $\check{c} = \check{c}(a, \varepsilon)$ a-uniformly approximated by*

$$\check{c} = \sqrt{2}\left(\frac{1}{2} - a\right) + O(\varepsilon).$$

Furthermore, if we have in addition $\varepsilon > K^* a^2$, then the tail of the pulse is oscillatory.

Proof. We take $K^* > \frac{1}{4}$ and fix K, κ satisfying $K > K^*$ and $\kappa > 0$. From Theorem 3.3 we obtain constants a_0, ε_0 and a traveling pulse for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$, where the pulses for $K^* a^2 < \varepsilon < Ka^2$ have oscillatory tails. By shrinking $\varepsilon_0 > 0$ further if necessary, Theorem 3.2 yields the existence of pulse solutions for each $(a, \varepsilon) \in [a_0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$, where we use that $[a_0, \frac{1}{2} - \kappa]$ is compact to ensure $\varepsilon_0 > 0$ is independent of a. □

4 Pointwise estimates of pulse solutions

Our main result of this section, Theorem 4.3, provides pointwise estimates describing the closeness of the traveling pulse solution $\phi_{a,\varepsilon}$ and its singular limit $\phi_{a,0}$ in $\mathbb{R}^3$. In order to prove Theorem 4.3, we will use results from the existence analysis in §3, as well as two additional technical results regarding the flow in the neighborhood $\mathcal{U}_F$ of the upper right fold point. We begin with the analysis near this fold point, followed by the statement and proof of Theorem 4.3.

4.1 Analysis near the upper right fold point

In this section, we obtain more detailed estimates on the flow in the a- and ε-independent neighborhood $\mathcal{U}_F$ of the upper right fold point. These will be helpful both in proving Theorem 4.3 and in the forthcoming stability analysis, particularly the eigenvalue computations in §6.5.

As mentioned in §3.2.2 there exists a local change of coordinates in a neighborhood of the fold point $(u^*, 0, w^*)$, where

$$u^* = \frac{1}{3}\left(a + 1 + \sqrt{a^2 - a + 1}\right),$$

and $w^* = f(u^*)$, that brings system (3.1) into the canonical form for a fold point. More precisely, we can perform for any $r \in \mathbb{Z}_{>0}$ a C^r-change of coordinates $\Phi_\varepsilon \colon \mathcal{U}_F \to \mathbb{R}^3$ to (3.1), which is C^r-smooth in c, a and ε for (c, a, ε)-values restricted to the set $[\check{c}_0(a_0), \check{c}_0(-a_0)] \times [-a_0, a_0] \times [-\varepsilon_0, \varepsilon_0]$, where $a_0, \varepsilon_0 > 0$ are chosen sufficiently small and $\check{c}_0(a) = \sqrt{2}(\frac{1}{2} - a)$. Applying Φ_ε to the flow of (3.1) in the neighborhood $\mathcal{U}_F$ of the fold point yields the canonical form

$$\begin{aligned} x' &= \theta_0 \left(y + x^2 + h(x, y, \varepsilon; c, a)\right), \\ y' &= \theta_0 \varepsilon g(x, y, \varepsilon; c, a), \\ z' &= z\left(c + O(x, y, z, \varepsilon)\right), \end{aligned} \tag{4.1}$$

where

$$\theta_0 = \frac{1}{c}\left(a^2 - a + 1\right)^{1/6} (u^* - \gamma w^*)^{1/3} > 0, \tag{4.2}$$

uniformly in $|a| \le a_0$ and $c \in [\check{c}_0(a_0), \check{c}_0(-a_0)]$, and h, g are C^r-functions satisfying

$$\begin{aligned} h(x, y, \varepsilon; c, a) &= O(\varepsilon, xy, y^2, x^3), \\ g(x, y, \varepsilon; c, a) &= 1 + O(x, y, \varepsilon), \end{aligned}$$

uniformly in $|a| \le a_0$ and $c \in [\check{c}_0(a_0), \check{c}_0(-a_0)]$. The coordinate transform Φ_ε can be decomposed in a linear and nonlinear part

$$\Phi_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathcal{N}\left[\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix}\right] + \tilde{\Phi}_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where the nonlinearity $\tilde{\Phi}_\varepsilon$ satisfies $\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = \partial\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = 0$ and the linear part $\mathcal{N}$ is given by

$$\mathcal{N} = \partial\Phi_\varepsilon \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix} = \begin{pmatrix} -\beta_1 & \frac{\beta_1}{c} & \frac{\beta_1}{c^2} \\ 0 & 0 & \frac{\beta_2}{c} \\ 0 & \frac{1}{c} & \frac{1}{c^2} \end{pmatrix},$$

where

$$\begin{aligned} \beta_1 &= \left(a^2 - a + 1\right)^{1/3} (u^* - \gamma w^*)^{-1/3} > 0, \\ \beta_2 &= c\left(a^2 - a + 1\right)^{1/6} (u^* - \gamma w^*)^{-2/3} > 0, \end{aligned}$$

uniformly in $|a| < a_0$ and $c \in [\check{c}_0(a_0), \check{c}_0(-a_0)]$. Finally, there exists a neighborhood $\mathcal{U}_F' \subset \mathbb{R}^3$ of 0, which is independent of c, a and ε, such that $\mathcal{U}_F' \subset \Phi_\varepsilon(\mathcal{U}_F)$.

In the transformed system (4.1), the x, y-dynamics is decoupled from the dynamics in the z-direction along the straightened out strong unstable fibers. Thus, the flow is fully described by the dynamics on the two-dimensional invariant manifold $z = 0$ and by the one-dimensional dynamics along the fibers in the z-direction. On this invariant manifold, for $\varepsilon = 0$ we see that the critical manifold is given by $\{(x, y) : y + x^2 + h(x, y, 0; c, a) = 0\}$, which is a approximately a downwards-opening parabola. The branch of this parabola for $x < 0$ is attracting and corresponds to the manifold $\mathcal{M}_0^r$. We define $\mathcal{M}_0^{r,+}$ to be the singular trajectory obtained by appending the fast trajectory given by the line $\{(x, 0) : x > 0\}$ to the attracting branch $\mathcal{M}_0^r$ of the critical manifold. We note that $\mathcal{M}_0^{r,+}$ can be represented as a graph $y = s_0(x)$. In [3] it was shown that, for sufficiently small $\varepsilon > 0$, $\mathcal{M}_0^{r,+}$ perturbs to a trajectory $\mathcal{M}_\varepsilon^{r,+}$ on $z = 0$, represented as a graph $y = s_\varepsilon(x)$, which is a-uniformly $C^0 - O\left(\varepsilon^{2/3}\right)$-close to $\mathcal{M}_0^{r,+}$ (see Figure 3).

We proceed by obtaining estimates on the flow in the invariant manifold $z = 0$ in (4.1). For sufficiently small $\rho, \sigma > 0$, we define the sections

$$\begin{aligned} \Sigma_\varepsilon^i &= \Sigma_\varepsilon^i(\rho, \sigma) := \{(\tilde{x}_\varepsilon(c, a) + x_0, -\rho^2) : 0 \le |x_0| < \sigma\rho\varepsilon\}, \\ \Sigma^o &= \Sigma^o(\rho) := \{(\rho, y) : y \in \mathbb{R}\}. \end{aligned}$$

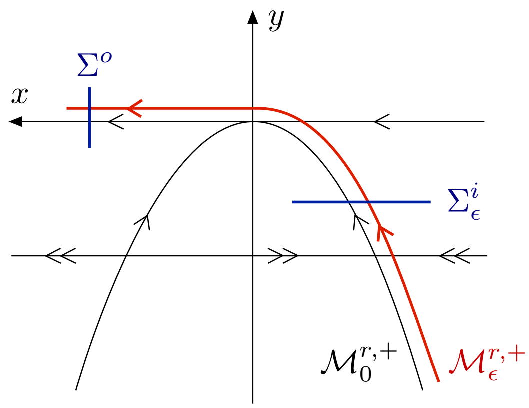


Figure 3: Shown is the flow on the invariant manifold $z = 0$ in the fold neighborhood $\mathcal{U}_F$. Note that x increases to the left.

where $\tilde{x}_\varepsilon(c, a)$ denotes the x-value at which the manifold $\mathcal{M}_\varepsilon^{r,+}$ intersects $y = -\rho^2$. In [3], using geometric blow-up techniques it was shown that between the sections Σ^i_ε and Σ^o, the manifold $\mathcal{M}_\varepsilon^{r,+}$ is $\mathcal{O}\left(\varepsilon^{2/3}\right)$-close to $\mathcal{M}_0^{r,+}$ and can be represented as the graph of an invertible function $y = s_\varepsilon(x)$.

Considering the flow of (4.1) on the invariant manifold $z = 0$, we rescale $\bar{t} = \theta_0 \xi$ and append an equation for ε, arriving at the system

$$\begin{aligned} \frac{dx}{d\bar{t}} &= y + x^2 + h(x, y, \varepsilon; c, a), \\ \frac{dy}{d\bar{t}} &= \varepsilon g(x, y, \varepsilon; c, a), \\ \frac{d\varepsilon}{d\bar{t}} &= 0. \end{aligned} \tag{4.3}$$

The blow-up analysis in [3] is based on [22] and makes use of three different rescalings in blow-up charts $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ to track solutions between Σ^i_ε and Σ^o. The chart $\mathcal{K}_1$ is described by the coordinates

$$x = r_1 x_1, \quad y = -r_1^2, \quad \varepsilon = r_1^3 \varepsilon_1, \tag{4.4}$$

the second chart $\mathcal{K}_2$ uses the coordinates

$$x = r_2 x_2, \quad y = -r_2^2 y_2, \quad \varepsilon = r_2^3, \tag{4.5}$$

and the third chart $\mathcal{K}_3$ uses the coordinates

$$x = r_3, \quad y = -r_3^2 y_3, \quad \varepsilon = r_3^3 \varepsilon_3. \tag{4.6}$$

In each of the charts $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$, we define entry/exit sections

$$\begin{aligned} \Sigma_1^{in} &:= \left\{(x_1, r_1, \varepsilon_1) : 0 < \varepsilon_1 < \delta, 0 \le \left|x_1 - \rho^{-1} s_\varepsilon^{-1}(-\rho^2)\right| < \sigma\rho^3\varepsilon_1, r_1 = \rho\right\}, \\ \Sigma_1^{out} &:= \left\{(x_1, r_1, \varepsilon_1) : \varepsilon_1 = \delta, 0 \le \left|x_1 - r_1^{-1} s_\varepsilon^{-1}(-r_1^2)\right| < \sigma r_1^3 \delta, 0 < r_1 \le \rho\right\}, \\ \Sigma_2^{in} &:= \left\{(x_2, y_2, r_2) : 0 \le \left|x_2 - r_2^{-1} s_\varepsilon^{-1}(-\delta^{-2/3} r_2^2)\right| < \sigma\rho^3\delta^{2/3}, y_2 = \delta^{-2/3}, 0 < r_2 \le \rho\delta^{1/3}\right\}, \\ \Sigma_2^{out} &:= \left\{(x_2, y_2, r_2) : x_2 = \delta^{-1/3}, 0 < r_2 \le \rho\delta^{1/3}\right\}, \\ \Sigma_3^{in} &:= \{(r_3, y_3, \varepsilon_3) : 0 < r_3 < \rho, y_3 \in [-\beta, \beta], \varepsilon_3 = \delta\}, \\ \Sigma_3^{out} &:= \{(r_3, y_3, \varepsilon_3) : r_3 = \rho, y_3 \in [-\beta, \beta], \varepsilon_3 \in (0, \delta)\}, \end{aligned}$$

for sufficiently small $\beta, \delta, \sigma, \rho > 0$ satisfying $2\Omega_0\delta^{2/3} < \beta$, where Ω_0 is the smallest positive zero of

$$J_{-1/3}\left(\tfrac{2}{3}z^{3/2}\right) + J_{1/3}\left(\tfrac{2}{3}z^{3/2}\right),$$

with J_r Bessel functions of the first kind. The set $\{(x, y, \varepsilon) \in \mathbb{R}^3 : (x, y) \in \Sigma^i_\varepsilon(\rho, \sigma), \varepsilon \in (0, \rho^3\delta)\}$ equals Σ_1^{in} in the $\mathcal{K}_1$ coordinates (4.4). Moreover, Σ_3^{out} is contained in the set $\{(x, y, \varepsilon) \in \mathbb{R}^3 : (x, y) \in \Sigma^o\}$, when converting to the $\mathcal{K}_3$ coordinates (4.6). In [3, §4], it was shown that the flow of (4.3) maps Σ_1^{in} into Σ_3^{out} via the sequence

$$\Sigma_1^{in} \longrightarrow \Sigma_1^{out} = \Sigma_2^{in} \longrightarrow \Sigma_2^{out} = \Sigma_3^{in} \longrightarrow \Sigma_3^{out},$$

taking into account the different coordinate systems to represent Σ_i^{in} and Σ_i^{out} for $i = 1, 2, 3$. The estimates on the flow between the various sections obtained in [3] enable us to prove the following.

Proposition 4.1. *For each sufficiently small $\rho, \sigma > 0$, there exists $a_0, \varepsilon_0 > 0$ such that for $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ the following holds. The flow of* (4.1) *on the invariant manifold $z = 0$ maps $\Sigma^i_\varepsilon(\rho, \sigma)$ into $\Sigma^o(\rho)$. In addition, a trajectory Γ starting at $x = \tilde{x}_\varepsilon(c, a) + x_0$ in Σ^i_ε satisfies*

(i) Between Σ^i_ε and Σ^o we have that Γ is $O(x_0)$-close to the manifold $\mathcal{M}^{r,+}_\varepsilon$. In particular, we have, along Γ between Σ^i_ε and Σ^o, the bound $|y - s_\varepsilon(x)| < C|x_0|$ for some constant $C > 0$ independent of a and ε.

(ii) There exist constants $k, \tilde{k} > 0$, independent of ρ, σ, a and ε, such that, along Γ between Σ^i_ε and Σ^o, we have $x' > (\tilde{k}/\rho)\varepsilon$. Furthermore, define the function $\Theta\colon (-\Omega_0, \infty) \to \mathbb{R}$ by

$$\Theta(\zeta) = \begin{cases} \sqrt{\zeta}\dfrac{I_{-2/3}\left(\frac{2}{3}\zeta^{3/2}\right) - I_{2/3}\left(\frac{2}{3}\zeta^{3/2}\right)}{I_{1/3}\left(\frac{2}{3}\zeta^{3/2}\right) - I_{-1/3}\left(\frac{2}{3}\zeta^{3/2}\right)}, & \text{if } \zeta > 0 \\ \\ \sqrt{-\zeta}\dfrac{J_{2/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right) - J_{-2/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right)}{J_{1/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}(-\zeta)^{3/2}\right)}, & \text{if } \zeta \le 0 \end{cases} \tag{4.7}$$

where J_r and I_r denote Bessel functions and modified Bessel functions of the first kind, respectively, and Ω_0 denotes the first positive zero of $J_{1/3}\left(\frac{2}{3}\zeta^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\zeta^{3/2}\right)$. Then, Θ is smooth, strictly decreasing and invertible and along Γ we approximate a-uniformly

$$x' = \theta_0\left(x^2 - \Theta^{-1}\left(x\varepsilon^{-1/3}\right)\varepsilon^{2/3}\right) + O(\varepsilon), \quad \text{for } 0 \le |x| < k\varepsilon^{1/3},$$

where θ_0 is defined in (4.2).

Proof. The proof of (i) follows from the proof of the estimates in [3, Corollary 4.1].

For (ii), we begin with the lower bound $x' > (\tilde{k}/\rho)\varepsilon$. Between the sections Σ_1^{in} and Σ_1^{out}, the existence of such a $\tilde{k} > 0$ follows from the proof of [3, Lemma 4.2]. In addition by [3, Lemmata 4.3, 4.4], by possibly taking $\tilde{k}$ smaller, the flow satisfies

$$x' = \theta_0 \frac{dx}{d\tilde{t}} > \tilde{k}\varepsilon^{2/3} > (\tilde{k}/\rho)\varepsilon,$$

between the sections Σ_2^{in} and Σ_3^{out}.

Finally, for any sufficiently small k, for $0 \le |x| < k\varepsilon^{1/3}$, we are concerned with the flow in the chart $\mathcal{K}_2$ between the sections Σ_2^{in} and Σ_2^{out}. In the $\mathcal{K}_2$ coordinates (4.5), the flow takes the form

$$\begin{aligned} \frac{dx_2}{dt_2} &= -y_2 + x_2^2 + O(r_2), \\ \frac{dy_2}{dt_2} &= -1 + O(r_2), \\ \frac{dr_2}{dt_2} &= 0, \end{aligned} \tag{4.8}$$

where $t_2 = r_2\bar{t}$. We quote a few facts from [3, §4.6]. Between the sections Σ_2^{in} and Σ_2^{out}, the manifold $\mathcal{M}_\varepsilon^{r,+}$ can be represented as the graph $(x_2, s_2(x_2; r_2))$ of a smooth invertible function $y_2 = s_2(x_2; r_2)$ smoothly parameterized by $r_2 = \varepsilon^{1/3}$ with $s_2(x_2; r_2) = s_2(x_2; 0) + O(r_2)$. Furthermore, using results from [26, § II.9], we have that $s_2(x_2; 0) = \Theta^{-1}(x_2)$, where the function Θ is defined in (4.7). The function Θ is smooth, strictly decreasing and maps $(-\Omega_0, \infty)$ bijectively onto $\mathbb{R}$. By part (i) above, we deduce that along Γ between Σ_2^{in} and Σ_2^{out}, we have $|y_2 - s_2(x_2; r_2)| = O(r_2)$. Hence we compute

$$x' = \theta_0 \frac{dx}{d\bar{t}} = \theta_0 r_2^2 \frac{dx_2}{dt_2} = \theta_0 r_2^2 \left(x_2^2 - y_2\right) + O\left(r_2^3\right) = \theta_0 r_2^2 \left(x_2^2 - \Theta^{-1}(x_2)\right) + O\left(r_2^3\right) = \theta_0 \left(x^2 - \varepsilon^{2/3}\Theta^{-1}\left(x\varepsilon^{-1/3}\right)\right) + O(\varepsilon),$$

which concludes the proof of assertion (ii). □

By studying the dynamics close to $\mathcal{M}_\varepsilon^{r,+}$, it is possible to track the pulse solution $\phi_{a,\varepsilon}(\xi)$ established in Proposition 3.5, which lies in the intersection of $\mathcal{W}_\varepsilon^u(0)$ and $\mathcal{W}_\varepsilon^{s,\ell}$, near the fold. For each sufficiently small $\rho, \sigma, z_0 > 0$, we define the sections

$$\begin{aligned} \Sigma_\varepsilon^{in} &= \Sigma_\varepsilon^{in}(\rho, \sigma, z_0) := \{(x, y, z) : (x, y) \in \Sigma_\varepsilon^i(\rho, \sigma), z \in [-z_0, z_0]\}, \\ \Sigma^{out} &= \Sigma^{out}(z_0) := \mathcal{U}_F' \cap \{z = z_0\}. \end{aligned} \tag{4.9}$$

We remark that for each sufficiently small $\rho, \sigma, z_0 > 0$, it is always possible to choose the fold neighborhood $\mathcal{U}_F$ and the Fenichel neighborhood $\mathcal{U}_E$ so that they intersect in a region containing the section Σ_ε^{in}. We have the following by [3, Proposition 4.1, Corollary 4.1 and §5.5].

Proposition 4.2. *For each sufficiently small $\sigma, \rho, z_0 > 0$ there exists $a_0, \varepsilon_0 > 0$ such that the following holds. For $(a, \varepsilon) \in (0, a_0)\times(0, \varepsilon_0)$ the solution $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ to system (4.1) enters the fold neighborhood $\mathcal{U}_F'$ via the section $\Sigma_\varepsilon^{in}(\rho, \sigma, z_0)$ and exits via $\Sigma^{out}(z_0)$. The intersection point of $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ with Σ^{out} is a-uniformly $O(\varepsilon^{2/3})$-close to the intersection point between Σ^{out} and the back solution $\Phi_0(\varphi_b(\xi), w_b^1)$ to system (4.1) at $\varepsilon = 0$.*

We note that by taking $\rho, \sigma, z_0 > 0$ smaller, it is possible to ensure that the solutions considered in Proposition 4.2 pass as close to the fold as desired, at the expense of possibly taking a_0, ε_0 smaller.

4.2 Main approximation result

In the stability analysis we need to approximate the pulse $\phi_{a,\varepsilon}$ pointwise by its singular limit $\phi_{a,0}$. More specifically, we will cover the real line by four intervals J_f, J_r, J_b and J_ℓ. For ξ-values in J_r or J_ℓ the pulse $\phi_{a,\varepsilon}(\xi)$ is close to the right or left branches $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$ of the slow manifold $\mathcal{M}_0$, respectively. For ξ-values in J_f or J_b the pulse $\phi_{a,\varepsilon}(\xi)$ is approximated by (some translate of) the front $(\phi_f(\xi), 0)$ or back $(\phi_b(\xi), w_b^1)$, respectively.

To determine suitable endpoints of the intervals J_f and J_b we need to find $\xi \in \mathbb{R}$ such that $\phi_{a,\varepsilon}(\xi)$ can be approximated by one of the four non-smooth corners of the concatenation $\phi_{a,0}$; see Figure 4. By translational invariance, we can define the $\varepsilon \to 0$ limit of $\phi_{a,\varepsilon}(0)$ to be $(\phi_f(0), 0)$. Intuitively, one expects that, since the dynamics on the slow manifold is of the order $O(\varepsilon)$, a point $\phi_{a,\varepsilon}(\Xi(\varepsilon))$ converges to the lower-right corner of $\phi_{a,0}$ as long as $\Xi(\varepsilon) \to \infty$ and $\varepsilon\Xi(\varepsilon) \to 0$ as $\varepsilon \to 0$; see also Theorem 4.5. This motivates to choose the upper endpoint of J_f to be an a- and ε-independent multiple of $-\log \varepsilon$. In a similar fashion one can determine endpoints for J_b.

We establish the following pointwise estimates for the traveling pulse $\phi_{a,\varepsilon}(\xi)$ along the front and back and along the right and left branches of the slow manifold.

Theorem 4.3. *For each sufficiently small $a_0, \sigma_0 > 0$ and each $\tau > 0$, there exists $\varepsilon_0 > 0$ and $C > 1$ such that the following holds. Let $\phi_{a,\varepsilon}(\xi)$ be a traveling-pulse solution as in Theorem 2.1 for $0 < \varepsilon < \varepsilon_0$, and define $\Xi_\tau(\varepsilon) := -\tau \log \varepsilon$. There exist $\xi_0, Z_{a,\varepsilon} > 0$ with ξ_0 independent of a and ε and $1/C \leq \varepsilon Z_{a,\varepsilon} \leq C$ such that:*

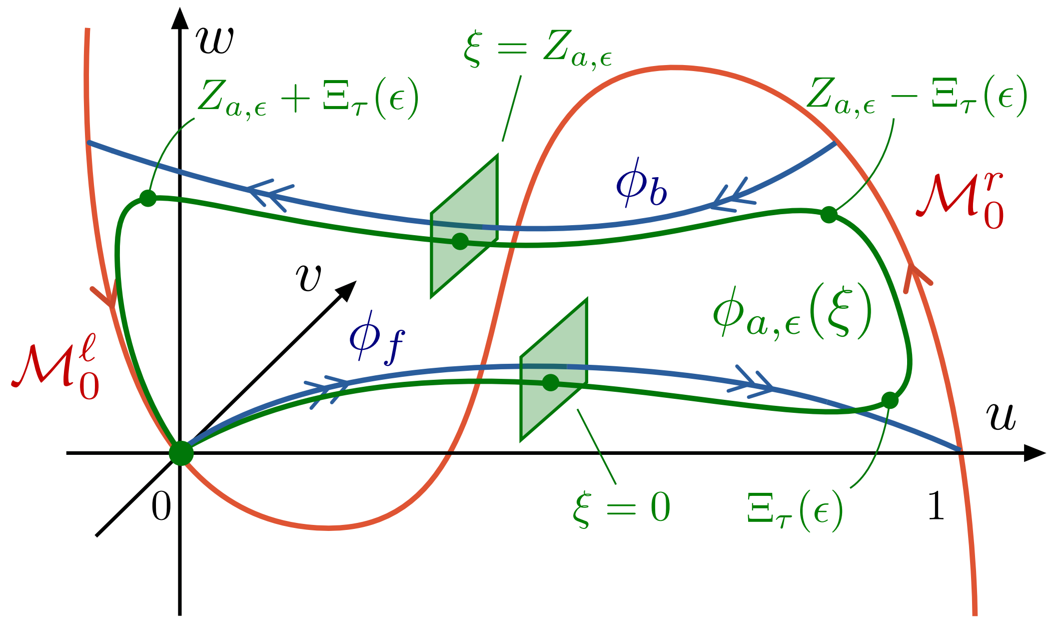


Figure 4: Shown is the pulse solution $\phi_{a,\varepsilon}(\xi)$ along with the singular Nagumo front ϕ_{f} and back ϕ_{b}. The points $\phi_{a,\varepsilon}(\Xi_\tau(\varepsilon))$ and $\phi_{a,\varepsilon}(Z_{a,\varepsilon} \pm \Xi_\tau(\varepsilon))$ approximate the intersection points of ϕ_{f} and ϕ_{b} with $\mathcal{M}_0^r$ and $\mathcal{M}_0^\ell$.

(i) For $\xi \in J_{\mathrm{f}} := (-\infty, \Xi_\tau(\varepsilon)]$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the front with

$$\left|\phi_{a,\varepsilon}(\xi) - \begin{pmatrix} \phi_{\mathrm{f}}(\xi) \\ 0 \end{pmatrix}\right| \leq C\varepsilon\Xi_\tau(\varepsilon).$$

(ii) For $\xi \in J_{\mathrm{b}} := [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} + \Xi_\tau(\varepsilon)]$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the back with

$$\left|\phi_{a,\varepsilon}(\xi) - \begin{pmatrix} \phi_{\mathrm{b}}(\xi - Z_{a,\varepsilon}) \\ w_{\mathrm{b}}^1 \end{pmatrix}\right| \leq C \begin{cases} \varepsilon\Xi_\tau(\varepsilon), & \textit{if } a \geq a_0, \\ \varepsilon^{2/3}\Xi_\tau(\varepsilon), & \textit{if } a < a_0. \end{cases}$$

(iii) For $\xi \in J_r := [\xi_0, Z_{a,\varepsilon} - \xi_0]$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the right slow manifold $\mathcal{M}_0^r$ with

$$d(\phi_{a,\varepsilon}(\xi), \mathcal{M}_0^r) \leq \sigma_0.$$

(iv) For $\xi \in J_\ell := [Z_{a,\varepsilon} + \xi_0, \infty)$, $\phi_{a,\varepsilon}(\xi)$ is approximated by the left slow manifold $\mathcal{M}_0^\ell$ with

$$d(\phi_{a,\varepsilon}(\xi), \mathcal{M}_0^\ell) \leq \sigma_0.$$

As an immediate corollary, we obtain

Corollary 4.4. *For each sufficiently small $\sigma_0 > 0$, there exists $\varepsilon_0 > 0$ such that the following holds. Let $\phi_{a,\varepsilon}$ denote a pulse solution to* (3.1) *in the setting of Theorem 2.1 with $0 < \varepsilon < \varepsilon_0$. The Hausdorff distance between $\phi_{a,\varepsilon}$ and $\phi_{a,0}$ as geometric objects in $\mathbb{R}^3$ is smaller than σ_0.*

Below we provide a proof of Theorem 4.3; we take care to separate the cases corresponding to Theorem 3.2 and that of Theorem 3.3 in which the pulse passes by upper fold. The estimates in Theorem 4.3 follow from standard Fenichel theory and the fold estimates along with the following argument from [6, 14]. Recall from §3 that in the ε-independent neighborhood $\mathcal{U}_E$ of $\mathcal{M}_\varepsilon^r$, there exists a C^r-change of coordinates $\Psi_\varepsilon\colon \mathcal{U}_E \to \mathbb{R}^3$ in which the flow is given by the Fenichel normal form (3.9). Here we have that $\mathcal{M}_\varepsilon^r$ is given by $U = V = 0$, $\mathcal{W}_\varepsilon^{u,r}$ and $\mathcal{W}_\varepsilon^{s,r}$ are given by $U = 0$ and $V = 0$, respectively, and the open Fenichel neighborhood $\Psi_\varepsilon(\mathcal{U}_E)$ contains a box $\{(U,V,W) : U,V \in [-\Delta,\Delta], W \in [-\Delta, W^*+\Delta]\}$ for $W^* > 0$ and some small $0 < \Delta \ll W^*$, both independent of ε. We define the following entry and exit manifolds

$$\begin{aligned} N_1 &:= \{(U,V,W) : U = \Delta, V \in [-\Delta,\Delta], W \in [-\Delta,\Delta]\},\\ N_2 &:= \{(U,V,W) : U,V \in [-\Delta,\Delta], W = W_0\}, \end{aligned}$$

for the flow around the corner where $0 < W_0 < W^*$. We make use of the following theorem, based on a result in [6].

Theorem 4.5 ([6, Theorem 4.1])**.** *Assume that $\Xi(\varepsilon)$ is a continuous function of ε into the reals satisfying*

$$\lim_{\varepsilon\to 0} \Xi(\varepsilon) = \infty, \quad \lim_{\varepsilon\to 0} \varepsilon\Xi(\varepsilon) = 0. \tag{4.10}$$

Moreover, assume that there is a one-parameter family of solutions $(U,V,W)(\xi,\cdot)$ to (3.9) *with $(U,V,W)(\xi_1,\varepsilon) \in N_1$, $(U,V,W)(\xi_2(\varepsilon),\varepsilon) \in N_2$ and $\lim_{\varepsilon\to 0} W(\xi_1,\varepsilon) = 0$ for some $\xi_1, \xi_2(\varepsilon) \in \mathbb{R}$. Let $U_0(\xi)$ denote the solution to*

$$U' = -\Lambda(U,0,0;c,a,0)U, \tag{4.11}$$

satisfying $U_0(\xi_1) = \Delta + \tilde{U}_0$ where $|\tilde{U}_0| \ll \Delta$. Then, for $\varepsilon > 0$ sufficiently small, we have that

$$\|(U,V,W)(\xi,\varepsilon) - (U_0(\xi),0,0)\| \le C\left(\varepsilon\Xi(\varepsilon) + |\tilde{U}_0| + |W(\xi_1,\varepsilon)|\right), \quad \text{for } \xi \in [\xi_1, \Xi(\varepsilon)],$$

where $C > 0$ is independent of a and ε.

Remark 4.6. We note that Theorem 4.5 extends the result [6, Theorem 4.1] to account for the following minor technicalities. Firstly, the estimates obtained along the singular $\varepsilon = 0$ solution are shown to hold along the entire interval $[\xi_1, \Xi(\varepsilon)]$ rather than just at the endpoint $\xi = \Xi(\varepsilon)$. Second, we allow for an error $\tilde{U}_0$ in the case that the solution in question does not arrive in N_1 at the same time ξ_1 as the singular solution U_0. Finally, no assumptions are made on the entry height $W(\xi_1,\varepsilon)$ other than continuity in ε with $\lim_{\varepsilon\to 0} W(\xi_1,\varepsilon) = 0$. This is necessary to deal with the $\mathcal{O}\left(\varepsilon^{2/3}\right)$ estimates along the back arising from Proposition 4.2 in the nonhyperbolic regime. A proof of Theorem 4.5 is given in Appendix A.

Proof of Theorem 4.3. We note that $\Xi_\tau(\varepsilon) := -\tau\log\varepsilon$ satisfies condition (4.10) in Theorem 4.5 for every $\tau > 0$.

We begin by showing (i). By standard geometric perturbation theory and the stable manifold theorem, the solution $\phi_{a,\varepsilon}(\xi)$ is a-uniformly $\mathcal{O}(\varepsilon)$-close to $(\phi_\mathrm{f}(\xi),0)$ upon entry in N_1 at $\xi_\mathrm{f} = \mathcal{O}(1)$. We apply the coordinate transform Ψ_ε in the neighborhood $\mathcal{U}_E$ of $\mathcal{M}^r_\varepsilon$, which brings system (3.1) into Fenichel normal form (3.9). For $\varepsilon = 0$, the orbit $(\phi_\mathrm{f}(\xi),0)$ converges exponentially to the equilibrium $(p^1_\mathrm{f},0)$ and hence lies in $\mathcal{W}^s(\mathcal{M}^r_0)$. Therefore, we have that $\Psi_0(\phi_\mathrm{f}(\xi),0) = (U_0(\xi),0,0)$, where $U_0(\xi)$ solves (4.11). We denote $(U_{a,\varepsilon}(\xi),V_{a,\varepsilon}(\xi),W_{a,\varepsilon}(\xi)) = \Psi_\varepsilon(\phi_{a,\varepsilon}(\xi))$. By Theorem 4.5 we have $\|(U_{a,\varepsilon}(\xi),V_{a,\varepsilon}(\xi),W_{a,\varepsilon}(\xi)) - (U_0(\xi),0,0)\| \le C\varepsilon\Xi_\tau(\varepsilon)$ for $\xi \in [\xi_\mathrm{f}, \Xi_\tau(\varepsilon)]$. Since the transform Ψ_ε to the Fenichel normal form is C^r-smooth in ε, we incur at most $\mathcal{O}(\varepsilon)$ errors when transforming back to the (u,v,w)-coordinates. Therefore, $\phi_{a,\varepsilon}(\xi)$ is a-uniformly $\mathcal{O}(\varepsilon\Xi_\tau(\varepsilon))$-close to $(\phi_\mathrm{f}(\xi),0)$ for $\xi \in [\xi_\mathrm{f}, \Xi_\tau(\varepsilon)]$ and we obtain the estimate (i).

We now prove (ii). From Proposition 4.2, for each sufficiently small $a_0 > 0$ we have that for $0 < a < a_0$ the solution $\phi_{a,\varepsilon}$ leaves the neighborhood $\mathcal{U}_E$ of the slow manifold $\mathcal{M}^r_\varepsilon$ after passing the section Σ^{in}_ε, defined in (4.9), where the flow enters the neighborhood $\mathcal{U}_F$ governed by the fold dynamics. With appropriate choice of the neighborhood $\mathcal{U}_E$, the case $a \ge a_0$ bounded away from zero is covered by standard geometric singular perturbation theory and the Exchange Lemma. Hence the estimate (ii) is split into two cases.

We first consider the case $a \ge a_0$ in which the classical arguments apply. In this case, the pulse leaves $\mathcal{M}^r_\varepsilon$ via the Fenichel neighborhood $\mathcal{U}_E$, where the flow is governed by the Fenichel normal form (3.9). By taking $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$ to be at leading order the time at which the pulse solution exits the Fenichel neighborhood $\mathcal{U}_E$ of $\mathcal{M}^r_\varepsilon$ along the back and treating the flow in a neighborhood of the left slow manifold $\mathcal{M}^\ell_\varepsilon$ in a similar manner, the estimate (ii) follows from a similar argument as (i).

We now consider the case $a < a_0$ in which $\phi_{a,\varepsilon}$ leaves $\mathcal{U}_E$ via the fold neighborhood $\mathcal{U}_F$. We apply the coordinate transform $\Phi_\varepsilon \colon \mathcal{U}_F \to \mathbb{R}^3$ in the neighborhood $\mathcal{U}_F$ bringing system (3.1) into the canonical form (4.1); see §4.1. Take $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$

to be at leading order the time at which the pulse solution exits the a- and ε-independent fold neighborhood $\mathcal{U}_F' \subset \Phi_\varepsilon(\mathcal{U}_F)$ via the section Σ^{out}, defined in (4.9); that is, we assume $\Phi_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_{\rm b})) \in \Sigma^{out}$, where $\xi_{\rm b} = O(1)$. We begin with establishing (ii) on the interval $J_{{\rm b},-} := [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} - \xi_{\rm b}]$.

The back solution $(\phi_{\rm b}(\xi), w_{\rm b}^1)$ to system (3.3) converges exponentially in backwards time to the equilibrium $(p_{\rm b}^1, w_{\rm b}^1) \in \mathcal{M}_0^r$ lying $O(a)$-close to the fold point $(u^*, 0, w^*)$. Therefore, the equilibrium $(p_{\rm b}^1, w_{\rm b}^1)$ is contained in $\mathcal{U}_F$, for $a > 0$ sufficiently small. Thus, transforming to system (4.1) for $\varepsilon = 0$ yields $\Phi_0(p_{\rm b}^1, w_{\rm b}^1) = (x_{\rm b}, y_{\rm b}, 0)$, where $x_{\rm b} < 0$ and the equilibrium $(x_{\rm b}, y_{\rm b})$ lies on the critical manifold $\mathcal{M}_0^r = \{(x, y) : x \leq 0, y + x^2 + h(x, y, 0, \check{c}, a) = 0\}$ of the invariant subspace $z = 0$. In addition, $\Phi_0(\phi_{\rm b}(\xi), w_{\rm b}^1)$ equals the solution $(x_{\rm b}, y_{\rm b}, z_{\rm b}(\xi))$ to (4.1) for $\varepsilon = 0$, where we gauge $z_{\rm b}(\xi)$ so that $(x_{\rm b}, y_{\rm b}, z_{\rm b}(-\xi_{\rm b})) \in \Sigma^{out}$.

Recall that by Proposition 4.2 $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ enters the fold neighborhood $\mathcal{U}_F'$ via the section Σ_ε^{in} and leaves via the section Σ^{out} at $\xi = Z_{a,\varepsilon} - \xi_{\rm b}$. Since the y-dynamics in (4.1) is $O(\varepsilon)$, one readily observes that $\phi_{a,\varepsilon}(\xi)$ lies in $\mathcal{U}_F$ for $\xi \in J_{{\rm b},-} = [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} - \xi_{\rm b}]$. We claim that the pulse solution $\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi)) = (x_{a,\varepsilon}(\xi), y_{a,\varepsilon}(\xi), z_{a,\varepsilon}(\xi))$ satisfies

$$\|\Phi_\varepsilon(\phi_{a,\varepsilon}(\xi)) - \Phi_0(\phi_{\rm b}(\xi - Z_{a,\varepsilon}), w_{\rm b}^1)\| \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon), \quad \text{for } \xi \in J_{{\rm b},-}. \tag{4.12}$$

By Proposition 4.2, $\Phi_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_{\rm b})) \in \Sigma^{out}$ lies a-uniformly $O\left(\varepsilon^{2/3}\right)$-close to $\Phi_0(\phi_{\rm b}(-\xi_{\rm b}), w_{\rm b}^1) \in \Sigma^{out}$. Hence, it holds

$$\Phi_\varepsilon\left(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_{\rm b})\right) = \left(x_{\rm b} + O\left(\varepsilon^{2/3}\right), y_{\rm b} + O\left(\varepsilon^{2/3}\right), z_0\right), \tag{4.13}$$

a-uniformly, for some $z_0 > 0$. First, since $(x_{\rm b}, y_{\rm b})$ lies on the critical manifold $\mathcal{M}_0^r$, we have $x_{\rm b} \leq 0$. So, by (4.13) it holds $x_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_{\rm b}) < C\varepsilon^{2/3}$. Second, Proposition 4.1 (ii) yields $x_{a,\varepsilon}'(\xi) > 0$ for $\xi \in J_{{\rm b},-}$. Combining these two observations, we establish $x_{a,\varepsilon}(\xi) < C\varepsilon^{2/3}$ for $\xi \in J_{{\rm b},-}$. Hence, by Proposition 4.1 (i) $(x_{a,\varepsilon}(\xi), y_{a,\varepsilon}(\xi))$ is $O(\varepsilon^{2/3})$-close to $\{(x, y) : y + x^2 + h(x, y, \varepsilon, \check{c}, a) = 0\}$ for $\xi \in J_{{\rm b},-}$. Thus, one observes directly from equation (4.1) that $|x_{a,\varepsilon}'(\xi)| < C\varepsilon^{2/3}$ and $|y_{a,\varepsilon}'(\xi)| < C\varepsilon$ for $\xi \in J_{{\rm b},-}$. Therefore, starting at $\xi = Z_{a,\varepsilon} - \xi_{\rm b}$ and integrating backwards, we have

$$\begin{aligned} |x_{a,\varepsilon}(\xi) - x_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_{\rm b})| &\leq \int_\xi^{Z_{a,\varepsilon}-\xi_{\rm b}} C\varepsilon^{2/3} dt \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon) \\ |y_{a,\varepsilon}(\xi) - y_{a,\varepsilon}(Z_{a,\varepsilon} - \xi_{\rm b})| &\leq \int_\xi^{Z_{a,\varepsilon}-\xi_{\rm b}} C\varepsilon dt \leq C\varepsilon\Xi_\tau(\varepsilon), \end{aligned} \tag{4.14}$$

for $\xi \in J_{{\rm b},-}$.

Define $\tilde{z}_{\rm b}(\xi) := z_{\rm b}(\xi - Z_{a,\varepsilon})$. In backwards time, trajectories in (4.1) are exponentially attracted to the invariant manifold $z = 0$ with rate greater than $\check{c}/2$ by taking $\mathcal{U}_F$ smaller if necessary. Note that $\check{c}$ is bounded from below away from 0 by an a-independent constant. Since $(x_{\rm b}, y_{\rm b}, z_{\rm b}(\xi))$ solves (4.1) for $\varepsilon = 0$ the difference $z_{a,\varepsilon}(\xi) - \tilde{z}_{\rm b}(\xi)$ satisfies on $J_{{\rm b},-}$

$$z_{a,\varepsilon}' - \tilde{z}_{\rm b}' = \left(\check{c} + O(x_{a,\varepsilon}, y_{a,\varepsilon}, z_{a,\varepsilon}, x_{\rm b}, y_{\rm b}, \tilde{z}_{\rm b}, \varepsilon)\right)(z_{a,\varepsilon} - \tilde{z}_{\rm b}) + O\left(\left(|x_{a,\varepsilon} - x_{\rm b}| + |y_{a,\varepsilon} - y_{\rm b}| + \varepsilon\right)\left(|z_{a,\varepsilon}| + |\tilde{z}_{\rm b}|\right)\right),$$

suppressing the ξ-dependence of terms. Hence, using (4.13), (4.14) and the fact that in backwards time $\tilde{z}_{\rm b}(\xi)$ and $z_{a,\varepsilon}(\xi)$ are exponentially decaying with rate $\check{c}/2$, we deduce that $z_{a,\varepsilon} - \tilde{z}_{\rm b}(\xi)$ satisfies a differential equation of the form

$$X' = b_1(\xi)X + b_2(\xi), \quad X(Z_{a,\varepsilon} - \xi_{\rm b}) = 0,$$

where $b_1(\xi) > \check{c}/2 > 0$ and

$$|b_2(\xi)| \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon)e^{-\check{c}(Z_{a,\varepsilon}-\xi)/2}$$

for $\xi \in J_{{\rm b},-}$. Hence, we estimate

$$|z_{a,\varepsilon}(\xi) - \tilde{z}_{\rm b}(\xi)| \leq C\varepsilon^{2/3}\Xi_\tau(\varepsilon).$$

for $\xi \in J_{{\rm b},-}$. Combining this with (4.13) and (4.14), we have that (4.12) holds. Hence, since the transform Φ_ε is C^r-smooth in a and ε, the pulse solution $\phi_{a,\varepsilon}(\xi)$ is a-uniformly $O\left(\varepsilon^{2/3}\Xi_\tau(\varepsilon)\right)$-close to the back $(\phi_{\rm b}(\xi), w_{\rm b}^1)$ and the estimate (ii) holds for $\xi \in J_{{\rm b},-} = [Z_{a,\varepsilon} - \Xi_\tau(\varepsilon), Z_{a,\varepsilon} - \xi_{\rm b}]$.

We now follow $\phi_{a,\varepsilon}$ along the back into a (Fenichel) neighborhood of $\mathcal{M}_\varepsilon^\ell$. Upon entry, $\phi_{a,\varepsilon}(\xi)$ is a-uniformly $O\left(\varepsilon^{2/3}\right)$-close to $(\phi_\mathrm{b}(\xi), w_\mathrm{b}^1)$. Combining this with another application of Theorem 4.5, the estimate (ii) follows for $\xi \in J_{\mathrm{b},+} = [Z_{a,\varepsilon} - \xi_\mathrm{b}, Z_{a,\varepsilon} + \Xi_\tau(\varepsilon)]$.

By taking the a- and ε-independent neighborhoods $\mathcal{U}_F$ and $\mathcal{U}_E$ smaller if necessary (and thus taking $a_0, \varepsilon_0 > 0$ smaller if necessary) and setting ξ_0 sufficiently large independent of a and ε, we have that $\phi_{a,\varepsilon}(\xi)$ lies in the union $\mathcal{U}_E \cup \mathcal{U}_F$ for $\xi \in [\xi_0, Z_{a,\varepsilon} - \xi_0]$. Hence we obtain (iii) along the right branch $\mathcal{M}_0^r$. Along the left branch $\mathcal{M}_0^\ell$, a similar argument combined with Proposition 3.6 gives the estimate (iv). □

5 Essential spectrum

In this section we prove that the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the left half plane and that it is bounded away from the imaginary axis. Moreover, we compute the intersection points of the essential spectrum with the real axis. Explicit expressions of these points are useful to determine whether there is a second eigenvalue of $\mathcal{L}_{a,\varepsilon}$ to the right of the essential spectrum.

Proposition 5.1. *In the setting of Theorem 2.1, let $\tilde{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to* (2.2) *with associated linear operator $\mathcal{L}_{a,\varepsilon}$. The essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half plane* $\mathrm{Re}(\lambda) \leq -\varepsilon\gamma$*. Moreover, for all $\lambda \in \mathbb{C}$ to the right of the essential spectrum the asymptotic matrix $\hat{A}_0(\lambda) = \hat{A}_0(\lambda; a, \varepsilon)$ of system* (2.3) *has precisely one (spatial) eigenvalue of positive real part. Finally, the essential spectrum intersects with the real axis at points*

$$\lambda = \begin{cases} -\frac{1}{2}a - \frac{1}{2}\varepsilon\gamma \pm \frac{1}{2}\sqrt{(\varepsilon\gamma - a)^2 - 4\varepsilon}, & \text{for } a > \varepsilon\gamma + 2\sqrt{\varepsilon}, \\ -\varepsilon\gamma + \check{c}^2 - \frac{1}{2}\sqrt{(2\check{c}^2 - \varepsilon\gamma + a)^2 - (\varepsilon\gamma - a)^2 + 4\varepsilon}, & \text{for } a \leq \varepsilon\gamma + 2\sqrt{\varepsilon}. \end{cases} \tag{5.1}$$

Proof. The essential spectrum is given by the λ-values for which the asymptotic matrix $\hat{A}_0(\lambda)$ of system (2.3) is nonhyperbolic. Thus we are looking for solutions $\lambda \in \mathbb{C}$ to

$$0 = \det(\hat{A}_0(\lambda) - i\tau) = \Delta\left(-i\tau - \tfrac{\lambda + \varepsilon\gamma}{\check{c}}\right) + \tfrac{\varepsilon}{\check{c}}, \tag{5.2}$$

with $\tau \in \mathbb{R}$ and $\Delta := -\tau^2 - \check{c}i\tau - a - \lambda$. For all $\tau \in \mathbb{R}$ and $\mathrm{Re}(\lambda) > -a$ we have that $\mathrm{Re}(\Delta) < 0$. For $\mathrm{Re}(\lambda) > -a$ we rewrite (5.2) as

$$\lambda = -\gamma\varepsilon + \varepsilon\Delta^{-1} - i\check{c}\tau.$$

Taking real parts in the latter equation yields $\mathrm{Re}(\lambda) < -\varepsilon\gamma$. This proves that the essential spectrum is confined to $\mathrm{Re}(\lambda) < -\min\{\varepsilon\gamma, a\}$. We now note that in the setting of Theorem 2.1, we have $\varepsilon < Ka^2$. This proves that the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half plane $\mathrm{Re}(\lambda) \leq -\varepsilon\gamma$.

One readily observes that for sufficiently large $\lambda > 0$, the asymptotic matrix $\hat{A}_0(\lambda)$ has precisely one unstable eigenvalue. By continuity this holds for all $\lambda \in \mathbb{C}$ to the right of the essential spectrum. This proves the second assertion.

For the third assertion we are interested in real solutions λ to the characteristic equation (5.2). Solving (5.2) yields

$$2\lambda = -\varepsilon\gamma - 2i\check{c}\tau - \tau^2 - a \pm \sqrt{(\varepsilon\gamma - a)^2 - 4\varepsilon + \tau^4 - 2(\varepsilon\gamma - a)\tau^2}. \tag{5.3}$$

Note that the square root in (5.3) is either real or purely imaginary. If the square root in (5.3) is real, it holds $0 = \mathrm{Im}(\lambda) = \check{c}\tau$ yielding $\tau = 0$. We obtain two real solutions given by (5.1) if and only if $(\varepsilon\gamma - a)^2 - 4\varepsilon > 0$. If the square root in (5.3) is purely imaginary it holds

$$\begin{aligned} 0 = 2\mathrm{Im}(\lambda) &= -2\check{c}\tau \pm \sqrt{-(\varepsilon\gamma - a)^2 + 4\varepsilon - \tau^4 + 2(\varepsilon\gamma - a)\tau^2}, \\ 2\lambda = 2\mathrm{Re}(\lambda) &= -\varepsilon\gamma - \tau^2 - a, \end{aligned}$$

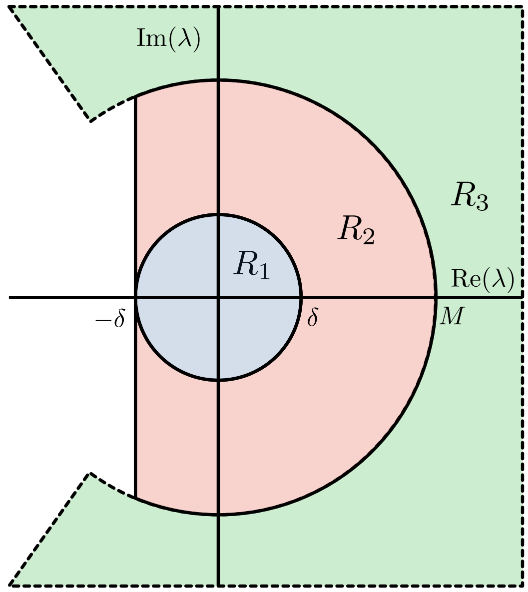


Figure 5: Shown are the regions $R_1(\delta), R_2(\delta, M), R_3(M)$ considered in the point spectrum analysis.

yielding

$$\tau^2 = -2\check{c}^2 + \varepsilon\gamma - a \pm \sqrt{(2\check{c}^2 - \varepsilon\gamma + a)^2 - (\varepsilon\gamma - a)^2 + 4\varepsilon}.$$

Since we have $\tau^2 \geq 0$, we obtain one real solution given by (5.1) if and only if $(\varepsilon\gamma - a)^2 - 4\varepsilon \leq 0$. □

6 Point spectrum

In order to prove Theorem 2.2, we need to show that the point spectrum of $\mathcal{L}_{a,\varepsilon}$ to the right of the essential spectrum consists at most of two eigenvalues. One of these eigenvalues is the simple translational eigenvalue $\lambda = 0$. The other eigenvalue is real and strictly negative. We will establish that this second eigenvalue is bounded away from the imaginary axis by εb_0 for some $b_0 > 0$. Moreover, we aim to provide a leading order expression of this eigenvalue in the hyperbolic and nonhyperbolic regimes to prove Theorem 2.4.

We cover the critical point spectrum by the following three regions (see Figure 5),

$$\begin{aligned} R_1 &= R_1(\delta) := B(0, \delta), \\ R_2 &= R_2(\delta, M) := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq -\delta, \delta \leq |\lambda| \leq M\}, \\ R_3 &= R_3(M) := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq 2\pi/3, |\lambda| > M\}, \end{aligned}$$

where $\delta, M > 0$ are a- and ε-independent constants. Recall that the point spectrum of $\mathcal{L}_{a,\varepsilon}$ is given by the eigenvalues λ of the linear problem (2.3), i.e. the λ-values such that (2.3) has an exponentially localized solution.

We start by showing that for $M > 0$ sufficiently large, the region $R_3(M)$ contains no point spectrum by rescaling the eigenvalue problem (2.3). The analysis in the regions R_1 and R_2 is more elaborate. The first step is to shift the essential spectrum away from the imaginary axis by introducing an exponential weight $\eta > 0$. The eigenvalues λ of system (2.3) and its shifted counterpart coincide to the right of the essential spectrum. Thus, it is sufficient to look at the eigenvalues λ of the shifted system to determine the critical point spectrum of $\mathcal{L}_{a,\varepsilon}$. We proceed by constructing a piecewise continuous eigenfunction for any prospective eigenvalue λ to the shifted problem. Finding eigenvalues then reduces to identifying the values of λ for which the discontinuous jumps vanish.

6.1 The region R_3

In this section we show that R_3 contains no point spectrum of $\mathcal{L}_{a,\varepsilon}$. Our approach is to prove that for $\lambda \in R_3(M)$, provided $M > 0$ is sufficiently large, a rescaled version of system (2.3) either has an exponential dichotomy on $\mathbb{R}$ or an exponential trichotomy on $\mathbb{R}$ with one-dimensional center direction. We proceed by showing that a system that admits such an exponential separation and that converges to the same asymptotic system as $\xi \to \infty$ and $\xi \to -\infty$ can not have nontrivial exponentially localized solutions. For the definition of exponential dichotomies and trichotomies we refer to Appendix B. We note that exponential dichotomies and trichotomies persist under small perturbations of the underlying ODE, a property referred to as roughness [5, 29].

Proposition 6.1. *In the setting of Theorem 2.1, let $\hat{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to* (2.2) *with associated linear operator $\mathcal{L}_{a,\varepsilon}$. There exists $M > 0$, independent of a and ε, such that the region $R_3(M)$ contains no point spectrum of $\mathcal{L}_{a,\varepsilon}$.*

Proof. Let $\lambda \in R_3$. We rescale system (2.3) by putting $\tilde{\xi} = \sqrt{|\lambda|}\xi$, $\tilde{u} = u$, $\sqrt{|\lambda|}\tilde{v} = v$ and $\tilde{w} = w$. The resulting system is of the form

$$\psi_\xi = \check{A}(\xi,\lambda)\psi, \quad \check{A}(\xi,\lambda) = \check{A}(\xi,\lambda;a,\varepsilon) := \check{A}_1(\lambda) + \tfrac{1}{\sqrt{|\lambda|}}\check{A}_2(\xi,\lambda), \tag{6.1}$$

$$\check{A}_1(\lambda) = \check{A}_1(\lambda;a,\varepsilon) := \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\lambda}{|\lambda|} & 0 & 0 \\ 0 & 0 & -\dfrac{\lambda}{\check{c}\sqrt{|\lambda|}} \end{pmatrix}, \quad \check{A}_2(\xi,\lambda) = \check{A}_2(\xi,\lambda;a,\varepsilon) := \begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{f'(u)}{\sqrt{|\lambda|}} & \check{c} & \dfrac{1}{\sqrt{|\lambda|}} \\ \dfrac{\varepsilon}{\check{c}} & 0 & -\dfrac{\varepsilon\gamma}{\check{c}} \end{pmatrix},$$

where we dropped the tildes. Note that $\check{A}_2$ is bounded on $\mathbb{R} \times R_3$ uniformly in $(a,\varepsilon) \in [0, \frac{1}{2} - \kappa] \times [0, \varepsilon_0]$. Our goal is to show that (6.1), and thus (2.3), admits no nontrivial exponentially localized solutions for $\lambda \in R_3$.

Since we have $|\arg(\lambda)| < 2\pi/3$ for all $\lambda \in R_3$, it holds $\mathrm{Re}(\sqrt{\lambda/|\lambda|}) > 1/2$. We distinguish between the cases $4|\mathrm{Re}(\lambda)| > \check{c}\sqrt{|\lambda|}$ and $4|\mathrm{Re}(\lambda)| \leq \check{c}\sqrt{|\lambda|}$. First, suppose $4|\mathrm{Re}(\lambda)| > \check{c}\sqrt{|\lambda|}$, then $\check{A}_1(\lambda)$ is hyperbolic with spectral gap larger than $1/4$. Thus, by roughness [5, p. 34] system (6.1) has an exponential dichotomy on $\mathbb{R}$ for $M > 0$ sufficiently large (with lower bound independent of a, ε and λ). Hence, (6.1) admits no nontrivial exponentially localized solutions and λ is not in the point spectrum of $\mathcal{L}_{a,\varepsilon}$.

Second, suppose $4|\mathrm{Re}(\lambda)| \leq \check{c}\sqrt{|\lambda|}$, then $\check{A}_1(\lambda)$ has one (spatial) eigenvalue with absolute real part $\leq 1/4$ and two eigenvalues with absolute real part $\geq 1/2$. By roughness system (6.1) has an exponential trichotomy on $\mathbb{R}$ for $M > 0$ sufficiently large (with lower bound independent of a, ε and λ). Hence, all exponentially localized solution must be contained in the one-dimensional center subspace. Fix $0 < k < 1/8$. By continuity the eigenvalues of the asymptotic matrix $\check{A}_\infty(\lambda) := \lim_{\xi\to\pm\infty} \check{A}(\xi,\lambda)$ are separated in one eigenvalue υ with absolute real part $\leq 1/4 + k$ and two eigenvalues with absolute real part $\geq 1/2 - k$ provided $M > 0$ is sufficiently large (with lower bound independent of a, ε and λ). Let β be the eigenvector associated with υ. Using [24, Theorem 1] we conclude that any solution $\psi(\xi)$ in the center subspace of (6.1) satisfies $\lim_{\xi\to\pm\infty} \psi(\xi)e^{-\upsilon\xi} = b_\pm\beta$ for some $b_\pm \in \mathbb{C} \setminus \{0\}$ and is therefore only exponentially localized in case it is trivial. Therefore, λ is not in the point spectrum of $\mathcal{L}_{a,\varepsilon}$. □

6.2 Setup for the regions R_1 and R_2

As described at the start of this section, we introduce a weight $\eta > 0$ and study the shifted system

$$\psi_\xi = A(\xi,\lambda)\psi, \quad A(\xi,\lambda) = A(\xi,\lambda;a,\varepsilon) := A_0(\xi,\lambda;a,\varepsilon) - \eta, \tag{6.2}$$

instead of the original eigenvalue problem (2.3) to determine the point spectrum of $\mathcal{L}_{a,\varepsilon}$ on the right hand side of the essential spectrum in the region $R_1 \cup R_2$. In this section we describe the approach in more detail and fully formulate the shifted eigenvalue problem.

6.2.1 Approach

The structure (3.8) of the singular limit $\phi_{a,0}$ of the pulse $\phi_{a,\varepsilon}$ leads to our framework for the construction of exponentially localized solutions to (6.2) in the regions R_1 and R_2. More specifically, depending on the value of $\xi \in \mathbb{R}$ the pulse $\phi_{a,\varepsilon}(\xi)$ is to leading order described by the front ϕ_f, the back ϕ_b or the left or right slow manifolds $\mathcal{M}_\varepsilon^\ell$ and $\mathcal{M}_\varepsilon^r$ (see Theorem 4.3). This leads to a partition of the real line in four intervals given by

$$I_\mathrm{f} = (-\infty, L_\varepsilon], \quad I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon], \quad I_\mathrm{b} = [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon} + L_\varepsilon], \quad I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty),$$

where $Z_{a,\varepsilon} = \mathcal{O}_s(\varepsilon^{-1})$ is defined in Theorem 4.3 and stands for the time the traveling-pulse solution spends near the right slow manifold $\mathcal{M}_\varepsilon^r$, and L_ε is given by

$$L_\varepsilon := -\nu \log \varepsilon, \tag{6.3}$$

with $\nu > 0$ an a- and ε-independent constant. The endpoints of the above intervals correspond to the ξ-values for which $\phi_{a,\varepsilon}(\xi)$ converges to one of the four non-smooth corners of the singular concatenation $\phi_{a,0}$; see §4 and Figure 4. Recall from Theorem 4.3 that the pulse $\phi_{a,\varepsilon}(\xi)$ is for ξ in I_r or I_ℓ close to the right or left slow manifold, respectively. Moreover, for ξ in I_f or I_b the pulse $\phi_{a,\varepsilon}(\xi)$ is approximated by the front or the back, respectively; see also Remark 6.2.

When the weight $\eta > 0$ is chosen appropriately, the spectrum of the coefficient matrix $A(\xi, \lambda)$ of system (6.2) has for ξ-values in I_r and I_ℓ a consistent splitting into one unstable and two stable eigenvalues. This splitting along the slow manifolds guarantees the existence of exponential dichotomies on the intervals I_r and I_ℓ. Solutions to (6.2) can be decomposed in terms of these dichotomies. To obtain suitable expressions for the solutions in the other two intervals I_f and I_b we have to distinguish between the regions R_1 and R_2.

We start with describing the set-up for the region R_1. For $\xi \in I_\mathrm{f}$ we establish a reduced eigenvalue problem by setting ε and λ to 0 in system (6.2), while approximating $\phi_{a,\varepsilon}(\xi)$ with the front $\phi_\mathrm{f}(\xi)$. The reduced eigenvalue problem admits exponential dichotomies on both half-lines. The full eigenvalue problem (6.2) can be seen as a (λ, ε)-perturbation of the reduced eigenvalue problem. Hence, one can construct solutions to (6.2) using a variation of constants approach on intervals

$$I_{\mathrm{f},-} := (-\infty, 0], \quad I_{\mathrm{f},+} := [0, L_\varepsilon],$$

which partition I_f and correspond to the positive and negative half-lines in the singular limit. The perturbation term is kept under control by taking $\delta > 0$ and $\varepsilon > 0$ sufficiently small. Similarly, we establish a reduced eigenvalue problem along the back and one can construct solutions to (6.2) using a variation of constants approach on intervals

$$I_{\mathrm{b},-} := [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon}], \quad I_{\mathrm{b},+} := [Z_{a,\varepsilon}, Z_{a,\varepsilon} + L_\varepsilon].$$

In summary, we obtain variation of constants formulas for the solutions to (6.2) on the four intervals $I_{\mathrm{f},\pm}$ and $I_{\mathrm{b},\pm}$ and expressions for the solutions to (6.2) in terms of exponential dichotomies on the two intervals I_r and I_ℓ. Matching of these expressions yields for any $\lambda \in R_1$ a piecewise continuous, exponentially localized solution to (6.2) which has jumps at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$. Finding eigenvalues then reduces to locating $\lambda \in R_1$ for which the two jumps vanish. Equating the jumps to zero leads to an analytic matching equation that is to leading order a quadratic in λ. The two solutions to this equation are the two eigenvalues of the shifted eigenvalue problem (6.2) in $R_1(\delta)$.

We know a priori that $\lambda = 0$ is a solution to the matching equation by translational invariance. The associated eigenfunction of (6.2) is the weighted derivative $e^{-\eta\xi}\phi_{a,\varepsilon}'(\xi)$ of the pulse. This information can be used to simplify some of the expressions in the matching equation. In the hyperbolic regime, this leads to a leading order expression of the second nonzero eigenvalue. In the nonhyperbolic regime the expressions in the matching equations relate to the dynamics at the fold point. One needs detailed information about the dynamics in the blow-up coordinates to determine the sign and magnitude of these expressions, which eventually yield that the second eigenvalue is strictly negative and smaller than $b_0\varepsilon$ for some $b_0 > 0$ independent of a and ε. In the regime $K_0 a^3 < \varepsilon$, a leading order expression for the second eigenvalue can be determined, which is of the order $\mathcal{O}(\varepsilon^{2/3})$.

Finally, we describe the set-up in the region R_2. Again our approach is to construct an exponentially localized solution to (6.2). We establish reduced eigenvalue problems for $\lambda \in R_2$ by setting ε to 0 in (6.2), while approximating $\phi_{a,\varepsilon}(\xi)$ with (a translate of) the front $\phi_{\mathrm{f}}(\xi)$ or the back $\phi_{\mathrm{b}}(\xi)$. However, we do keep the λ-dependence in contrast to the reduction done in the region R_1. Since λ is bounded away from the origin, the reduced eigenvalue problems admit exponential dichotomies on the *whole* real line: a fundamental difference with the region R_1. By roughness these dichotomies transfer to exponential dichotomies of (6.2) on the two intervals I_{f} and I_{b}. Thus, the real line is partitioned in four intervals $I_{\mathrm{f}}, I_{\mathrm{b}}, I_r$ and I_ℓ such that on each interval system (6.2) admits an exponential dichotomy governing the solutions. An exponentially localized solution can now be obtained by matching expressions for solutions on these four intervals. Notice that in contrast to the region R_1, we do not obtain matching conditions at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$. By comparing the dichotomy projections at the endpoints of the four intervals, we will show that such matching conditions can only lead to the trivial solution. Thus, for $\lambda \in R_2(\delta, M)$ the shifted eigenvalue problem (6.2) admits no nontrivial exponentially localized solution for any $M > 0$ and each $\delta > 0$ sufficiently small.

Remark 6.2. We emphasize that the intervals $I_{\mathrm{f}}, I_{\mathrm{b}}, I_r$ and I_ℓ, *partitioning* the real line, are strictly contained in the intervals $J_{\mathrm{f}}, J_{\mathrm{b}}, J_r$ and J_ℓ introduced in Proposition 4.3 *covering* the real line. The reason for this is a technical one: to estimate the dichotomy projections on the endpoints of the I-intervals we need a ξ-region where the pulse solution $\phi_{a,\varepsilon}(\xi)$ can be estimated both by its distance to one of to slow manifolds and by its distance to the front or back; see Proposition 6.5.

6.2.2 Formulation of the shifted eigenvalue problem

In this section we determine $\eta, \nu > 0$ such that the shifted system (6.2) admits exponential dichotomies on the intervals $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$, where L_ε is given by (6.3) and $Z_{a,\varepsilon}$ is as in Theorem 4.3. Recall that for ξ-values in I_r and I_ℓ the pulse $\phi_{a,\varepsilon}(\xi)$ is close to the right and left slow manifold, respectively. The following technical result shows that for appropriate values of η the spectrum of the coefficient matrix $A(\xi, \lambda)$ of system (6.2) has for ξ-values in I_r and I_ℓ a consistent splitting into one unstable and two stable eigenvalues.

Lemma 6.3. *Let $\kappa, M > 0$ and define for $\sigma_0 > 0$*

$$\mathcal{U}(\sigma_0, \kappa) := \left\{(a,u) \in \mathbb{R}^2 : a \in \left[0, \tfrac{1}{2} - \kappa\right], u \in \left[\tfrac{1}{3}(2a-1) - \sigma_0, \sigma_0\right] \cup \left[\tfrac{2}{3}(a+1) - \sigma_0, 1 + \sigma_0\right]\right\}.$$

Take $\eta = \frac{1}{2}\sqrt{2}\kappa > 0$. For $\sigma_0, \delta > 0$ sufficiently small, there exists $\varepsilon_0 > 0$ and $0 < \mu \leq \eta$ such that the matrix

$$\hat{A} = \hat{A}(u, \lambda, a, \varepsilon) := \begin{pmatrix} -\eta & 1 & 0 \\ \lambda - f'(u) & \check{c} - \eta & 1 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon\gamma}{\check{c}} - \eta \end{pmatrix},$$

has for $(a,u) \in \mathcal{U}(\sigma_0, \kappa), \lambda \in (R_1(\delta) \cup R_2(\delta, M))$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a uniform spectral gap larger than $\mu > 0$ and precisely one eigenvalue of positive real part.

Proof. The matrix $\hat{A}(u, \lambda, a, \varepsilon)$ is nonhyperbolic if and only if

$$0 = \det(\hat{A}(u, \lambda, a, \varepsilon) - i\tau) = \left(\eta^2 - \tau^2 + 2i\tau\eta - \check{c}i\tau + f'(u) - \lambda - \check{c}\eta\right)\left(-i\tau - \tfrac{\lambda + \varepsilon\gamma}{\check{c}} - \eta\right) + \tfrac{\varepsilon}{\check{c}},$$

is satisfied for some $\tau \in \mathbb{R}$. Thus, all λ-values for which $\hat{A}(u, \lambda, a, 0)$ is nonhyperbolic are given by the union of a line and a parabola

$$\{-\check{c}_0\eta + i\check{c}_0\tau : \tau \in \mathbb{R}\} \cup \left\{\eta^2 - \tau^2 + 2i\tau\eta - \check{c}_0 i\tau + f'(u) - \check{c}_0\eta : \tau \in \mathbb{R}\right\}. \tag{6.4}$$

Recall that $\check{c}_0 = \check{c}_0(a)$ is given by $\sqrt{2}\left(\frac{1}{2} - a\right)$. For any $(a,u) \in \mathcal{U}(\sigma_0, \kappa)$, it holds $\check{c}_0 = \check{c}_0(a) \geq \sqrt{2}\kappa$ and $f'(u) = -3u^2 + 2(a+1)u - a \leq 3\sigma_0$. Hence, for $(a,u) \in \mathcal{U}(\sigma_0, \kappa)$ the union (6.4) lies in the half plane

$$\mathrm{Re}(\lambda) \leq \max\left\{-\check{c}_0\eta, \eta^2 - \sqrt{2}\kappa\eta + 3\sigma_0\right\}.$$

Take $\eta = \frac{1}{2}\sqrt{2}\kappa$ and $3\sigma_0 < \frac{1}{4}\kappa^2$. We deduce that (6.4) is contained in $\mathrm{Re}(\lambda) \leq -\frac{1}{4}\kappa^2 < 0$ for any $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$. Hence, provided $\delta > 0$ is sufficiently small, the union (6.4) does not intersect the compact set $R_1(\delta) \cup R_2(\delta, M)$ for any (a, u) in the compact set $\mathcal{U}(\sigma_0, \kappa)$. By continuity we conclude that there exists $\varepsilon_0 > 0$ such that the matrix $\hat{A}(u, \lambda, a, \varepsilon)$ has for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$, $\lambda \in (R_1(\delta) \cup R_2(\delta, M))$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a uniform spectral gap larger than some $\mu > 0$. Note that $-\eta$ is in the spectrum of $\hat{A}(0, 0, a, 0)$. Therefore, we must have $\mu \leq \eta$.

In addition, one readily observes that for sufficiently large $\lambda > 0$ the matrix $\hat{A}(u, \lambda, a, 0)$ has precisely one eigenvalue of positive real part. On the other hand, the union (6.4) lies in the half plane $\mathrm{Re}(\lambda) \leq -\frac{1}{4}\kappa^2 < 0$ for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$. So, by continuity $\hat{A}(u, \lambda, a, 0)$ has precisely one eigenvalue of positive real part for $\lambda \in \mathbb{C}$ lying to the right of (6.4). Taking $\delta, \varepsilon_0 > 0$ sufficiently small, we conclude that $\hat{A}(u, \lambda, a, \varepsilon)$ has precisely one eigenvalue of positive real part for $(a, u) \in \mathcal{U}(\sigma_0, \kappa)$, $\lambda \in (R_1(\delta) \cup R_2(\delta, M))$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. □

We are now able to state a suitable version of the shifted eigenvalue problem (6.2). Thus, we started with $\kappa > 0$ and $K > K^*$, where $K^* > 0$ is as in Theorem 2.1. Then, Theorem 2.1 provided us with an $\varepsilon_0 > 0$ such that for any $(a, \varepsilon) \in [0, \frac{1}{2} - \kappa] \times (0, \varepsilon_0)$ satisfying $\varepsilon < Ka^2$ there exists a traveling-pulse solution $\tilde{\phi}_{a,\varepsilon}(\xi)$ to (2.2). In Proposition 6.1 we obtained $M > 0$, independent of a and ε, such that the region $R_3(M)$ contains no point spectrum of the associated linear operator $\mathcal{L}_{a,\varepsilon}$. We fix

$$\eta := \frac{1}{2}\sqrt{2}\kappa > 0,$$

and take $\nu > 0$ an a- and ε-independent constant satisfying

$$\nu \geq \max\left\{\frac{2}{\mu}, 2\sqrt{2}\right\} > 0, \tag{6.5}$$

where $\mu > 0$ is as in Lemma 6.3. The shifted eigenvalue problem is given by

$$\psi_\xi = A(\xi, \lambda)\psi, \qquad A(\xi, \lambda) = A(\xi, \lambda; a, \varepsilon) := \begin{pmatrix} -\eta & 1 & 0 \\ \lambda - f'(u_{a,\varepsilon}(\xi)) & \check{c} - \eta & 1 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda + \varepsilon\gamma}{\check{c}} - \eta \end{pmatrix}, \tag{6.6}$$
$$(\lambda, a, \varepsilon) \in (R_1(\delta) \cup R_2(\delta, M)) \times [0, \tfrac{1}{2} - \kappa] \times (0, \varepsilon_0), \quad \varepsilon < Ka^2,$$

where $u_{a,\varepsilon}(\xi)$ denotes the u-component of the pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$ and $\delta > 0$ is as in Lemma 6.3. In the next section we will show that with the above choice of η, δ, M and ν system (6.6) admits for $\lambda \in R_1(\delta) \cup R_2(\delta, M)$ exponential dichotomies on the intervals $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$, where L_ε is given by (6.3) and $Z_{a,\varepsilon}$ is as in Theorem 4.3. However, before establishing these dichotomies, we prove that it is indeed sufficient to study the shifted eigenvalue problem (6.6) to determine the critical point spectrum of $\mathcal{L}_{a,\varepsilon}$ in $R_1 \cup R_2$.

Proposition 6.4. *In the setting of Theorem 2.1, let $\tilde{\phi}_{a,\varepsilon}(x, t)$ denote a traveling-pulse solution to* (2.2) *with associated linear operator $\mathcal{L}_{a,\varepsilon}$. A point $\lambda \in R_1 \cup R_2$ lying to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is in the point spectrum of $\mathcal{L}_{a,\varepsilon}$ if and only if it is an eigenvalue of the shifted eigenvalue problem* (6.6).

Proof. The spectra of the asymptotic matrices $\hat{A}_0(\lambda; a, \varepsilon)$ and $\hat{A}(0, \lambda, a, \varepsilon)$ of systems (2.3) and (6.6), respectively, are related via $\sigma(\hat{A}(0, \lambda, a, \varepsilon)) = \sigma(\hat{A}_0(\lambda; a, \varepsilon)) - \eta$. Moreover, both $\hat{A}(0, \lambda, a, \varepsilon)$ and $\hat{A}_0(\lambda; a, \varepsilon)$ have precisely one (spatial) eigenvalue of positive real part for $\lambda \in R_1 \cup R_2$ to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ by Proposition 5.1 and Lemma 6.3. Therefore, for $\lambda \in R_1 \cup R_2$ to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$, system (2.3) admits a nontrivial exponentially localized solution $\psi(\xi)$ if and only if system (6.6) admits one given by $e^{-\eta\xi}\psi(\xi)$. □

6.2.3 Exponential dichotomies along the right and left slow manifolds

For ξ-values in I_ℓ or I_r the pulse $\phi_{a,\varepsilon}(\xi)$ is by Theorem 4.3 close to the right or left slow manifolds on which the dynamics is of the order $\mathcal{O}(\varepsilon)$. Hence, for $\xi \in I_\ell \cup I_r$ the coefficient matrix $A(\xi, \lambda)$ of the shifted eigenvalue problem (6.6) has slowly varying coefficients and is pointwise hyperbolic by Lemma 6.3. It is well-known that such systems admit exponential dichotomies; see [5, Proposition 6.1]. We will prove below that the associated projections can be chosen to depend analytically on λ and are close to the spectral projections on the (un)stable eigenspaces of $A(\xi,\lambda)$. As described in §6.2.1 the exponential dichotomies provide the framework for the construction of solutions to (6.6) on I_r and I_ℓ. The approximations of the dichotomy projections by the spectral projections are needed to match solutions to (6.6) on I_r and I_ℓ to solutions on the other two intervals I_f and I_b.

Proposition 6.5. *For each sufficiently small $a_0 > 0$, there exists $\varepsilon_0 > 0$ such that system* (6.6) *admits for $0 < \varepsilon < \varepsilon_0$ exponential dichotomies on the intervals $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$ with constants $C, \mu > 0$, where $\mu > 0$ is as in Lemma 6.3. The associated projections $Q^{u,s}_{r,\ell}(\xi,\lambda) = Q^{u,s}_{r,\ell}(\xi,\lambda; a, \varepsilon)$ are analytic in λ on $R_1 \cup R_2$ and are approximated at the endpoints $L_\varepsilon, Z_{a,\varepsilon} \pm L_\varepsilon$ by*

$$\left\|[Q^s_r - \mathcal{P}](L_\varepsilon,\lambda)\right\| \leq C\varepsilon|\log\varepsilon|,$$
$$\left\|[Q^s_r - \mathcal{P}](Z_{a,\varepsilon} - L_\varepsilon,\lambda)\right\|, \left\|[Q^s_\ell - \mathcal{P}](Z_{a,\varepsilon} + L_\varepsilon,\lambda)\right\| \leq C\varepsilon^{\rho(a)}|\log\varepsilon|,$$

where $\rho(a) = 1$ for $a \geq a_0$, $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\mathcal{P}(\xi,\lambda) = \mathcal{P}(\xi,\lambda; a,\varepsilon)$ are the spectral projections onto the stable eigenspace of the coefficient matrix $A(\xi,\lambda)$ of (6.6)*. In the above $C > 0$ is a constant independent of λ, a and ε.*

Proof. We begin by proving the existence of the desired exponential dichotomy on the interval I_r. The construction on the interval I_ℓ is similar, and we outline the differences only. Denote $\hat{L}_\varepsilon := L_\varepsilon/2 = -\frac{\nu}{2}\log\varepsilon$. We introduce a smooth partition of unity $\chi_i \colon \mathbb{R} \to [0,1]$, $i = 1,2,3$, satisfying

$$\sum_{i=1}^{3} \chi_i(\xi) = 1, \quad |\chi_i'(\xi)| \leq 2, \quad \xi \in \mathbb{R},$$
$$\mathrm{supp}(\chi_1) \subset (-\infty, \hat{L}_\varepsilon), \quad \mathrm{supp}(\chi_2) \subset (\hat{L}_\varepsilon - 1, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1), \quad \mathrm{supp}(\chi_3) \subset (Z_{a,\varepsilon} - \hat{L}_\varepsilon, \infty).$$

The equation

$$\psi_\xi = \mathcal{A}(\xi,\lambda)\psi, \tag{6.7}$$

with

$$\mathcal{A}(\xi,\lambda) = \mathcal{A}(\xi,\lambda; a,\varepsilon) := \chi_1(\xi)A(\hat{L}_\varepsilon,\lambda) + \chi_2(\xi)A(\xi,\lambda) + \chi_3(\xi)A(Z_{a,\varepsilon} - \hat{L}_\varepsilon,\lambda),$$

coincides with (6.6) on I_r. By Theorem 4.3 (iii) there exists, for any $\sigma_0 > 0$ sufficiently small, a constant $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ it holds

$$\|u_{a,\varepsilon}'(\xi)\| \leq \sigma_0, \quad u_{a,\varepsilon}(\xi) \in \left[u_\mathrm{b}^1 - \sigma_0, 1 + \sigma_0\right] = \left[\tfrac{2}{3}(a+1) - \sigma_0, 1 + \sigma_0\right]. \tag{6.8}$$

for $\xi \in [\hat{L}_\varepsilon - 1, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1]$. We calculate

$$\partial_\xi \mathcal{A}(\xi,\lambda) = \begin{cases} \chi_2(\xi)\partial_\xi A(\xi,\lambda), & \xi \in (\hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon), \\ \chi_2'(\xi)(A(\xi,\lambda) - A(\hat{L}_\varepsilon,\lambda)) + \chi_2(\xi)\partial_\xi A(\xi,\lambda), & \xi \in [\hat{L}_\varepsilon - 1, \hat{L}_\varepsilon], \\ \chi_2'(\xi)(A(\xi,\lambda) - A(Z_{a,\varepsilon} - \hat{L}_\varepsilon,\lambda)) + \chi_2(\xi)\partial_\xi A(\xi,\lambda), & \xi \in [Z_{a,\varepsilon} - \hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1], \\ 0, & \text{otherwise.} \end{cases} \tag{6.9}$$

First, we have that $\|\partial_\xi \mathcal{A}(\xi,\lambda)\| \leq C\sigma_0$ on $\mathbb{R} \times (R_1 \cup R_2)$ by the mean value theorem and identities (6.8) and (6.9). Second, by Lemma 6.3 and (6.8) the matrix $\mathcal{A}(\xi,\lambda)$ is hyperbolic on $\mathbb{R} \times (R_1 \cup R_2)$ with a- and ε-uniform spectral gap larger than $\mu > 0$. Third, $\mathcal{A}(\xi,\lambda)$ can be bounded on $\mathbb{R} \times (R_1 \cup R_2)$ uniformly in a and ε. Combining these three items with [5, Proposition 6.1] gives that system (6.7) has, provided $\sigma_0 > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$, independent of λ, a and ε, and projections $Q^{u,s}_r(\xi,\lambda) = Q^{u,s}_r(\xi,\lambda; a,\varepsilon)$. Since (6.7) coincides with (6.6) on $[\hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon]$, we have established the desired exponential dichotomy of (6.6) on I_r with constants $C, \mu > 0$ and projections $Q^{u,s}_r(\xi,\lambda)$.

The next step is to prove that the projections $Q_r^{u,s}(\xi,\lambda)$ are analytic in λ on $R_1 \cup R_2$. Any solution to the constant coefficient system $\psi_\xi = A(\hat{L}_\varepsilon,\lambda)\psi$ that converges to 0 as $\xi \to -\infty$ must be in the kernel of the spectral projection $\mathcal{P}(\hat{L}_\varepsilon,\lambda)$ on the stable eigenspace of $A(\hat{L}_\varepsilon,\lambda)$. Hence, it holds $R(1-\mathcal{P}(\hat{L}_\varepsilon,\lambda)) = R(Q_r^u(\hat{L}_\varepsilon - 1,\lambda))$ by construction of (6.7). Moreover, the spectral projection $\mathcal{P}(\hat{L}_\varepsilon,\lambda)$ is analytic in λ, since $A(\hat{L}_\varepsilon,\lambda)$ is analytic in λ. Thus, $R(Q_r^u(\hat{L}_\varepsilon - 1,\lambda))$ and similarly $R(Q_r^s(Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1,\lambda))$ must be analytic subspaces in λ. Denote by $\mathcal{T}(\xi,\hat{\xi},\lambda) = \mathcal{T}(\xi,\hat{\xi},\lambda;a,\varepsilon)$ the evolution of (6.7), which is analytic in λ. We conclude that both $\ker(Q_r^s(\hat{L}_\varepsilon - 1,\lambda))$ and

$$R(Q_r^s(\hat{L}_\varepsilon - 1,\lambda)) = R(\mathcal{T}(\hat{L}_\varepsilon - 1, Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1,\lambda)Q_r^s(Z_{a,\varepsilon} - \hat{L}_\varepsilon + 1,\lambda)),$$

are analytic subspaces. Therefore, the projection $Q_r^s(\hat{L}_\varepsilon - 1,\lambda)$ (and thus any projection $Q_r^{u,s}(\xi,\lambda)$, $\xi \in \mathbb{R}$) is analytic in λ on $R_1 \cup R_2$.

Finally, we shall prove that the projections $Q_r^s(\xi,\lambda)$ are close to the spectral projections $\mathcal{P}(\xi,\lambda)$ on the stable eigenspace of $A(\xi,\lambda)$ at the points $\xi = L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon$. First, observe that we have,

$$\begin{aligned} |u_{a,\varepsilon}'(\xi)| &\leq C\varepsilon|\log\varepsilon|, \quad \xi \in [\hat{L}_\varepsilon, 3\hat{L}_\varepsilon], \\ |u_{a,\varepsilon}'(\xi)| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \quad \xi \in [Z_{a,\varepsilon} - 3\hat{L}_\varepsilon, Z_{a,\varepsilon} - \hat{L}_\varepsilon], \end{aligned} \tag{6.10}$$

by Theorem 4.3 (i)-(ii). Consider the family of constant coefficient systems

$$\psi_\xi = \hat{A}(u,\lambda)\psi, \tag{6.11}$$

parameterized over $u \in \mathbb{R}$, where $\hat{A}(u,\lambda) = \hat{A}(u,\lambda;a,\varepsilon)$ is defined in Lemma 6.3. Denote by $\hat{\mathcal{P}}(u,\lambda) = \hat{\mathcal{P}}(u,\lambda;a,\varepsilon)$ the spectral projection on the stable eigenspace of $\hat{A}(u,\lambda)$ and by $\hat{\mathcal{T}}(\xi,\hat{\xi},u,\lambda) = \hat{\mathcal{T}}(\xi,\hat{\xi},u,\lambda;a,\varepsilon)$ the evolution operator of (6.11). Thus, we have $\hat{A}(u_{a,\varepsilon}(\xi),\lambda) = A(\xi,\lambda)$ and $\hat{\mathcal{P}}(u_{a,\varepsilon}(\xi),\lambda) = \mathcal{P}(\xi,\lambda)$ for $\xi \in \mathbb{R}$. Let $b_1 \in R(\mathcal{P}(L_\varepsilon,\lambda))$. Observe that

$$\hat{\psi}(\xi) := \mathcal{P}(\xi,\lambda)\hat{\mathcal{T}}(\xi,L_\varepsilon,u_{a,\varepsilon}(\xi),\lambda)b_1,$$

satisfies the inhomogeneous equation

$$\psi_\xi = A(\xi,\lambda)\psi + \hat{g}(\xi), \quad \hat{g}(\xi) := \partial_u\, \hat{\mathcal{P}}(u,\lambda)\hat{\mathcal{T}}(\xi,L_\varepsilon,u,\lambda)\Big|_{u=u_{a,\varepsilon}(\xi)} u_{a,\varepsilon}'(\xi)b_1.$$

By the variation of constants formula there exists $b_2 \in \mathbb{C}^3$ such that

$$\hat{\psi}(\xi) = \mathcal{T}(\xi,L_\varepsilon+\hat{L}_\varepsilon,\lambda)b_2 + \int_{L_\varepsilon}^{\xi} Q_r^s(\xi,\lambda)\mathcal{T}(\xi,\hat{\xi},\lambda)\hat{g}(\hat{\xi})d\hat{\xi} + \int_{L_\varepsilon+\hat{L}_\varepsilon}^{\xi} Q_r^u(\xi,\lambda)\mathcal{T}(\xi,\hat{\xi},\lambda)\hat{g}(\hat{\xi})d\hat{\xi}, \tag{6.12}$$

for $\xi \in [L_\varepsilon, L_\varepsilon + \hat{L}_\varepsilon]$. By [29, Lemma 1.1] and (6.10) we have

$$\|\hat{\psi}(\xi)\| \leq Ce^{-\mu(\xi-L_\varepsilon)}\|b_1\|, \quad \|\hat{g}(\xi)\| \leq C\varepsilon|\log\varepsilon|e^{-\mu(\xi-L_\varepsilon)}\|b_1\|, \tag{6.13}$$

for $\xi \in [L_\varepsilon, L_\varepsilon + \hat{L}_\varepsilon]$. Evaluating (6.12) at $L_\varepsilon + \hat{L}_\varepsilon$ while using (6.13), we derive $\|b_2\| \leq C\varepsilon|\log\varepsilon|\|b_1\|$, since $\nu \geq \mu/2$ by (6.5). Thus, applying $Q_r^u(L_\varepsilon,\lambda)$ to (6.12) at L_ε yields the bound $\|Q_r^u(L_\varepsilon,\lambda)b_1\| \leq C\varepsilon|\log\varepsilon|\|b_1\|$ for every $b_1 \in R(\mathcal{P}(L_\varepsilon,\lambda))$ by (6.13). Similarly, one shows that for every $b_1 \in \ker(\mathcal{P}(L_\varepsilon,\lambda))$ we have $\|Q_r^s(L_\varepsilon,\lambda)b_1\| \leq C\varepsilon|\log\varepsilon|\|b_1\|$. Thus, we obtain

$$\left\|[Q_r^s - \mathcal{P}](L_\varepsilon,\lambda)\right\| \leq \left\|[Q_r^u\mathcal{P}](L_\varepsilon,\lambda)\right\| + \left\|[Q_r^s(1-\mathcal{P})](L_\varepsilon,\lambda)\right\| \leq C\varepsilon|\log\varepsilon|.$$

The bound at $Z_{a,\varepsilon} - L_\varepsilon$ is obtain analogously.

In a similar way one obtains for $\lambda \in R_1 \cup R_2$ the desired exponential dichotomy for (6.6) on I_ℓ with constants $C,\mu > 0$ and projections $Q_\ell^{u,s}(\xi,\lambda)$. The only fundamental difference in the analysis is that the analyticity of the range of $Q_\ell^s(\xi,\lambda)$ is immediate, since the asymptotic system $\lim_{\xi\to\infty} A(\xi,\lambda)$ is analytic in λ, see [32, Theorem 1]. □

6.3 The region $R_1(\delta)$

6.3.1 A reduced eigenvalue problem

As described in §6.2.1 we establish for ξ in I_f or I_b a reduced eigenvalue problem by setting ε and λ to 0 in system (6.6), while approximating $\phi_{a,\varepsilon}(\xi)$ with (a translate of) the front $\phi_\mathrm{f}(\xi)$ or the back $\phi_\mathrm{b}(\xi)$, respectively. Thus, the reduced eigenvalue problem reads

$$\psi_\xi = A_j(\xi)\psi, \quad A_j(\xi) = A_j(\xi;a) := \begin{pmatrix} -\eta & 1 & 0 \\ -f'(u_j(\xi)) & \check{c}_0 - \eta & 1 \\ 0 & 0 & -\eta \end{pmatrix}, \quad j = \mathrm{f}, \mathrm{b}, \tag{6.14}$$

where $u_j(\xi)$ denotes the u-component of $\phi_j(\xi)$ and a is in $[0, \frac{1}{2} - \kappa]$. Now, for ξ-values in $I_\mathrm{f} = (-\infty, L_\varepsilon]$, problem (6.6) can be written as the perturbation

$$\psi_\xi = (A_\mathrm{f}(\xi) + B_\mathrm{f}(\xi,\lambda))\,\psi, \quad B_\mathrm{f}(\xi,\lambda) = B_\mathrm{f}(\xi,\lambda;a,\varepsilon) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda - [f'(u_{a,\varepsilon}(\xi)) - f'(u_\mathrm{f}(\xi))] & \check{c} - \check{c}_0 & 0 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda+\varepsilon\gamma}{\check{c}} \end{pmatrix}. \tag{6.15}$$

To define (6.6) as a proper perturbation of (6.14) along the back, we introduce the translated version of (6.6)

$$\psi_\xi = A(\xi + Z_{a,\varepsilon}, \lambda)\psi. \tag{6.16}$$

For ξ-values in $[-L_\varepsilon, L_\varepsilon]$ problem (6.16) can be written as the perturbation

$$\psi_\xi = (A_\mathrm{b}(\xi) + B_\mathrm{b}(\xi,\lambda))\,\psi, \quad B_\mathrm{b}(\xi,\lambda) = B_\mathrm{b}(\xi,\lambda;a,\varepsilon) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda - [f'(u_{a,\varepsilon}(\xi + Z_{a,\varepsilon})) - f'(u_\mathrm{b}(\xi))] & \check{c} - \check{c}_0 & 0 \\ \frac{\varepsilon}{\check{c}} & 0 & -\frac{\lambda+\varepsilon\gamma}{\check{c}} \end{pmatrix}. \tag{6.17}$$

The reduced eigenvalue problem (6.14) has an upper triangular block structure. Consequently, system (6.14) leaves the subspace $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$ invariant and the dynamics of (6.14) on that space is given by

$$\varphi_\xi = C_j(\xi)\varphi, \quad C_j(\xi) = C_j(\xi;a) := \begin{pmatrix} -\eta & 1 \\ -f'(u_j(\xi)) & \check{c}_0 - \eta \end{pmatrix}, \quad j = \mathrm{f}, \mathrm{b}. \tag{6.18}$$

Before studying the full reduced eigenvalue problem (6.14) we study the dynamics on the invariant subspace. We observe that system (6.18) has a one-dimensional space of bounded solution spanned by

$$\varphi_j(\xi) = \varphi_j(\xi;a) := e^{-\eta\xi}\phi_j'(\xi), \quad j = \mathrm{f}, \mathrm{b}.$$

Therefore, the adjoint system

$$\varphi_\xi = -C_j(\xi)^*\varphi, \quad j = \mathrm{f}, \mathrm{b}, \tag{6.19}$$

also has a one-dimensional space of bounded solution spanned by

$$\varphi_{j,\mathrm{ad}}(\xi) = \varphi_{j,\mathrm{ad}}(\xi;a) := \begin{pmatrix} v_j'(\xi) \\ -u_j'(\xi) \end{pmatrix} e^{(\eta - \check{c}_0)\xi}, \quad j = \mathrm{f}, \mathrm{b}. \tag{6.20}$$

We emphasize that φ_j and $\varphi_{j,\mathrm{ad}}$ can be determined explicitly using the expressions in (3.6) for ϕ_j, $j = \mathrm{f}, \mathrm{b}$. We establish exponential dichotomies for subsystem (6.18) on both half-lines.

Proposition 6.6. *Let $\kappa > 0$. For each $a \in [0, \frac{1}{2} - \kappa]$, system* (6.18) *admits exponential dichotomies on both half-lines $\mathbb{R}_\pm$ with a-independent constants $C, \mu > 0$ and projections $\Pi^{u,s}_{j,\pm}(\xi) = \Pi^{u,s}_{j,\pm}(\xi;a)$, $j = \mathrm{f}, \mathrm{b}$. Here, $\mu > 0$ is as in Lemma 6.3 and the projections can be chosen in such a way that*

$$R(\Pi^s_{j,+}(0)) = \mathrm{Span}(\varphi_j(0)) = R(\Pi^u_{j,-}(0)), \quad R(\Pi^u_{j,+}(0)) = \mathrm{Span}(\varphi_{j,\mathrm{ad}}(0)) = R(\Pi^s_{j,-}(0)), \quad j = \mathrm{f}, \mathrm{b}. \tag{6.21}$$

Proof. Define the asymptotic matrices $C_{j,\pm\infty} = C_{j,\pm\infty}(a) := \lim_{\xi\to\pm\infty} C_j(\xi)$ of (6.18) for $j = \mathrm{f,b}$. Consider the matrix $\hat{A}(u,\lambda,a,\varepsilon)$ from Lemma 6.3. The spectra of $C_{\mathrm{f},-\infty}$ and $C_{\mathrm{f},\infty}$ are contained in the spectra of $\hat{A}(0,0,a,0)$ and $\hat{A}(1,0,a,0)$, respectively. Similarly, we have the spectral inclusions $\sigma(C_{\mathrm{b},-\infty}) \subset \sigma(\hat{A}(u_{\mathrm{b}}^1,0,a,0))$ and $\sigma(C_{\mathrm{b},\infty}) \subset \sigma(\hat{A}(u_{\mathrm{b}}^0,0,a,0))$. By Lemma 6.3 the matrices $\hat{A}(u,0,a,0)$ have for $u = 0,1,u_{\mathrm{b}}^0,u_{\mathrm{b}}^1$ and $a \in [0,\frac{1}{2}-\kappa]$ a uniform spectral gap larger than $\mu > 0$. Thus, the same holds for the asymptotic matrices $C_{j,\pm\infty}$, $j = \mathrm{f,b}$. Hence, it follows from [29, Lemmata 1.1 and 1.2] that system (6.18) admits exponential dichotomies on both half-lines with constants $C,\mu > 0$ and projections as in (6.21). By compactness of $[0,\frac{1}{2}-\kappa]$ the constant $C > 0$ can be chosen independent of a. □

We shift our focus to the full reduced eigenvalue problem (6.14). One readily observes that

$$\omega_j(\xi) = \omega_j(\xi;a) := \begin{pmatrix} \varphi_j(\xi) \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\eta\xi}\phi_j'(\xi) \\ 0 \end{pmatrix}, \quad j = \mathrm{f,b}, \tag{6.22}$$

is a bounded solution to (6.14). Moreover, using variation of constants formulas the exponential dichotomies of the subsystem (6.18) can be transferred to the full system (6.14).

Corollary 6.7. *Let $\kappa > 0$. For each $a \in [0,\frac{1}{2}-\kappa]$ system* (6.14) *admits exponential dichotomies on both half-lines $\mathbb{R}_\pm$ with a-independent constants $C,\mu > 0$ and projections $Q_{j,\pm}^{u,s}(\xi) = Q_{j,\pm}^{u,s}(\xi;a)$, $j = \mathrm{f,b}$, given by*

$$\begin{aligned} Q_{j,+}^s(\xi) &= \begin{pmatrix} \Pi_{j,+}^s(\xi) & \displaystyle\int_\infty^\xi e^{\eta(\xi-\hat{\xi})}\Phi_{j,+}^u(\xi,\hat{\xi})F d\hat{\xi} \\ 0 & 1 \end{pmatrix} = 1 - Q_{j,+}^u(\xi), \quad \xi \geq 0, \\ Q_{j,-}^s(\xi) &= \begin{pmatrix} \Pi_{j,-}^s(\xi) & \displaystyle\int_0^\xi e^{\eta(\xi-\hat{\xi})}\Phi_{j,-}^u(\xi,\hat{\xi})F d\hat{\xi} \\ 0 & 1 \end{pmatrix} = 1 - Q_{j,-}^u(\xi), \quad \xi \leq 0, \end{aligned} \tag{6.23}$$

where F is the vector $\binom{0}{1}$ and $\Phi_{j,\pm}^{u,s}(\xi,\hat{\xi}) = \Phi_{j,\pm}^{u,s}(\xi,\hat{\xi};a)$ denotes the (un)stable evolution of system (6.18) *under the exponential dichotomies established in Proposition 6.6. Here, $\mu > 0$ is as in Lemma 6.3 and the projections satisfy*

$$\begin{aligned} R(Q_{j,+}^u(0)) &= \mathrm{Span}(\Psi_{1,j}), \quad R(Q_{j,+}^s(0)) = \mathrm{Span}(\omega_j(0),\Psi_2), \\ R(Q_{j,-}^u(0)) &= \mathrm{Span}(\omega_j(0)), \quad R(Q_{j,-}^s(0)) = \mathrm{Span}(\Psi_{1,j},\Psi_2), \end{aligned} \tag{6.24}$$

where ω_j is defined in (6.22) *and*

$$\Psi_{1,j} = \Psi_{1,j}(a) := \begin{pmatrix} \varphi_{j,\mathrm{ad}}(0) \\ 0 \end{pmatrix}, \quad \Psi_2 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad j = \mathrm{f,b}, \tag{6.25}$$

with $\varphi_{j,\mathrm{ad}}(\xi)$ defined in (6.20).

Proof. By variation of constants, the evolution $T_j(\xi,\hat{\xi}) = T_j(\xi,\hat{\xi};a)$ of the triangular block system (6.14) is given by

$$T_j(\xi,\hat{\xi}) = \begin{pmatrix} \Phi_j(\xi,\hat{\xi}) & \displaystyle\int_{\hat{\xi}}^\xi \Phi_j(\xi,z)Fe^{-\eta(z-\hat{\xi})}dz \\ 0 & e^{-\eta(\xi-\hat{\xi})} \end{pmatrix}, \quad j = \mathrm{f,b}.$$

Hence, using Proposition 6.6, one readily observes that the projections defined in (6.23) yield exponential dichotomies on both half-lines for (6.14) with constants $C, \min\{\mu,\eta\} > 0$, where $C > 0$ is independent of a. The result follows, since $\mu \leq \eta$ by Lemma 6.3. □

6.3.2 Along the front

In the previous section we showed that the eigenvalue problem (6.6) can be written as a (λ,ε)-perturbation (6.15) of the reduced eigenvalue problem (6.14). Moreover, we established an exponential dichotomy of (6.14) on $(-\infty,0]$ in

Corollary 6.7. Hence, solutions to (6.6) can be expressed by a variation of constant formula on $(-\infty, 0]$. This leads to an exit condition at $\xi = 0$ for exponentially decaying solutions to (6.6) in backward time.

Eventually, our plan is to also obtain entry and exit conditions for solutions to (6.6) on $[0, Z_{a,\varepsilon}]$ and for exponentially decaying solutions to (6.6) in forward time on $[Z_{a,\varepsilon}, \infty)$. As outlined in §6.2.1 equating these exit and entry conditions at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$ leads to a system of equations that can be reduced to a single analytic matching equation, whose solutions are λ-values for which (6.6) admits an exponentially localized solution.

Simultaneously, we evaluate the obtained exit condition at $\lambda = 0$ using that we know a priori that the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ of the pulse is the eigenfunction of (6.6) at $\lambda = 0$. As described in §6.2.1 this leads to extra information needed to simplify the expressions in the final matching equation.

Proposition 6.8. *Let B_f be as in* (6.15) *and ω_f as in* (6.22)*. Denote by $T^{u,s}_{\mathrm{f},-}(\xi,\hat{\xi}) = T^{u,s}_{\mathrm{f},-}(\xi,\hat{\xi};a)$ the (un)stable evolution of system* (6.14) *under the exponential dichotomy on $I_{\mathrm{f},-} = (-\infty, 0]$ established in Corollary 6.7 and by $Q^{u,s}_{\mathrm{f},-}(\xi) = Q^{u,s}_{\mathrm{f},-}(\xi;a)$ the associated projections.*

(i) There exists $\delta, \varepsilon_0 > 0$ such that for $\lambda \in R_1(\delta)$ and $\varepsilon \in (0, \varepsilon_0)$ any solution $\psi_{\mathrm{f},-}(\xi,\lambda)$ to (6.6) *decaying exponentially in backward time satisfies*

$$\psi_{\mathrm{f},-}(0,\lambda) = \beta_{\mathrm{f},-}\omega_\mathrm{f}(0) + \beta_{\mathrm{f},-}\int_{-\infty}^{0} T^s_{\mathrm{f},-}(0,\hat{\xi})B_\mathrm{f}(\hat{\xi},\lambda)\omega_\mathrm{f}(\hat{\xi})d\hat{\xi} + \mathcal{H}_{\mathrm{f},-}(\beta_{\mathrm{f},-}), \quad Q^u_{\mathrm{f},-}(0)\psi_{\mathrm{f},-}(0,\lambda) = \beta_{\mathrm{f},-}\omega_\mathrm{f}(0), \tag{6.26}$$

for some $\beta_{\mathrm{f},-} \in \mathbb{C}$, where $\mathcal{H}_{\mathrm{f},-}$ is a linear map satisfying the bound

$$\|\mathcal{H}_{\mathrm{f},-}(\beta_{\mathrm{f},-})\| \le C(\varepsilon|\log\varepsilon| + |\lambda|)^2|\beta_{\mathrm{f},-}|,$$

with $C > 0$ independent of λ, a and ε. Moreover, $\psi_{\mathrm{f},-}(\xi,\lambda)$ is analytic in λ.

(ii) The derivative $\phi'_{a,\varepsilon}$ of the pulse solution satisfies

$$Q^s_{\mathrm{f},-}(0)\phi'_{a,\varepsilon}(0) = \int_{-\infty}^{0} T^s_{\mathrm{f},-}(0,\hat{\xi})B_\mathrm{f}(\hat{\xi},0)e^{-\eta\hat{\xi}}\phi'_{a,\varepsilon}(\hat{\xi})d\hat{\xi}. \tag{6.27}$$

Proof. We begin with (i). Take $0 < \hat{\mu} < \mu$ with $\mu > 0$ as in Lemma 6.3. Denote by $C_{\hat{\mu}}(I_{\mathrm{f},-}, \mathbb{C}^3)$ the space of $\hat{\mu}$-exponentially decaying, continuous functions $I_{\mathrm{f},-} \to \mathbb{C}^3$ endowed with the norm $\|\psi\|_{\hat{\mu}} = \sup_{\xi\le 0}\|\psi(\xi)\|e^{\hat{\mu}|\xi|}$. By Theorem 4.3 (i) we bound the perturbation matrix B_f by

$$\|B_\mathrm{f}(\xi,\lambda;a,\varepsilon)\| \le C(\varepsilon|\log\varepsilon| + |\lambda|), \tag{6.28}$$

for $\xi \in I_{\mathrm{f},-}$. Let $\beta \in \mathbb{C}$ and $\lambda \in R_1(\delta)$. Combining (6.28) with Corollary 6.7 the function $\mathcal{G}_{\beta,\lambda} \colon C_{\hat{\mu}}(I_{\mathrm{f},-}, \mathbb{C}^3) \to C_{\hat{\mu}}(I_{\mathrm{f},-}, \mathbb{C}^3)$ given by

$$\mathcal{G}_{\beta,\lambda}(\psi)(\xi) = \beta\omega_\mathrm{f}(\xi) + \int_0^{\xi} T^u_{\mathrm{f},-}(\xi,\hat{\xi})B_\mathrm{f}(\hat{\xi},\lambda)\psi(\hat{\xi})d\hat{\xi} + \int_{-\infty}^{\xi} T^s_{\mathrm{f},-}(\xi,\hat{\xi})B_\mathrm{f}(\hat{\xi},\lambda)\psi(\hat{\xi})d\hat{\xi},$$

is a well-defined contraction mapping for each $\delta, \varepsilon > 0$ sufficiently small (with upper bound independent of β and a). By the Banach Contraction Theorem there exists a unique fixed point $\psi_{\mathrm{f},-} \in C_{\hat{\mu}}(I_{\mathrm{f},-}, \mathbb{C}^3)$ satisfying

$$\psi_{\mathrm{f},-} = \mathcal{G}_{\beta,\lambda}(\psi_{\mathrm{f},-}), \quad \xi \in I_{f,-}. \tag{6.29}$$

Observe that $\psi_{\mathrm{f},-}(\xi,\lambda)$ is analytic in λ, because the perturbation matrix $B_\mathrm{f}(\xi,\lambda)$ is analytic in λ. Moreover, $\psi_{\mathrm{f},-}$ is linear in β by construction. Hence, using estimate (6.28) we derive the bound

$$\|\psi_{\mathrm{f},-}(\xi,\lambda) - \beta\omega_\mathrm{f}(\xi)\| \le C|\beta|(\varepsilon|\log\varepsilon| + |\lambda|), \tag{6.30}$$

for $\xi \in I_{\mathrm{f},-}$.

The solutions to the family of fixed point equations (6.29) parameterized over $\beta \in \mathbb{C}$ form a one-dimensional space of exponentially decaying solutions as $\xi \to -\infty$ to (6.6). By Lemma 6.3 the asymptotic matrix $\hat{A}(0,\lambda,a,\varepsilon)$ of system (6.6) has precisely one eigenvalue of positive real part. Therefore, the space of decaying solutions in backward time to (6.6) is one-dimensional. This proves that any solution $\psi_{\mathrm{f},-}(\xi,\lambda)$ to (6.6) that converges to 0 as $\xi \to -\infty$, satisfies (6.29) for some $\beta \in \mathbb{C}$. Evaluating (6.29) at $\xi = 0$ and using estimates (6.28) and (6.30) yields (6.26).

For (ii), note that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an eigenfunction of (6.6) at $\lambda = 0$. Therefore, $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ satisfies the fixed point identity (6.29) at $\lambda = 0$ for some $\beta \in \mathbb{C}$ and identity (6.27) follows. □

6.3.3 Passage near the right slow manifold

Using the exponential dichotomies of system (6.14) established in Corollary 6.7 one can construct expressions for solutions to (6.6) via a variation of constants approach on the intervals $I_{\mathrm{f},+} = [0, L_\varepsilon]$ and $I_{\mathrm{b},-} = [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon}]$. Moreover, the exponential dichotomies established in Proposition 6.5 govern the solutions to (6.6) on $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$. Matching the solutions on these three intervals we obtain the following entry and exit conditions at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$.

Proposition 6.9. *Let B_j be as in* (6.15) *and* (6.17)*, Ψ_2 as in* (6.25) *and ω_j as in* (6.22) *for $j = \mathrm{f}, \mathrm{b}$. Denote by $T^{u,s}_{j,\pm}(\xi,\hat{\xi}) = T^{u,s}_{j,\pm}(\xi,\hat{\xi};a)$ the (un)stable evolution of system* (6.14) *under the exponential dichotomies established in Corollary 6.7 and by $Q^{u,s}_{j,\pm}(\xi) = Q^{u,s}_{j,\pm}(\xi;a)$ the associated projections for $j = \mathrm{f}, \mathrm{b}$.*

(i) For each sufficiently small $a_0 > 0$, there exists $\delta, \varepsilon_0 > 0$ such that for $\lambda \in R_1(\delta)$ and $\varepsilon \in (0,\varepsilon_0)$ any solution $\psi^{\mathrm{sl}}(\xi,\lambda)$ to (6.6) *satisfies*

$$\begin{aligned} \psi^{\mathrm{sl}}(0,\lambda) &= \beta_{\mathrm{f}}\omega_{\mathrm{f}}(0) + \zeta_{\mathrm{f}} Q^s_{\mathrm{f},+}(0)\Psi_2 + \beta_{\mathrm{f}} \int_{L_\varepsilon}^0 T^u_{\mathrm{f},+}(0,\hat{\xi}) B_{\mathrm{f}}(\hat{\xi},\lambda)\omega_{\mathrm{f}}(\hat{\xi}) d\hat{\xi} + \mathcal{H}_{\mathrm{f}}(\beta_{\mathrm{f}},\zeta_{\mathrm{f}},\beta_{\mathrm{b}}), \\ Q^u_{\mathrm{f},-}(0)\psi^{\mathrm{sl}}(0,\lambda) &= \beta_{\mathrm{f}}\omega_{\mathrm{f}}(0), \end{aligned} \tag{6.31}$$

$$\begin{aligned} \psi^{\mathrm{sl}}(Z_{a,\varepsilon},\lambda) &= \beta_{\mathrm{b}}\omega_{\mathrm{b}}(0) + \beta_{\mathrm{b}} \int_{-L_\varepsilon}^0 T^s_{\mathrm{b},-}(0,\hat{\xi}) B_{\mathrm{b}}(\hat{\xi},\lambda)\omega_{\mathrm{b}}(\hat{\xi}) d\hat{\xi} + \mathcal{H}_{\mathrm{b}}(\beta_{\mathrm{f}},\zeta_{\mathrm{f}},\beta_{\mathrm{b}}), \\ Q^u_{\mathrm{b},-}(0)\psi^{\mathrm{sl}}(Z_{a,\varepsilon},\lambda) &= \beta_{\mathrm{b}}\omega_{\mathrm{b}}(0), \end{aligned} \tag{6.32}$$

for some $\beta_{\mathrm{f}}, \beta_{\mathrm{b}}, \zeta_{\mathrm{f}} \in \mathbb{C}$, where $\mathcal{H}_{\mathrm{f}}$ and $\mathcal{H}_{\mathrm{b}}$ are linear maps satisfying the bounds

$$\begin{aligned} \|\mathcal{H}_{\mathrm{f}}(\beta_{\mathrm{f}},\zeta_{\mathrm{f}},\beta_{\mathrm{b}})\| &\le C\left((\varepsilon|\log\varepsilon| + |\lambda|)|\zeta_{\mathrm{f}}| + (\varepsilon|\log\varepsilon| + |\lambda|)^2|\beta_{\mathrm{f}}| + e^{-q/\varepsilon}|\beta_{\mathrm{b}}|\right), \\ \|\mathcal{H}_{\mathrm{b}}(\beta_{\mathrm{f}},\zeta_{\mathrm{f}},\beta_{\mathrm{b}})\| &\le C\left((\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)^2|\beta_{\mathrm{b}}| + e^{-q/\varepsilon}(|\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|)\right), \end{aligned}$$

where $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\rho(a) = 1$ for $a \ge a_0$ and $q, C > 0$ independent of λ, a and ε. Moreover, $\psi^{\mathrm{sl}}(\xi,\lambda)$ is analytic in λ.

(ii) The derivative $\phi'_{a,\varepsilon}$ of the pulse solution satisfies

$$\begin{aligned} Q^u_{\mathrm{f},+}(0)\phi'_{a,\varepsilon}(0) &= T^u_{\mathrm{f},+}(0,L_\varepsilon)e^{-\eta L_\varepsilon}\phi'_{a,\varepsilon}(L_\varepsilon) + \int_{L_\varepsilon}^0 T^u_{\mathrm{f},+}(0,\hat{\xi}) B_{\mathrm{f}}(\hat{\xi},0) e^{-\eta\hat{\xi}}\phi'_{a,\varepsilon}(\hat{\xi}) d\hat{\xi}, \\ Q^s_{\mathrm{b},-}(0)\phi'_{a,\varepsilon}(Z_{a,\varepsilon}) &= T^s_{\mathrm{b},-}(0,-L_\varepsilon)e^{\eta L_\varepsilon}\phi'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) + \int_{-L_\varepsilon}^0 T^s_{\mathrm{b},-}(0,\hat{\xi}) B_{\mathrm{f}}(\hat{\xi},0) e^{-\eta\hat{\xi}}\phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \hat{\xi}) d\hat{\xi}. \end{aligned} \tag{6.33}$$

Proof. We begin with (i). For the matching procedure, we need to compare projections $Q^{u,s}_{\mathrm{f},+}(\xi)$ of the exponential dichotomies of (6.14) established in Corollary 6.7 with the projections $Q^{u,s}_r(\xi,\lambda)$ of the dichotomy of (6.6) on I_r established in Proposition 6.5. First, recall that the front $\phi_{\mathrm{f}}(\xi)$ is a heteroclinic to the fixed point $(1,0)$ of (3.4). By looking at the linearization of (3.4) about $(1,0)$ we deduce that $\phi_{\mathrm{f}}(\xi)$, and thus the coefficient matrix $A_{\mathrm{f}}(\xi)$ of (6.14), converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to some asymptotic matrix $A_{\mathrm{f},\infty}$ as $\xi \to \infty$. Hence, by [28, Lemma 3.4] and its proof the projections $Q^{u,s}_{\mathrm{f},+}$ associated with the exponential dichotomy of system (6.14) satisfy for $\xi \geq 0$

$$\|Q^{u,s}_{\mathrm{f},+}(\xi) - P^{u,s}_{\mathrm{f}}\| \leq C\left(e^{-\frac{1}{2}\sqrt{2}\xi} + e^{-\mu\xi}\right), \tag{6.34}$$

where $P^{u,s}_{\mathrm{f}} = P^{u,s}_{\mathrm{f}}(a)$ denotes the spectral projection on the (un)stable eigenspace of the asymptotic matrix $A_{\mathrm{f},\infty}$. Moreover, the coefficient matrix $A(\xi,\lambda)$ of (6.6) is approximated at $L_\varepsilon = -\nu\log\varepsilon$ by

$$\|A(L_\varepsilon,\lambda) - A_{\mathrm{f},\infty}\| \leq C(\varepsilon|\log\varepsilon| + |\lambda|),$$

by Theorem 4.3 (i) and the fact that $A_{\mathrm{f}}(\xi)$ converges to $A_{\mathrm{f},\infty}$ at an exponential rate $\frac{1}{2}\sqrt{2}$ as $\xi \to \infty$, using that ν is chosen larger than $2\sqrt{2}$ in (6.5). By continuity the same bound holds for the spectral projections associated with the matrices $A(L_\varepsilon,\lambda)$ and $A_{\mathrm{f},\infty}$. Combining the latter facts with (6.34) and the bounds in Proposition 6.5 we obtain

$$\|Q^{u,s}_r(L_\varepsilon,\lambda) - Q^{u,s}_{\mathrm{f},+}(L_\varepsilon)\| \leq C(\varepsilon|\log\varepsilon| + |\lambda|), \tag{6.35}$$

using $\nu \geq \max\{\frac{2}{\mu}, 2\sqrt{2}\}$. In a similar way we obtain an estimate at $Z_{a,\varepsilon} - L_\varepsilon$

$$\|Q^{u,s}_r(Z_{a,\varepsilon} - L_\varepsilon,\lambda) - Q^{u,s}_{\mathrm{b},-}(-L_\varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|). \tag{6.36}$$

using Theorem 4.3 (ii).

By the variation of constants formula, any solution $\psi^{\mathrm{sl}}_{\mathrm{f}}(\xi,\lambda)$ to (6.6) must satisfy on $I_{\mathrm{f},+}$

$$\begin{aligned}\psi^{\mathrm{sl}}_{\mathrm{f}}(\xi,\lambda) = T^u_{\mathrm{f},+}(\xi,L_\varepsilon)\alpha_{\mathrm{f}} + \beta_{\mathrm{f}}\omega_{\mathrm{f}}(\xi) + \zeta_{\mathrm{f}}T^s_{\mathrm{f},+}(\xi,0)\Psi_2 + \int_0^\xi T^s_{\mathrm{f},+}(\xi,\hat\xi)B_{\mathrm{f}}(\hat\xi,\lambda)\psi^{\mathrm{sl}}_{\mathrm{f}}(\hat\xi,\lambda)d\hat\xi \\ + \int_{L_\varepsilon}^\xi T^u_{\mathrm{f},+}(\xi,\hat\xi)B_{\mathrm{f}}(\hat\xi,\lambda)\psi^{\mathrm{sl}}_{\mathrm{f}}(\hat\xi,\lambda)d\hat\xi,\end{aligned} \tag{6.37}$$

for some $\beta_{\mathrm{f}},\zeta_{\mathrm{f}} \in \mathbb{C}$ and $\alpha_{\mathrm{f}} \in R(Q^u_{\mathrm{f},+}(L_\varepsilon))$. By Theorem 4.3 (i) we bound the perturbation matrix B_{f} as

$$\|B_{\mathrm{f}}(\xi,\lambda;a,\varepsilon)\| \leq C(\varepsilon|\log\varepsilon| + |\lambda|), \tag{6.38}$$

for $\xi \in I_{\mathrm{f},+}$. Hence, for all sufficiently small $|\lambda|,\varepsilon > 0$, there exists a unique solution $\psi^{\mathrm{sl}}_{\mathrm{f}}$ to (6.37) by the contraction mapping principle. Note that $\psi^{\mathrm{sl}}_{\mathrm{f}}$ is linear in $(\alpha_{\mathrm{f}},\beta_{\mathrm{f}},\zeta_{\mathrm{f}})$ and satisfies the bound

$$\sup_{\xi\in[0,L_\varepsilon]} \|\psi^{\mathrm{sl}}_{\mathrm{f}}(\xi,\lambda)\| \leq C(\|\alpha_{\mathrm{f}}| + |\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|), \tag{6.39}$$

by estimate (6.38), taking $\delta,\varepsilon_0 > 0$ smaller if necessary.

Denote by $\mathcal{T}^{u,s}_r(\xi,\hat\xi,\lambda) = \mathcal{T}^{u,s}_r(\xi,\hat\xi,\lambda;a,\varepsilon)$ the (un)stable evolution of system (6.6) under the exponential dichotomy on I_r established in Proposition 6.5. Any solution ψ_r to (6.6) on I_r is of the form

$$\psi_r(\xi,\lambda) = \mathcal{T}^u_r(\xi,Z_{a,\varepsilon} - L_\varepsilon,\lambda)\alpha_r + \mathcal{T}^s_r(\xi,L_\varepsilon,\lambda)\beta_r, \tag{6.40}$$

for some $\alpha_r \in R(Q^u_r(Z_{a,\varepsilon} - L_\varepsilon,\lambda))$ and $\beta_r \in R(Q^s_r(L_\varepsilon,\lambda))$. Applying the projection $Q^u_r(L_\varepsilon,\lambda)$ to the difference $\psi_r(L_\varepsilon,\lambda) - \psi^{\mathrm{sl}}_{\mathrm{f}}(L_\varepsilon,\lambda)$ yields the matching condition

$$\begin{aligned}\alpha_{\mathrm{f}} &= \mathcal{H}_1(\alpha_{\mathrm{f}},\beta_{\mathrm{f}},\alpha_r), \\ \|\mathcal{H}_1(\alpha_{\mathrm{f}},\beta_{\mathrm{f}},\alpha_r)\| &\leq C((\varepsilon|\log\varepsilon| + |\lambda|)(\|\alpha_{\mathrm{f}}\| + |\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|) + e^{-q/\varepsilon}\|\alpha_r\|),\end{aligned} \tag{6.41}$$

where we use (6.35), (6.38), (6.39) and the fact that $Z_{a,\varepsilon} = O_s(\varepsilon^{-1})$ (see Theorem 4.3) to obtain the bound on the linear map $\mathcal{H}_1$. Similarly, applying the projection $Q^s_r(L_\varepsilon,\lambda)$ to the difference $\psi_r(L_\varepsilon,\lambda) - \psi^{\mathrm{sl}}_{\mathrm{f}}(L_\varepsilon,\lambda)$ yields the matching condition

$$\begin{aligned}\beta_r &= \mathcal{H}_2(\alpha_{\mathrm{f}},\beta_{\mathrm{f}},\zeta_{\mathrm{f}}), \\ \|\mathcal{H}_2(\alpha_{\mathrm{f}},\beta_{\mathrm{f}},\zeta_{\mathrm{f}})\| &\leq C(\varepsilon|\log\varepsilon| + |\lambda|)(\|\alpha_{\mathrm{f}}\| + |\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|),\end{aligned} \tag{6.42}$$

where we use (6.35), (6.38), (6.39) and $\nu \geq 2/\mu$ to obtain the bound on the linear map $\mathcal{H}_2$.

Consider the translated version (6.16) of system (6.6). By the variation of constants formula, any solution $\psi_{\mathrm{b}}^{\mathrm{sl}}(\xi,\lambda)$ to (6.16) on $[-L_\varepsilon, 0]$ must satisfy

$$\psi_{\mathrm{b}}^{\mathrm{sl}}(\xi,\lambda) = T_{\mathrm{b},-}^s(\xi,-L_\varepsilon)\alpha_{\mathrm{b}} + \beta_{\mathrm{b}}\omega_{\mathrm{b}}(\xi) + \int_0^\xi T_{\mathrm{b},-}^u(\xi,\hat{\xi})B_{\mathrm{b}}(\hat{\xi},\lambda)\psi_{\mathrm{b}}^{\mathrm{sl}}(\hat{\xi},\lambda)d\hat{\xi} + \int_{-L_\varepsilon}^\xi T_{\mathrm{b},-}^s(\xi,\hat{\xi})B_{\mathrm{b}}(\hat{\xi},\lambda)\psi_{\mathrm{b}}^{\mathrm{sl}}(\hat{\xi},\lambda)d\hat{\xi}, \tag{6.43}$$

for some $\beta_{\mathrm{b}} \in \mathbb{C}$ and $\alpha_{\mathrm{b}} \in R(Q_{\mathrm{b},-}^s(-L_\varepsilon))$. By Theorem 4.3 (ii) we estimate

$$\|B_{\mathrm{b}}(\xi,\lambda;a,\varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|), \tag{6.44}$$

for $\xi \in [-L_\varepsilon, 0]$. For all sufficiently small $|\lambda|, \varepsilon > 0$, there exists a unique solution $\psi_{\mathrm{b}}^{\mathrm{sl}}$ of (6.43). Note that $\psi_{\mathrm{b}}^{\mathrm{sl}}$ is linear in $(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}})$ and using (6.44) we obtain the bound

$$\sup_{\xi\in[-L_\varepsilon,0]} \|\psi_{\mathrm{b}}^{\mathrm{sl}}(\xi,\lambda)\| \leq C(\|\alpha_{\mathrm{b}}\| + |\beta_{\mathrm{b}}|), \tag{6.45}$$

taking $\delta, \varepsilon_0 > 0$ smaller if necessary. The matching of $\psi_{\mathrm{b}}^{\mathrm{sl}}(-L_\varepsilon,\lambda)$ with $\psi_r(Z_{a,\varepsilon} - L_\varepsilon, \lambda)$ is completely similar to the matching of $\psi_{\mathrm{f}}^{\mathrm{sl}}(L_\varepsilon,\lambda)$ with $\psi_r(L_\varepsilon,\lambda)$ in the previous paragraph using (6.45) instead of (6.39) and (6.36) instead of (6.35). Hence we give only the resulting matching conditions

$$\begin{aligned} \alpha_r &= \mathcal{H}_3(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}}), \\ \|\mathcal{H}_3(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}})\| &\leq C(\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)(\|\alpha_{\mathrm{b}}\| + |\beta_{\mathrm{b}}|), \end{aligned} \tag{6.46}$$

$$\begin{aligned} \alpha_{\mathrm{b}} &= \mathcal{H}_4(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}}, \beta_r), \\ \|\mathcal{H}_4(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}}, \beta_r)\| &\leq C\left((\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)(\|\alpha_{\mathrm{b}}\| + |\beta_{\mathrm{b}}|) + e^{-q/\varepsilon}\|\beta_r\|\right), \end{aligned} \tag{6.47}$$

where $\mathcal{H}_3$ and $\mathcal{H}_4$ are again linear maps.

We now combine the above results regarding the solution on $[0, Z_{a,\varepsilon}]$ to obtain the relevant conditions satisfied at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$. Combining equations (6.42) and (6.47), we obtain a linear map $\mathcal{H}_5$ satisfying

$$\begin{aligned} \alpha_{\mathrm{b}} &= \mathcal{H}_5(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}}, a_{\mathrm{f}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}}), \\ \|\mathcal{H}_5(\alpha_{\mathrm{b}}, \beta_{\mathrm{b}}, \beta_r, c_r)\| &\leq C\left((\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)(\|\alpha_{\mathrm{b}}\| + |\beta_{\mathrm{b}}|) + e^{-q/\varepsilon}(\|\alpha_{\mathrm{f}}\| + |\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|)\right). \end{aligned} \tag{6.48}$$

Thus, solving (6.48) for α_{b}, we obtain for all sufficiently small $|\lambda|, \varepsilon > 0$

$$\begin{aligned} \alpha_{\mathrm{b}} &= \alpha_{\mathrm{b}}(\alpha_{\mathrm{f}}, \beta_{\mathrm{b}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}}), \\ \|\alpha_{\mathrm{b}}(\alpha_{\mathrm{f}}, \beta_{\mathrm{b}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}})\| &\leq C\left((\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)|\beta_{\mathrm{b}}| + e^{-q/\varepsilon}(\|\alpha_{\mathrm{f}}\| + |\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|)\right). \end{aligned} \tag{6.49}$$

From (6.41), (6.46) and (6.49) we obtain a linear map $\mathcal{H}_6$ satisfying

$$\begin{aligned} \alpha_{\mathrm{f}} &= \mathcal{H}_6(\alpha_{\mathrm{f}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}}, \beta_{\mathrm{b}}), \\ \|\mathcal{H}_6(\alpha_{\mathrm{f}}, \beta_{\mathrm{b}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}})\| &\leq C\left((\varepsilon|\log\varepsilon| + |\lambda|)(\|\alpha_{\mathrm{f}}\| + |\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|) + e^{-q/\varepsilon}|\beta_{\mathrm{b}}|\right), \end{aligned} \tag{6.50}$$

We solve (6.50) for α_{f} for each sufficiently small $|\lambda|, \varepsilon > 0$ and obtain

$$\begin{aligned} \alpha_{\mathrm{f}} &= \alpha_{\mathrm{f}}(\beta_{\mathrm{b}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}}), \\ \|\alpha_{\mathrm{f}}(\beta_{\mathrm{b}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}})\| &\leq C\left((\varepsilon|\log\varepsilon| + |\lambda|)(|\beta_{\mathrm{f}}| + |\zeta_{\mathrm{f}}|) + e^{-q/\varepsilon}|\beta_{\mathrm{b}}|\right). \end{aligned} \tag{6.51}$$

Substituting (6.51) into (6.37) at $\xi = 0$ we deduce, using $\nu \geq \mu/2$ and identities (6.24), (6.38) and (6.39), that any solution $\psi^{\mathrm{sl}}(\xi,\lambda)$ to (6.6) satisfies the entry condition (6.31). Similarly, we substitute (6.51) into (6.49) and substitute the resulting expression for α_{b} into (6.43) at $\xi = 0$. Using estimates (6.44) and (6.45) and we obtain the exit condition (6.32). Since the perturbation matrices $B_j(\xi,\lambda)$, $j = \mathrm{f}, \mathrm{b}$, the evolution $\mathcal{T}(\xi,\hat{\xi},\lambda)$ of system (6.6) and the projections $Q_r^{u,s}(\xi,\lambda)$ associated with the exponential dichotomy of (6.6) are analytic in λ, all quantities occurring in this proof depend analytically on λ. Thus, $\psi^{\mathrm{sl}}(\xi,\lambda)$ is analytic in λ.

For (ii), we note that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an eigenfunction of (6.6) at $\lambda = 0$. Therefore, there exists $\beta_{\mathrm{f},0}, \zeta_{\mathrm{f},0} \in \mathbb{C}$ and $\alpha_{\mathrm{f},0} \in R(Q^u_{\mathrm{f},+}(L_\varepsilon))$ such that (6.37) is satisfied at $\lambda = 0$ with $\psi^{\mathrm{sl}}_{\mathrm{f}}(\xi,0) = e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ and $(\alpha_{\mathrm{f}},\beta_{\mathrm{f}},\zeta_{\mathrm{f}}) = (\alpha_{\mathrm{f},0},\beta_{\mathrm{f},0},\zeta_{\mathrm{f},0})$. We derive $\alpha_{\mathrm{f},0} = Q^u_{\mathrm{f},+}(L_\varepsilon)e^{-\eta L_\varepsilon}\phi'_{a,\varepsilon}(L_\varepsilon)$ by applying $Q^u_{\mathrm{f},+}(L_\varepsilon)$ to (6.37) at $\xi = L_\varepsilon$. Therefore, the first identity in (6.33) follows by applying $Q^u_{\mathrm{f},+}(0)$ to (6.37) at $\xi = 0$. The second identity in (6.33) follows in a similar fashion using that there exists $\beta_{\mathrm{b},0} \in \mathbb{C}$ and $\alpha_{\mathrm{b},0} \in R(Q^s_{\mathrm{b},-}(-L_\varepsilon))$ such that (6.43) is satisfied at $\lambda = 0$ with $\psi^{\mathrm{sl}}_{\mathrm{b}}(\xi,0) = e^{-\eta(\xi+Z_{a,\varepsilon})}\phi'_{a,\varepsilon}(Z_{a,\varepsilon}+\xi)$ and $(\alpha_{\mathrm{b}},\beta_{\mathrm{b}}) = (\alpha_{\mathrm{b},0},\beta_{\mathrm{b},0})$. □

6.3.4 Along the back

Finally, we establish an entry condition for exponentially decaying solution to (6.6) on the interval $[Z_{a,\varepsilon},\infty)$.

Proposition 6.10. *Let B_{b} be as in* (6.17)*, Ψ_2 as in* (6.25) *and ω_{b} as in* (6.22)*. Denote by $T^{u,s}_{\mathrm{b},\pm}(\xi,\hat\xi) = T^{u,s}_{\mathrm{b},\pm}(\xi,\hat\xi;a)$ the (un)stable evolution of system* (6.14) *under the exponential dichotomies established in Corollary 6.7 and by $Q^{u,s}_{\mathrm{b},\pm}(\xi) = Q^{u,s}_{\mathrm{b},\pm}(\xi;a)$ the associated projections.*

(i) For each sufficiently small $a_0 > 0$, there exists $\delta,\varepsilon_0 > 0$ such that for $\lambda \in R_1(\delta)$ and $\varepsilon \in (0,\varepsilon_0)$ any solution $\psi_{\mathrm{b},+}(\xi,\lambda)$ to (6.6)*, which is exponentially decaying in forward time, satisfies*

$$\begin{aligned}\psi_{\mathrm{b},+}(Z_{a,\varepsilon},\lambda) &= \beta_{\mathrm{b},+}\omega_{\mathrm{b}}(0) + \zeta_{\mathrm{b},+}Q^s_{\mathrm{b},+}(0)\Psi_2 + \beta_{\mathrm{b},+}\int_{L_\varepsilon}^0 T^u_{\mathrm{b},+}(0,\hat\xi)B_{\mathrm{b}}(\hat\xi,\lambda)\omega_{\mathrm{b}}(\hat\xi)d\hat\xi + \mathcal{H}_{\mathrm{b},+}(\beta_{\mathrm{b},+},\zeta_{\mathrm{b},+}),\\ Q^u_{\mathrm{b},-}(0)\psi_{\mathrm{b},+}(Z_{a,\varepsilon},\lambda) &= \beta_{\mathrm{b},+}\omega_{\mathrm{b}}(0),\end{aligned} \tag{6.52}$$

for some $\beta_{\mathrm{b},+},\zeta_{\mathrm{b},+} \in \mathbb{C}$, where $\mathcal{H}_{\mathrm{b},+}$ is a linear map satisfying the bound

$$\|\mathcal{H}_{\mathrm{b},+}(\beta_{\mathrm{b},+},\zeta_{\mathrm{b},+})\| \leq C\left((\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)|\zeta_{\mathrm{b},+}| + (\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)^2|\beta_{\mathrm{b}}|\right),$$

with $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\rho(a) = 1$ for $a \geq a_0$ and $C > 0$ independent of λ, a and ε. Moreover, $\psi_{\mathrm{b},+}(\xi,\lambda)$ is analytic in λ.

(ii) The derivative $\phi'_{a,\varepsilon}$ of the pulse solution satisfies

$$Q^u_{\mathrm{b},+}(0)\phi'_{a,\varepsilon}(Z_{a,\varepsilon}) = T^u_{\mathrm{b},+}(0,L_\varepsilon)e^{-\eta L_\varepsilon}\phi'_{a,\varepsilon}(Z_{a,\varepsilon}+L_\varepsilon) + \int_{L_\varepsilon}^0 T^u_{\mathrm{b},+}(0,\hat\xi)B_{\mathrm{b}}(\hat\xi,0)e^{-\eta\hat\xi}\phi'_{a,\varepsilon}(Z_{a,\varepsilon}+\hat\xi)d\hat\xi. \tag{6.53}$$

Proof. We begin with (i). Consider the translated version (6.16) of system (6.6). By the variation of constants formula, any solution $\hat\psi_{\mathrm{b},+}(\xi,\lambda)$ to (6.16) on $[0,L_\varepsilon]$ must satisfy

$$\begin{aligned}\hat\psi_{\mathrm{b},+}(\xi,\lambda) = T^u_{\mathrm{b},+}(\xi,L_\varepsilon)\alpha_{\mathrm{b},+} + \beta_{\mathrm{b},+}\omega_{\mathrm{b}}(\xi) + \zeta_{\mathrm{b},+}T^s_{\mathrm{b},+}(\xi,0)\Psi_2 + \int_0^\xi T^s_{\mathrm{b},+}(\xi,\hat\xi)B_{\mathrm{b}}(\hat\xi,\lambda)\hat\psi_{\mathrm{b},+}(\hat\xi,\lambda)d\hat\xi\\ + \int_{L_\varepsilon}^\xi T^u_{\mathrm{b},+}(\xi,\hat\xi)B_{\mathrm{b}}(\hat\xi,\lambda)\hat\psi_{\mathrm{b},+}(\hat\xi,\lambda)d\hat\xi,\end{aligned} \tag{6.54}$$

for some $\beta_{\mathrm{b},+},\zeta_{\mathrm{b},+} \in \mathbb{C}$ and $\alpha_{\mathrm{b},+} \in R(Q^u_{\mathrm{b},+}(L_\varepsilon))$. By Theorem 4.3 (ii) we estimate

$$\|B_{\mathrm{b}}(\xi,\lambda;a,\varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|), \tag{6.55}$$

for $\xi \in [0,L_\varepsilon]$. For all sufficiently small $|\lambda|,\varepsilon > 0$, there exists a unique solution $\hat\psi_{\mathrm{b},+}$ of (6.54). Note that $\hat\psi_{\mathrm{b},+}$ is linear in $(\alpha_{\mathrm{b},+},\beta_{\mathrm{b},+},\zeta_{\mathrm{b},+})$ and using (6.55) we obtain the bound,

$$\sup_{\xi\in[0,L_\varepsilon]}\|\hat\psi_{\mathrm{b},+}(\xi,\lambda)\| \leq C(\|\alpha_{\mathrm{b},+}\| + |\beta_{\mathrm{b},+}| + |\zeta_{\mathrm{b},+}|), \tag{6.56}$$

taking $\delta, \varepsilon_0 > 0$ smaller if necessary.

Consider the exponential dichotomies of (6.6) on $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$ established in Proposition 6.5 with associated projections $Q_\ell^{u,s}(\xi, \lambda)$. Completely analogous to the derivation of (6.36) in the proof of Proposition 6.9 we establish

$$\|Q_\ell^{u,s}(Z_{a,\varepsilon} + L_\varepsilon, \lambda) - Q_{\mathrm{b},+}^{u,s}(L_\varepsilon)\| \leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|). \tag{6.57}$$

The image of any exponentially decaying solution to (6.6) at $Z_{a,\varepsilon} + L_\varepsilon$ under $Q_\ell^u(Z_{a,\varepsilon} + L_\varepsilon, \lambda)$ must be 0, i.e. any solution $\psi_\ell(\xi, \lambda)$ to (6.6) decaying in forward time can be written as

$$\psi_\ell(\xi, \lambda) = \mathcal{T}_\ell^s(\xi, Z_{a,\varepsilon} + L_\varepsilon, \lambda)\beta_\ell, \tag{6.58}$$

for some $\beta_\ell \in R(Q_\ell^s(Z_{a,\varepsilon} + L_\varepsilon, \lambda))$, where $\mathcal{T}_\ell^s(\xi, \hat{\xi}, \lambda)$ denotes the stable evolution of system (6.6). Thus, by applying $Q_\ell^u(Z_{a,\varepsilon} + L_\varepsilon, \lambda)$ to $\hat{\psi}_{\mathrm{b},+}(L_\varepsilon, \lambda)$ we obtain a linear map $\mathcal{H}_1$ satisfying

$$\begin{aligned} \alpha_{\mathrm{b},+} &= \mathcal{H}_1(\alpha_{\mathrm{b},+}, \beta_{\mathrm{b},+}, \zeta_{\mathrm{b},+}), \\ \|\mathcal{H}_1(\alpha_{\mathrm{b},+}, \beta_{\mathrm{b},+}, \zeta_{\mathrm{b}}, \beta_r)\| &\leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(\|\alpha_{\mathrm{b},+}\| + |\beta_{\mathrm{b},+}| + |\zeta_{\mathrm{b},+}|), \end{aligned} \tag{6.59}$$

where we have used (6.55), (6.56) and (6.57). So, for sufficiently small $|\lambda|, \varepsilon > 0$, solving (6.59) for $\alpha_{\mathrm{b},+}$ yields

$$\begin{aligned} \alpha_{\mathrm{b},+} &= \alpha_{\mathrm{b},+}(\beta_{\mathrm{b},+}, \zeta_{\mathrm{b},+}) \\ \|\alpha_{\mathrm{b},+}(\beta_{\mathrm{b},+}, \zeta_{\mathrm{b},+})\| &\leq C(\varepsilon^{\rho(a)}|\log \varepsilon| + |\lambda|)(|\beta_{\mathrm{b},+}| + |\zeta_{\mathrm{b},+}|). \end{aligned} \tag{6.60}$$

Substituting (6.60) into (6.54) we deduce with the aid of (6.24), (6.55) and (6.56) that any exponentially decaying solution $\psi_{\mathrm{b},+}(\xi, \lambda) = \hat{\psi}_{\mathrm{b},+}(\xi - Z_{a,\varepsilon}, \lambda)$ to (6.6) satisfies the entry condition (6.52) at $\xi = Z_{a,\varepsilon}$. Moreover, analyticity of $\psi_{\mathrm{b},+}(\xi, \lambda)$ in λ follows from the analyticity of $B_{\mathrm{b}}(\xi, \lambda)$, of the evolution $\mathcal{T}(\xi, \hat{\xi}, \lambda)$ and of the projections $Q_\ell^{u,s}(\xi, \lambda)$.

We now prove (ii). Identity (6.53) follows in a similar fashion as (6.27) in the proof of Proposition 6.9 using that there exists $\beta_{\mathrm{b},+}, \zeta_{\mathrm{b},+} \in \mathbb{C}$ and $\alpha_{\mathrm{b},+} \in R(Q_{\mathrm{b},+}^u(L_\varepsilon))$ such that (6.43) is satisfied at $\lambda = 0$ with $\hat{\psi}_{\mathrm{b},+}(\xi, 0) = e^{-\eta(\xi + Z_{a,\varepsilon})}\phi'_{a,\varepsilon}(Z_{a,\varepsilon} + \xi)$. □

6.3.5 The matching procedure

In the previous sections we constructed a piecewise continuous, exponentially localized solution to the shifted eigenvalue problem (6.6) for any $\lambda \in R_1(\delta)$. At the two discontinuous jumps at $\xi = 0$ and $\xi = Z_{a,\varepsilon}$ we obtained expressions for the left and right limits of the solution; these are the so-called exit and entry conditions. Finding eigenvalues now reduces to locating $\lambda \in R_1$ for which the exit and entry conditions match up. Equating the exit and entry conditions leads, after reduction, to a single analytic matching equation in λ.

During the matching process we simplify terms in the following way. Recall that we evaluated the obtained exit and entry conditions at $\lambda = 0$ using that the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ of the pulse is an eigenfunction of (6.6) at $\lambda = 0$. This leads to identities that can be substituted in the matching equations; see Remark 6.12.

Since the final analytic matching equation is to leading order a quadratic in λ, it has precisely two solutions in R_1. These solutions are the eigenvalues of $\mathcal{L}_{a,\varepsilon}$ in R_1. A priori we know that $\lambda_0 = 0$ must be one of these two eigenvalues by translational invariance. In the next section 6.4 we show that λ_0 is in fact a simple eigenvalue of $\mathcal{L}_{a,\varepsilon}$. The other eigenvalue λ_1 can be determined to leading order. Section 6.5 is devoted to the calculation of this second eigenvalue, which differs between the hyperbolic and nonhyperbolic regime.

Thus, our aim is to prove the following result.

Theorem 6.11. *For each sufficiently small $a_0 > 0$, there exists $\delta, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ system* (6.6) *has precisely two different eigenvalues $\lambda_0, \lambda_1 \in R_1(\delta)$. The eigenvalue λ_0 equals* 0 *and the corresponding eigenspace is spanned by the solution $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ to* (6.6). *The other eigenvalue λ_1 is a-uniformly approximated by*

$$\lambda_1 = -\frac{M_{\mathrm{b},2}}{M_{\mathrm{b},1}} + O\left(\left|\varepsilon^{\rho(a)} \log \varepsilon\right|^2\right),$$

with

$$M_{\mathrm{b},1} := \int_{-\infty}^{\infty} \left(u_{\mathrm{b}}'(\xi)\right)^2 e^{-c_0\xi} d\xi, \quad M_{\mathrm{b},2} := \left\langle \Psi_*, \phi_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \right\rangle, \quad \Psi_* := \begin{pmatrix} e^{c_0 L_\varepsilon} v_{\mathrm{b}}'(-L_\varepsilon) \\ -e^{c_0 L_\varepsilon} u_{\mathrm{b}}'(-L_\varepsilon) \\ \displaystyle\int_{\infty}^{-L_\varepsilon} e^{-c_0\hat{\xi}} u_{\mathrm{b}}'(\hat{\xi}) d\hat{\xi} \end{pmatrix}, \tag{6.61}$$

where $(u_{\mathrm{b}}(\xi), v_{\mathrm{b}}(\xi)) = \phi_{\mathrm{b}}(\xi)$ *denotes the heteroclinic back solution to the Nagumo system* (3.5) *and the exponent* $\rho(a)$ *equals* $\frac{2}{3}$ *for* $a < a_0$ *and* 1 *for* $a \geq a_0$. *The corresponding eigenspace is spanned by a solution* $\psi_1(\xi)$ *to* (6.6) *satisfying*

$$\begin{aligned} \|\psi_1(\xi + Z_{a,\varepsilon}) - \omega_{\mathrm{b}}(\xi)\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \quad \xi \in [-L_\varepsilon, L_\varepsilon], \\ \|\psi_1(\xi + Z_{a,\varepsilon})\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \quad \xi \in \mathbb{R} \setminus [-L_\varepsilon, L_\varepsilon], \end{aligned} \tag{6.62}$$

where ω_{b} *is as in* (6.22) *and* $C > 1$ *is independent of* a *and* ε. *Finally, the quantities* $M_{\mathrm{b},1}$ *and* $M_{\mathrm{b},2}$ *satisfy the bounds*

$$1/C \leq M_{\mathrm{b},1} \leq C, \quad |M_{\mathrm{b},2}| \leq C\varepsilon^{\rho(a)}|\log\varepsilon|.$$

Proof. We start the proof with some estimates from the existence problem. By Theorem 4.3 (i)-(ii) we have the bounds

$$\begin{aligned} \|B_{\mathrm{f}}(\xi,\lambda;a,\varepsilon)\| &\leq C(\varepsilon|\log\varepsilon| + |\lambda|), \quad \xi \in (-\infty, L_\varepsilon], \\ \|B_{\mathrm{b}}(\xi,\lambda;a,\varepsilon)\| &\leq C(\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|), \quad \xi \in [-L_\varepsilon, L_\varepsilon]. \end{aligned} \tag{6.63}$$

where B_{f} and B_{b} are as in (6.15) and (6.17). Moreover, we use the equations (3.4) and (3.5) for ϕ_{f} and ϕ_{b} and the equation (3.1) for $\phi_{a,\varepsilon}$ in combination with Theorem 4.3 (i)-(ii) to estimate the difference between the derivatives

$$\begin{aligned} \left\| \begin{pmatrix} \phi_{\mathrm{f}}'(\xi) \\ 0 \end{pmatrix} - \phi_{a,\varepsilon}'(\xi) \right\| &\leq C\varepsilon|\log\varepsilon|, \quad \xi \in (-\infty, L_\varepsilon], \\ \left\| \begin{pmatrix} \phi_{\mathrm{b}}'(\xi) \\ 0 \end{pmatrix} - \phi_{a,\varepsilon}'(Z_{a,\varepsilon} + \xi) \right\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \quad \xi \in [-L_\varepsilon, L_\varepsilon]. \end{aligned} \tag{6.64}$$

We outline the matching procedure that yields the two λ-values for which (6.6) admits nontrivial exponentially localized solutions. By Proposition 6.8 any solution $\psi_{\mathrm{f},-}(\xi,\lambda)$ to (6.6) decaying exponentially in backward time satisfies (6.26) at $\xi = 0$ for some constant $\beta_{\mathrm{f},-} \in \mathbb{C}$. Moreover, by Proposition 6.9 any solution $\psi^{\mathrm{sl}}(\xi,\lambda)$ to (6.6) satisfies (6.31) at $\xi = 0$ for some $\beta_{\mathrm{f}}, \zeta_{\mathrm{f}} \in \mathbb{C}$ and (6.32) at $\xi = Z_{a,\varepsilon}$ for some $\beta_{\mathrm{b}} \in \mathbb{C}$. Finally, by Proposition 6.10 any solution $\psi_{\mathrm{b},+}(\xi,\lambda)$ to (6.6) decaying exponentially in forward time satisfies (6.52) at $\xi = Z_{a,\varepsilon}$ for some $\beta_{\mathrm{b},+}, \zeta_{\mathrm{b},+} \in \mathbb{C}$. To obtain an exponentially localized solution to (6.6) we match the solutions $\psi_{\mathrm{f},-}, \psi^{\mathrm{sl}}$ and $\psi_{\mathrm{b},+}$ at $\xi = 0$ and at $\xi = Z_{a,\varepsilon}$. It suffices to require that the differences $\psi_{\mathrm{f},-}(0,\lambda) - \psi^{\mathrm{sl}}(0,\lambda)$ and $\psi^{\mathrm{sl}}(Z_{a,\varepsilon},\lambda) - \psi_{\mathrm{b},+}(Z_{a,\varepsilon},\lambda)$ vanish under the projections $Q^{u,s}_{\mathrm{f},-}(0)$ and $Q^{u,s}_{\mathrm{b},-}(0)$ associated with the exponential dichotomy of (6.14) established in Corollary 6.7.

We first apply the projections $Q^u_{j,-}(0)$, $j = \mathrm{f},\mathrm{b}$ to the differences $\psi_{\mathrm{f},-}(0,\lambda) - \psi^{\mathrm{sl}}(0,\lambda)$ and $\psi^{\mathrm{sl}}(Z_{a,\varepsilon},\lambda) - \psi_{\mathrm{b},+}(Z_{a,\varepsilon},\lambda)$ and immediately obtain $\beta_{\mathrm{f}} = \beta_{\mathrm{f},-}$ and $\beta_{\mathrm{b}} = \beta_{\mathrm{b},+}$ using (6.26), (6.31), (6.32) and (6.52). For the remaining matching conditions, consider the vectors $\Psi_{1,j}$ and Ψ_2 defined in (6.25) and the bounded solution $\varphi_{j,\mathrm{ad}}$, given by (6.20), to the adjoint equation (6.19) of the reduced eigenvalue problem (6.14). By (6.24) the vectors Ψ_2 and

$$\Psi_{j,\perp} := \Psi_{1,j} - \int_{\infty}^{0} e^{-\eta\xi} \left\langle \varphi_{j,\mathrm{ad}}(\xi), F \right\rangle d\xi\, \Psi_2, \quad F = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad j = \mathrm{f},\mathrm{b},$$

span $R(Q^s_{j,-}(0))$ and $\Psi_{j,\perp}$ is contained in $\ker(Q^s_{j,+}(0)^*) = R(Q^u_{j,+}(0)^*) \subset R(Q^s_{j,-}(0)^*)$ for $j = \mathrm{f},\mathrm{b}$. Thus, we obtain four other matching conditions by requiring that the inner products of the differences $\psi_{\mathrm{f},-}(0,\lambda) - \psi^{\mathrm{sl}}(0,\lambda)$ and $\psi^{\mathrm{sl}}(Z_{a,\varepsilon},\lambda) - \psi_{\mathrm{b},+}(Z_{a,\varepsilon},\lambda)$ with Ψ_2 and $\Psi_{j,\perp}$ vanish for $j = \mathrm{f},\mathrm{b}$. With the aid of the identities (6.26), (6.31), (6.32) and (6.52) we obtain the first two matching conditions by pairing with Ψ_2

$$\begin{aligned} 0 &= \left\langle \Psi_2, \psi_{\mathrm{f},-}(0,\lambda) - \psi^{\mathrm{sl}}(0,\lambda) \right\rangle = -\zeta_{\mathrm{f}} + \mathcal{H}_1(\beta_{\mathrm{b}}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}}), \\ 0 &= \left\langle \Psi_2, \psi^{\mathrm{sl}}(Z_{a,\varepsilon},\lambda) - \psi_{\mathrm{b},+}(Z_{a,\varepsilon},\lambda) \right\rangle = -\zeta_{\mathrm{b},+} + \mathcal{H}_2(\beta_{\mathrm{b}}, \zeta_{\mathrm{b},+}, \beta_{\mathrm{f}}, \zeta_{\mathrm{f}}), \end{aligned} \tag{6.65}$$

where the linear maps $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy by (6.63) the bounds

$$\begin{aligned}|\mathcal{H}_1(\beta_{\rm b},\beta_{\rm f},\zeta_{\rm f})| &\leq C\left((\varepsilon|\log\varepsilon|+|\lambda|)(|\beta_{\rm f}|+|\zeta_{\rm f}|)+e^{-q/\varepsilon}|\beta_{\rm b}|\right),\\ |\mathcal{H}_2(\beta_{\rm b},\zeta_{\rm b,+},\beta_{\rm f},\zeta_{\rm f})| &\leq C\left(\left(\varepsilon^{\rho(a)}|\log\varepsilon|+|\lambda|\right)(|\beta_{\rm b}|+|\zeta_{\rm b,+}|)+e^{-q/\varepsilon}(|\beta_{\rm f}|+|\zeta_{\rm f}|)\right),\end{aligned}$$

with $q>0$ independent of λ, a and ε. Hence, we can solve system (6.65) for $\zeta_{\rm f}$ and $\zeta_{\rm b,+}$, provided $|\lambda|,\varepsilon>0$ are sufficiently small, and obtain

$$\begin{aligned}\zeta_{\rm f} &= \zeta_{\rm f}(\beta_{\rm b},\beta_{\rm f}), & \zeta_{\rm b,+} &= \zeta_{\rm b,+}(\beta_{\rm b},\beta_{\rm f}),\\ |\zeta_{\rm f}(\beta_{\rm b},\beta_{\rm f})| &\leq C\left((\varepsilon|\log\varepsilon|+|\lambda|)|\beta_{\rm f}|+e^{-q/\varepsilon}|\beta_{\rm b}|\right), & |\zeta_{\rm b,+}(\beta_{\rm b},\beta_{\rm f})| &\leq C\left((\varepsilon^{\rho(a)}|\log\varepsilon|+|\lambda|)|\beta_{\rm b}|+e^{-q/\varepsilon}|\beta_{\rm f}|\right).\end{aligned} \tag{6.66}$$

For the last two matching conditions we substitute (6.66) into the identities (6.26), (6.31), (6.32) and (6.52). Moreover, we estimate the tail of the integral in (6.26), i.e. the part from $-\infty$ to $-L_\varepsilon$, using that the exponential dichotomy of (6.14) on $\mathbb{R}_-$ has exponent μ by Corollary 6.7 and it holds $\nu \geq \mu/2$. Thus, we obtain the last two matching conditions by pairing with $\Psi_{\rm f,\perp} \in \ker(Q^s_{\rm f,+}(0)^*)$ and $\Psi_{\rm b,\perp} \in \ker(Q^s_{\rm b,+}(0)^*)$

$$0=\left\langle \Psi_{\rm f,\perp}, \psi_{\rm f,-}(0,\lambda)-\psi^{\rm sl}(0,\lambda)\right\rangle = \beta_{\rm f}\int_{-L_\varepsilon}^{L_\varepsilon}\langle T_{\rm f}(0,\xi)^*\Psi_{\rm f,\perp}, B_{\rm f}(\xi,\lambda)\omega_{\rm f}(\xi)\rangle\, d\xi + \mathcal{H}_3(\beta_{\rm b},\beta_{\rm f}), \tag{6.67}$$

$$0=\left\langle \Psi_{\rm b,\perp}, \psi^{\rm sl}(Z_{a,\varepsilon},\lambda)-\psi_{\rm b,+}(Z_{a,\varepsilon},\lambda)\right\rangle = \beta_{\rm b}\int_{-L_\varepsilon}^{L_\varepsilon}\langle T_{\rm b}(0,\xi)^*\Psi_{\rm b,\perp}, B_{\rm b}(\xi,\lambda)\omega_{\rm b}(\xi)\rangle\, d\xi + \mathcal{H}_4(\beta_{\rm b},\beta_{\rm f}), \tag{6.68}$$

where the linear maps $\mathcal{H}_3$ and $\mathcal{H}_4$ satisfy the bounds

$$\begin{aligned}|\mathcal{H}_3(\beta_{\rm b},\beta_{\rm f})| &\leq C\left((\varepsilon|\log\varepsilon|+|\lambda|)^2|\beta_{\rm f}|+e^{-q/\varepsilon}|\beta_{\rm b}|\right),\\ |\mathcal{H}_4(\beta_{\rm b},\beta_{\rm f})| &\leq C\left(\left(\varepsilon^{\rho(a)}|\log\varepsilon|+|\lambda|\right)^2|\beta_{\rm b}|+e^{-q/\varepsilon}|\beta_{\rm f}|\right).\end{aligned}$$

The same procedure can be done using the expressions (6.27), (6.33) and (6.53) instead. We approximate a-uniformly

$$\begin{aligned}0 &= \left\langle \Psi_{\rm f,\perp}, \phi_{a,\varepsilon}'(0)-\phi_{a,\varepsilon}'(0)\right\rangle = \left\langle \Psi_{\rm f,\perp}, Q^s_{\rm f,-}(0)\phi_{a,\varepsilon}'(0)-Q^u_{\rm f,+}(0)\phi_{a,\varepsilon}'(0)\right\rangle\\ &= \int_{-L_\varepsilon}^{L_\varepsilon}\left\langle e^{-\xi\eta}T_{\rm f}(0,\xi)^*\Psi_{\rm f,\perp}, B_{\rm f}(\xi,0)\phi_{a,\varepsilon}'(\xi)\right\rangle d\xi + O\left(\varepsilon^2\right),\end{aligned} \tag{6.69}$$

$$\begin{aligned}0 &= \left\langle \Psi_{\rm b,\perp}, \phi_{a,\varepsilon}'(0)-\phi_{a,\varepsilon}'(0)\right\rangle = \left\langle \Psi_{\rm b,\perp}, Q^s_{\rm b,-}(0)\phi_{a,\varepsilon}'(0)-Q^u_{\rm b,+}(0)\phi_{a,\varepsilon}'(0)\right\rangle\\ &= \int_{-L_\varepsilon}^{L_\varepsilon}\left\langle e^{-\xi\eta}T_{\rm b}(0,\xi)^*\Psi_{\rm f,\perp}, B_{\rm b}(\xi,0)\phi_{a,\varepsilon}'(Z_{a,\varepsilon}+\xi)\right\rangle d\xi + \left\langle e^{\eta L_\varepsilon}T_{\rm b}(0,-L_\varepsilon)^*\Psi_{\rm b,\perp}, \phi_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\right\rangle + O\left(\varepsilon^2\right),\end{aligned} \tag{6.70}$$

using $\nu \geq \mu/2 \geq \eta/2$ (see (6.5) and Lemma 6.3).

Our plan is to use the identities (6.69) and (6.70) to simplify the expressions in (6.67) and (6.68). First, we calculate

$$e^{-\eta\xi}T_j(0,\xi)^*\Psi_{j,\perp} = \begin{pmatrix} e^{-\eta\xi}\varphi_{j,\rm ad}(\xi) \\ -\displaystyle\int_\infty^\xi e^{-\eta\hat\xi}\left\langle\varphi_{j,\rm ad}(\hat\xi),F\right\rangle d\hat\xi\end{pmatrix} = \begin{pmatrix} e^{-\check c_0\xi}v_j'(\xi)\\ -e^{-\check c_0\xi}u_j'(\xi)\\ \displaystyle\int_\infty^\xi e^{-\check c_0\hat\xi}u_j'(\hat\xi)d\hat\xi\end{pmatrix}, \quad \xi\in\mathbb{R}, \quad j={\rm f},{\rm b}, \tag{6.71}$$

where $(u_j(\xi),v_j(\xi))=\phi_j(\xi)$. Recall that the front $\phi_{\rm f}$ is a heteroclinic connection between the fixed points $(0,0)$ and $(1,0)$ of the Nagumo system (3.4). By looking at the linearization of (3.4) about $(0,0)$ and $(1,0)$ we deduce that $\phi_{\rm f}'(\xi)$ converges to 0 at an exponential rate $\frac{1}{2}\sqrt{2}$ as $\xi\to\pm\infty$. The same holds for $\phi_{\rm b}'(\xi)$ by symmetry. Recall that $\check c_0$ is given by $\sqrt{2}(\frac{1}{2}-a)$. So, for all $a\geq 0$, the upper two entries of (6.71) are bounded on $\mathbb{R}$ by some constant $C>0$, independent of a, whereas the last entry is bounded by $C|\log\varepsilon|$ on $[-L_\varepsilon,L_\varepsilon]$. Further, by (6.63) the upper two rows of $B_{\rm f}(\xi,0)$ are bounded by $C\varepsilon|\log\varepsilon|$ on $[-L_\varepsilon,L_\varepsilon]$, whereas the last row is bounded by $C\varepsilon$ as can be observed from (6.15). Combining these bounds with

$\nu \geq 2\sqrt{2}$, (6.64) and (6.69) we approximate a-uniformly

$$\begin{aligned}\int_{-L_\varepsilon}^{L_\varepsilon} \langle T_{\mathrm{f}}(0,\xi)^*\Psi_{\mathrm{f},\perp}, B_{\mathrm{f}}(\xi,\lambda)\omega_{\mathrm{f}}(\xi)\rangle\, d\xi &= \int_{-L_\varepsilon}^{L_\varepsilon} \left\langle e^{-\xi\eta} T_{\mathrm{f}}(0,\xi)^*\Psi_{\mathrm{f},\perp}, B_{\mathrm{f}}(\xi,0)\phi_{a,\varepsilon}'(\xi)\right\rangle d\xi \\ &\quad - \lambda \int_{-L_\varepsilon}^{L_\varepsilon} e^{-\check{c}_0\xi}\left(u_{\mathrm{f}}'(\xi)\right)^2 d\xi + O\left(|\varepsilon \log\varepsilon|^2\right) \\ &= -\lambda \int_{-\infty}^{\infty} e^{-\check{c}_0\xi}\left(u_{\mathrm{f}}'(\xi)\right)^2 d\xi + O\left(|\varepsilon\log\varepsilon|^2\right).\end{aligned} \tag{6.72}$$

Similarly, we estimate a-uniformly

$$\begin{aligned}\int_{-L_\varepsilon}^{L_\varepsilon} \langle T_{\mathrm{b}}(0,\xi)^*\Psi_{\mathrm{b},\perp}, B_{\mathrm{b}}(\xi,\lambda)\omega_{\mathrm{b}}(\xi)\rangle\, d\xi &= -\left\langle e^{\eta L_\varepsilon} T_{\mathrm{b}}(0,-L_\varepsilon)^*\Psi_{\mathrm{b},\perp}, \phi_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\right\rangle \\ &\quad - \lambda\int_{-\infty}^{\infty} e^{-\check{c}_0\xi}\left(u_{\mathrm{b}}'(\xi)\right)^2 d\xi + O\left(|\varepsilon^{\rho(a)}\log\varepsilon|^2\right),\end{aligned} \tag{6.73}$$

using (6.70) instead of (6.69). Substituting identities (6.72) and (6.73) into the remaining matching conditions (6.67) and (6.68) we arrive at the linear system

$$\begin{pmatrix} -\lambda \mathcal{M}_{\mathrm{f}} + O\left((\varepsilon|\log\varepsilon| + |\lambda|)^2\right) & O(e^{-q/\varepsilon}) \\ O(e^{-q/\varepsilon}) & -\lambda \mathcal{M}_{\mathrm{b},1} - \mathcal{M}_{\mathrm{b},2} + O\left((\varepsilon^{\rho(a)}|\log\varepsilon| + |\lambda|)^2\right)\end{pmatrix}\begin{pmatrix}\beta_{\mathrm{f}} \\ \beta_{\mathrm{b}}\end{pmatrix} = 0, \tag{6.74}$$

where the approximations are a-uniformly and with $\mathcal{M}_{\mathrm{b},1}$ and $\mathcal{M}_{\mathrm{b},2}$ as in (6.61) and

$$\mathcal{M}_f := \int_{-\infty}^{\infty}\left(u_{\mathrm{f}}'(\xi)\right)^2 e^{-\check{c}_0\xi} d\xi > 0. \tag{6.75}$$

Thus, any nontrivial solution $(\beta_{\mathrm{b}}, \beta_{\mathrm{f}})$ to (6.74) corresponds to an eigenfunction of (6.6).

Since the perturbation matrices $B_j(\xi,\lambda)$, $j = \mathrm{f},\mathrm{b}$, the evolution $\mathcal{T}(\xi,\hat{\xi},\lambda)$ of system (6.6) and the projections $Q^{u,s}_{r,\ell}(\xi,\lambda)$ associated with the exponential dichotomy of (6.6) established in Proposition 6.5 are analytic in λ, all quantities occurring in this section are analytic in λ. Thus, the matrix in (6.74) and its determinant $D(\lambda) = D(\lambda; a, \varepsilon)$ are analytic in λ.

Observe that the ε-independent quantities $\mathcal{M}_f$ and $\mathcal{M}_{\mathrm{b},1}$ are to leading order bounded away from 0, i.e. it holds $1/C \leq \mathcal{M}_f, \mathcal{M}_{\mathrm{b},1} \leq C$, since $u_j'(\xi)$ converges to 0 as $\xi \to \pm\infty$ at an exponential rate $\frac{1}{2}\sqrt{2}$; see also (3.6). Second, we estimate a-uniformly $\mathcal{M}_{\mathrm{b},2} = O(\varepsilon^{\rho(a)}|\log\varepsilon|)$ by combining (6.63) and (6.70). Hence, provided $\delta, \varepsilon > 0$ are sufficiently small, we have for $\lambda \in \partial R_1(\delta) = \{\lambda \in \mathbb{C} : |\lambda| = \delta\}$

$$|D(\lambda) - \lambda \mathcal{M}_f(\lambda \mathcal{M}_{\mathrm{b},1} + \mathcal{M}_{\mathrm{b},2})| < |\lambda \mathcal{M}_f(\lambda \mathcal{M}_{\mathrm{b},1} + \mathcal{M}_{\mathrm{b},2})|.$$

By Rouché's Theorem $D(\lambda)$ has in $R_1(\delta)$ precisely two roots λ_0, λ_1 that are a-uniformly $O(|\varepsilon^{\rho(a)}\log\varepsilon|^2)$-close to the roots of the quadratic $\lambda\mathcal{M}_f(\lambda\mathcal{M}_{\mathrm{b},1} + \mathcal{M}_{\mathrm{b},2})$ given by 0 and $-\mathcal{M}_{\mathrm{b},2}\mathcal{M}_{\mathrm{b},1}^{-1}$. We conclude that (6.6) has two eigenvalues λ_0, λ_1 in the region R_1.

We are interested in an eigenfunction $\psi_1(\xi)$ of (6.6) corresponding to the eigenvalue λ_1 that is a-uniformly $O(|\varepsilon^{\rho(a)}\log\varepsilon|^2)$-close to $-\mathcal{M}_{\mathrm{b},2}\mathcal{M}_{\mathrm{b},1}^{-1}$. The associated solution to (6.74) is given by the eigenvector $(\beta_{\mathrm{f}}, \beta_{\mathrm{b}}) = \left(O(e^{-q/\varepsilon}), 1\right)$. In the proofs of Propositions 6.8, 6.9 and 6.10 we established a piecewise continuous eigenfunction to (6.6) for any prospective eigenvalue $\lambda \in R_1$. Thus, the eigenfunction $\psi_1(\xi)$ to (6.6), corresponding to the eigenvalue λ_1, satisfies (6.29) on $I_{\mathrm{f},-}$, (6.37) on $I_{\mathrm{f},+}$, (6.40) on I_r, (6.43) on $I_{\mathrm{b},-}$, (6.54) on $I_{\mathrm{b},+}$ and (6.58) on I_ℓ. The variables occurring in these six expressions can all be expressed in $\beta_{\mathrm{f}} = O(e^{-q/\varepsilon})$ and $\beta_{\mathrm{b}} = 1$. This leads to the approximation (6.62) of $\psi_1(\xi)$.

By translational invariance we know a priori that $e^{-\eta\xi}\phi_{a,\varepsilon}'(\xi)$ is an eigenfunction of (6.6) at $\lambda = 0$. Therefore, $\lambda = 0$ is one of the two eigenvalues $\lambda_0, \lambda_1 \in R_1$ of (6.6). With the aid of the bounds (6.62) one observes that the eigenfunction $\psi_1(\xi)$ is not a multiple of $e^{-\eta\xi}\phi_{a,\varepsilon}'(\xi)$. On the other hand, the space of exponentially decaying solutions in backward time to (6.6) is one-dimensional, because the asymptotic matrix $\hat{A}(0,\lambda,a,\varepsilon)$ of system (6.6) has precisely one eigenvalue of positive real part by Lemma 6.3. Hence, the eigenfunctions $\psi_1(\xi)$ and $e^{-\eta\xi}\phi_{a,\varepsilon}'(\xi)$ must correspond to different eigenvalues. We conclude $\lambda_0 = 0$ and $\lambda_1 \neq \lambda_0$. □

Remark 6.12. In the proof of Theorem 6.11 we simplified the final matching equation by using that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an exponentially localized solution to (6.6) at $\lambda = 0$. More precisely, during the matching procedure we substituted the expressions

$$\int_{-L_\varepsilon}^{L_\varepsilon} \left\langle T_j(0,\xi)^*\Psi_{j,\perp}, B_j(\xi,0)\omega_j(\xi)\right\rangle d\xi, \quad j = \mathrm{f}, \mathrm{b}, \tag{6.76}$$

by (6.72) and (6.73). Alternatively, one could try to calculate (6.76) directly using (6.71). The most problematic term is the difference $f'(u_{a,\varepsilon}(\xi)) - f'(u_j(\xi))$ in $B_j(\xi,0)$. This difference can be calculated using an identity of the form

$$(\partial_\xi - C_j(\xi))\left[e^{-\eta\xi}\begin{pmatrix} u'_{a,\varepsilon}(\xi) \\ v'_{a,\varepsilon}(\xi)\end{pmatrix}\right] = e^{-\eta\xi}\begin{pmatrix} 0 \\ (c(\varepsilon)-c(0))v'_{a,\varepsilon}(\xi) - (f'(u_{a,\varepsilon}(\xi)) - f'(u_j(\xi)))u'_{a,\varepsilon}(\xi) + w'_{a,\varepsilon}(\xi)\end{pmatrix}, \quad j = \mathrm{f}, \mathrm{b},$$

where C_j is the coefficient matrix of (6.18). The equivalent of the latter is done in [16] in the context of the lattice Fitzhugh-Nagumo equations.

Remark 6.13. The proof of Theorem 6.11 shows that any eigenfunction of problem (6.6) corresponds to an eigenvector $(\beta_\mathrm{f}, \beta_\mathrm{b})$ of (6.74). Such an eigenfunction is obtained by pasting together the eigenfunctions $\omega_\mathrm{f}(\xi)$ and $\omega_\mathrm{b}(\xi)$ to the reduced eigenvalue problems (6.14) with amplitudes β_f and β_b, respectively.

The eigenvector $(\beta_\mathrm{f}, \beta_\mathrm{b}) = \left(1, O(e^{-q/\varepsilon})\right)$ of (6.74) corresponds to the eigenfunction $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ of (6.6) at $\lambda = 0$. Indeed, this eigenfunction is centered at the front and close to $\omega_\mathrm{f}(\xi)$. Switching back to the unshifted eigenvalue problem (2.3), we observe that the corresponding eigenfunction $\phi'_{a,\varepsilon}(\xi)$ to (2.3) is close to a concatenation of $\omega_\mathrm{f}(\xi)$ and $\omega_\mathrm{b}(\xi)$; see also Theorem 4.3.

The other eigenvector $(\beta_\mathrm{f}, \beta_\mathrm{b}) = \left(O(e^{-q/\varepsilon}), 1\right)$ of (6.74) corresponds to the eigenfunction $\psi_1(\xi)$ of (6.6) at $\lambda = \lambda_1$. The eigenfunction $\psi_1(\xi)$ is centered at the back and close to $\omega_\mathrm{b}(\xi)$; see also estimate (6.62). When λ_1 lies to the right of the essential spectrum of $\mathcal{L}_{a,\varepsilon}$, it is also an eigenvalue of the unshifted eigenvalue problem (2.3) by Proposition 6.4. An eigenfunction of (2.3) corresponding to this potential second eigenvalue λ_1 is given by $\tilde{\psi}_1(\xi) := e^{\eta(\xi - Z_{a,\varepsilon})}\psi_1(\xi)$. Using the estimate (6.62) we conclude that $\tilde{\psi}_1(\xi)$ is centered at the back and the left slow manifold and close to $\omega_\mathrm{b}(\xi)$ along the back, i.e. it holds

$$\begin{aligned} \|\tilde{\psi}_1(\xi)\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon| e^{-\eta(Z_{a,\varepsilon}-\xi)}, \quad \xi \in (-\infty, Z_{a,\varepsilon} - L_\varepsilon], \\ \|\tilde{\psi}_1(\xi + Z_{a,\varepsilon}) - \omega_\mathrm{b}(\xi)\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon| e^{\eta\xi}, \quad \xi \in [-L_\varepsilon, L_\varepsilon]. \end{aligned}$$

We emphasize that in contrast to the shifted eigenvalue problem, we do not obtain that the eigenfunction $\tilde{\psi}_1(\xi)$ is small along the left slow manifold, i.e. for $\xi \in I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$. This observation agrees with the figures obtained numerically in §8.

6.4 The translational eigenvalue is simple

In this section we prove that $\lambda_0 = 0$ is a simple eigenvalue of $\mathcal{L}_{a,\varepsilon}$. This is an essential ingredient to establish nonlinear stability of the traveling pulse $\tilde{\phi}_{a,\varepsilon}(\xi)$; see [7, 8] and Theorem 2.3. By Theorem 6.11 λ_0 has geometric multiplicity one. To prove that λ_0 also has algebraic multiplicity one we consider the associated shifted generalized eigenvalue problem at $\lambda = \lambda_0$. Particular solutions to this inhomogeneous problem are given by the λ-derivatives of solutions $\psi(\xi, \lambda)$ to the shifted eigenvalue problem (6.6). By differentiating the exit and entry conditions at $\xi = 0$ and at $\xi = Z_{a,\varepsilon}$ established in Propositions 6.8, 6.9 and 6.10 we obtain exit and entry conditions for exponentially localized solutions to the generalized eigenvalue problem. Matching of these expression leads to a contradiction showing that λ_0 also has algebraic multiplicity one.

Proposition 6.14. *In the setting of Theorem 2.1, let $\tilde{\phi}_{a,\varepsilon}(\xi)$ denote a traveling-pulse solution to* (2.2) *with associated linear operator $\mathcal{L}_{a,\varepsilon}$. The translational eigenvalue $\lambda_0 = 0$ of $\mathcal{L}_{a,\varepsilon}$ is simple.*

Proof. By Theorem 6.11 the eigenspace of the shifted eigenvalue problem (6.6) at $\lambda = \lambda_0$ is spanned by the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$. Translating back to the original system (2.3) we deduce $\ker(\mathcal{L}_{a,\varepsilon})$ is one-dimensional and spanned by $\tilde{\phi}'_{a,\varepsilon}(\xi)$. So the geometric multiplicity of λ_0 equals one. Regarding the algebraic multiplicity of the eigenvalue λ_0 we are interested in exponentially localized solutions $\tilde{\psi}$ to the generalized eigenvalue problem $\mathcal{L}_{a,\varepsilon}\tilde{\psi} = \tilde{\phi}'_{a,\varepsilon}(\xi)$. This problem can be represented by the inhomogeneous ODE

$$\check{\psi}_\xi = A_0(\xi,0)\check{\psi} + [\partial_\lambda A_0](\xi,0)\phi'_{a,\varepsilon}(\xi), \tag{6.77}$$

where $A_0(\xi,\lambda)$ is the coefficient matrix of (2.3). The asymptotic matrices of (2.3) and the shifted version (6.6) have precisely one eigenvalue of positive real part at $\lambda = 0$ by Proposition 5.1 and Lemma 6.3. Moreover, the weighted derivative $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is exponentially localized. Therefore, $\check{\psi}(\xi)$ is an exponentially localized solution to (6.77) if and only if $\psi(\xi) = e^{-\eta\xi}\check{\psi}(\xi)$ is an exponentially localized solution to

$$\psi_\xi = A(\xi,0)\psi + e^{-\eta\xi}[\partial_\lambda A](\xi,0)\phi'_{a,\varepsilon}(\xi), \tag{6.78}$$

where $A(\xi,\lambda)$ is the coefficient matrix of the shifted eigenvalue problem (6.6).

Since $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is an exponentially localized solution to (6.6) at $\lambda = 0$, there exists by Propositions 6.8, 6.9 and 6.10 solutions $\psi_{\mathrm{f},-}(\xi,\lambda)$, $\psi^{\mathrm{sl}}(\xi,\lambda)$ and $\psi_{\mathrm{b},+}(\xi,\lambda)$ to (6.6), which are analytic in λ and satisfy (6.26), (6.31), (6.32) and (6.52) for some $\beta_{\mathrm{f},-},\beta_{\mathrm{f}},\zeta_{\mathrm{f}},\beta_{\mathrm{b}},\beta_{\mathrm{b},+},\zeta_{\mathrm{b},+} \in \mathbb{C}$, such that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ equals $\psi_{\mathrm{f},-}(\xi,0)$ on $(-\infty,0]$, $\psi^{\mathrm{sl}}(\xi,0)$ on $[0,Z_{a,\varepsilon}]$ and $\psi_{\mathrm{b},+}(\xi,0)$ on $[Z_{a,\varepsilon},\infty)$. As in the proof of Theorem 6.11 we match $\psi_{\mathrm{f},-}(0,0)$ to $\psi^{\mathrm{sl}}(0,0)$ and $\psi^{\mathrm{sl}}(Z_{a,\varepsilon},0)$ to $\psi_{\mathrm{b},-}(Z_{a,\varepsilon},0)$. Applying the projections $Q^u_{j,-}(0)$, $j = \mathrm{f},\mathrm{b}$ to the differences $\psi_{\mathrm{f},-}(0,0) - \psi^{\mathrm{sl}}(0,0)$ and $\psi^{\mathrm{sl}}(Z_{a,\varepsilon},0) - \psi_{\mathrm{b},-}(Z_{a,\varepsilon},0)$ yields $\beta_{\mathrm{f},-} = \beta_{\mathrm{f}}$ and $\beta_{\mathrm{b}} = \beta_{\mathrm{b},+}$. Taking the inner products $0 = \langle \Psi_2, \psi_{\mathrm{f},-}(0,0) - \psi^{\mathrm{sl}}(0,0)\rangle$ and $0 = \langle \Psi_2, \psi^{\mathrm{sl}}(Z_{a,\varepsilon},0) - \psi_{\mathrm{b},-}(Z_{a,\varepsilon},0)\rangle$ we obtain that ζ_{f} and $\zeta_{\mathrm{b},+}$ can be expressed in β_{b} and β_{f} as

$$\begin{aligned} &\zeta_{\mathrm{f}} = \zeta_{\mathrm{f}}(\beta_{\mathrm{b}},\beta_{\mathrm{f}}), && \zeta_{\mathrm{b},+} = \zeta_{\mathrm{b},+}(\beta_{\mathrm{b}},\beta_{\mathrm{f}}), \\ &|\zeta_{\mathrm{f}}(\beta_{\mathrm{b}},\beta_{\mathrm{f}})| \leq C\left(\varepsilon|\log\varepsilon||\beta_{\mathrm{f}}| + e^{-q/\varepsilon}|\beta_{\mathrm{b}}|\right), && |\zeta_{\mathrm{b},+}(\beta_{\mathrm{b}},\beta_{\mathrm{f}})| \leq C\left(\varepsilon^{2/3}|\log\varepsilon||\beta_{\mathrm{b}}| + e^{-q/\varepsilon}|\beta_{\mathrm{f}}|\right), \end{aligned} \tag{6.79}$$

where $C > 0$ is independent of a and ε.

Observe that the derivatives $[\partial_\lambda\psi_{\mathrm{f},-}](\xi,0)$, $[\partial_\lambda\psi^{\mathrm{sl}}](\xi,0)$ and $[\partial_\lambda\psi_{\mathrm{b},+}](\xi,0)$ are particular solutions to the equation (6.78) on $(-\infty,0]$, $[0,Z_{a,\varepsilon}]$ and $[Z_{a,\varepsilon},\infty)$, respectively. Moreover, $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ spans the space of exponentially localized solutions to the homogeneous problem (6.6) associated to (6.78). Now suppose that $\psi(\xi)$ is an exponentially localized solution to (6.78). By the previous two observations it holds

$$\begin{aligned} \psi(\xi) &= [\partial_\lambda\psi_{\mathrm{f},-}](\xi,0) + \alpha_1 e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi), \quad \xi \in (-\infty,0], \\ \psi(\xi) &= [\partial_\lambda\psi^{\mathrm{sl}}](\xi,0) + \alpha_2 e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi), \quad \xi \in [0,Z_{a,\varepsilon}], \\ \psi(\xi) &= [\partial_\lambda\psi_{\mathrm{b},+}](\xi,0) + \alpha_3 e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi), \quad \xi \in [Z_{a,\varepsilon},\infty), \end{aligned} \tag{6.80}$$

for some $\alpha_{1,2,3} \in \mathbb{C}$. We differentiate the analytic expressions (6.26) and (6.31) with respect to λ and obtain by the Cauchy estimates and (6.79)

$$\begin{aligned} [\partial_\lambda\psi_{\mathrm{f},-}](\xi,0) &= \beta_{\mathrm{f}}\int_{-\infty}^0 T^s_{\mathrm{f},-}(0,\hat{\xi})\tilde{B}\omega_{\mathrm{f}}(\hat{\xi})d\hat{\xi} + \mathcal{H}_1(\beta_{\mathrm{f}}), \quad \|\mathcal{H}_1(\beta_{\mathrm{f}})\| \leq C\varepsilon|\log\varepsilon||\beta_{\mathrm{f},-}|, \\ [\partial_\lambda\psi^{\mathrm{sl}}](\xi,0) &= \beta_{\mathrm{f}}\int_{L_\varepsilon}^0 T^u_{\mathrm{f},+}(0,\hat{\xi})\tilde{B}\omega_{\mathrm{f}}(\hat{\xi})d\hat{\xi} + \mathcal{H}_2(\beta_{\mathrm{f}},\beta_{\mathrm{b}}), \quad \|\mathcal{H}_2(\beta_{\mathrm{f}},\beta_{\mathrm{b}})\| \leq C\left(\varepsilon|\log\varepsilon||\beta_{\mathrm{f}}| + e^{-q/\varepsilon}|\beta_{\mathrm{b}}|\right), \end{aligned} \tag{6.81}$$

where ω_{f} is as in (6.22), $\mathcal{H}_{1,2}$ are linear maps and $\tilde{B}$ denotes the derivative of the perturbation matrix

$$\tilde{B} = \tilde{B}(a,\varepsilon) := [\partial_\lambda]B_{\mathrm{f}}(\xi,\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} \end{pmatrix}.$$

On the other hand, we estimate using Theorem 4.3 (i)

$$\left\|\Psi_{\mathrm{f},a,\varepsilon} - \Psi_{1,\mathrm{f}}\right\| \leq C\varepsilon|\log\varepsilon|, \qquad \text{where } \Psi_{\mathrm{f},a,\varepsilon} := \begin{pmatrix} v'_{a,\varepsilon}(0) \\ -u'_{a,\varepsilon}(0) \\ 0 \end{pmatrix}, \tag{6.82}$$

and $\Psi_{1,\mathrm{f}}$ is defined in (6.25). Note that $\Psi_{\mathrm{f},a,\varepsilon}$ is perpendicular to the derivative $\phi'_{a,\varepsilon}(0)$. As in the proof of Theorem 6.11 note that the front $\phi'_{\mathrm{f}}(\xi) = (u'_{\mathrm{f}}(\xi), v'_{\mathrm{f}}(\xi))$ decays to 0 as $\xi \to \pm\infty$ with an exponential rate $\frac{1}{2}\sqrt{2}$. Thus, we calculate using $\nu \geq 2\sqrt{2}$, (6.80), (6.81) and (6.82)

$$\begin{aligned} 0 &= \left\langle \Psi_{\mathrm{f},a,\varepsilon}, [\partial_\lambda \psi_{\mathrm{f},-}](0,0) - [\partial_\lambda \psi^{\mathrm{sl}}](0,0) + (\alpha_1 - \alpha_2)\phi'_{a,\varepsilon}(0)\right\rangle \\ &= \beta_{\mathrm{f}}\left(\int_{-\infty}^{L_\varepsilon} \left\langle T_{\mathrm{f}}(0,\xi)^*\Psi_{1,\mathrm{f}}, \tilde{B}\omega_{\mathrm{f}}(\xi)\right\rangle d\xi + O(\varepsilon|\log\varepsilon|)\right) + \beta_{\mathrm{b}} O\left(e^{-q/\varepsilon}\right) \\ &= \beta_{\mathrm{f}}\left(-M_{\mathrm{f}} + O(\varepsilon|\log\varepsilon|)\right) + \beta_{\mathrm{b}} O\left(e^{-q/\varepsilon}\right), \end{aligned} \tag{6.83}$$

a-uniformly, where M_{f} is defined in (6.75). Let $\Psi_{\mathrm{b},a,\varepsilon} = (v'_{a,\varepsilon}(Z_{a,\varepsilon}), -u'_{a,\varepsilon}(Z_{a,\varepsilon}), 0)$. A similar calculation shows

$$\begin{aligned} 0 &= \left\langle \Psi_{\mathrm{b},a,\varepsilon}, [\partial_\lambda \psi^{\mathrm{sl}}](Z_{a,\varepsilon},0) - [\partial_\lambda \psi_{\mathrm{b},+}](Z_{a,\varepsilon},0) + (\alpha_2 - \alpha_3)e^{-\eta Z_{a,\varepsilon}}\phi'_{a,\varepsilon}(Z_{a,\varepsilon})\right\rangle \\ &= \beta_{\mathrm{b}}\left(-M_{\mathrm{b},1} + O(\varepsilon^{2/3}|\log\varepsilon|)\right) + \beta_{\mathrm{f}} O\left(e^{-q/\varepsilon}\right), \end{aligned} \tag{6.84}$$

a-uniformly, where $M_{\mathrm{b},1}$ is defined in (6.61). The conditions (6.83) and (6.84) form a system of linear equations in β_{f} and β_{b}. The only solution to this system is $\beta_{\mathrm{f}} = \beta_{\mathrm{b}} = 0$, because $M_{\mathrm{f}}, M_{\mathrm{b},1} > 0$ are independent of ε and bounded below away from 0 uniformly in a. This is a contradiction with the fact that $e^{-\eta\xi}\phi'_{a,\varepsilon}(\xi)$ is not the zero solution to (6.6). We conclude that (6.78) has no exponentially localized solution and that also the algebraic multiplicity of the eigenvalue $\lambda = 0$ of $\mathcal{L}_{a,\varepsilon}$ equals one. □

6.5 Calculation of second eigenvalue

By Theorem 6.11 the second eigenvalue $\lambda_1 \in R_1$ of (6.6) is a-uniformly $O(|\varepsilon^{\rho(a)}\log\varepsilon|^2)$-close to the quotient $-M_{\mathrm{b},2}M_{\mathrm{b},1}^{-1}$. Thus, to prove our main stability results 2.2, we need to show $-M_{\mathrm{b},2}M_{\mathrm{b},1}^{-1} \leq -\varepsilon b_0$, where b_0 is independent of a and ε. Since $M_{\mathrm{b},1} > 0$ is independent of ε and bounded by an a-independent constant, the problem amounts to proving that $M_{\mathrm{b},2}$ is bounded below by $\varepsilon\tilde{b}_0$ for some $\tilde{b}_0 > 0$. We distinguish between the hyperbolic and nonhyperbolic regime.

In the hyperbolic regime, it is possible to determine the quantity $M_{\mathrm{b},2}$ to leading order. This relies on the fact that the solution $\varphi_{\mathrm{b,ad}}(\xi)$, defined in (6.20), to the adjoint system (6.19) converges exponentially to 0 as $\xi \to -\infty$ with rate $\sqrt{2}a$. Since a is bounded below in the hyperbolic regime, the first two coordinates of Ψ_*, defined in (6.61), are of higher order by choosing ν sufficiently large.

Therefore, the calculation for $M_{\mathrm{b},2}$ reduces to approximating the product $w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)\int_{-\infty}^{L_\varepsilon} u'_{\mathrm{b}}(\xi)e^{-\check{c}_0\xi}d\xi$. This leads to the following result.

Proposition 6.15. *For each $a_0 > 0$ there exists $\varepsilon_0 > 0$ such that for each $(a,\varepsilon) \in [a_0, \frac{1}{2} - \kappa] \times (0,\varepsilon_0)$ the quantity $M_{\mathrm{b},2}$ in Theorem 6.11 is approximated (a-uniformly) by*

$$M_{\mathrm{b},2} = \frac{\varepsilon}{\check{c}_0}\left(\gamma w_{\mathrm{b}}^1 - u_{\mathrm{b}}^1\right)\int_{-\infty}^{\infty} u'_{\mathrm{b}}(\xi)e^{-\check{c}_0\xi}d\xi + O\left(\varepsilon^2|\log\varepsilon|\right), \tag{6.85}$$

In particular, we have $M_{\mathrm{b},2} > \varepsilon/k_0$ for some $k_0 > 1$, independent of a and ε.

Proof. The Nagumo back solution $\phi_{\mathrm{b}}(\xi)$ to system (3.5) converges to the fixed point $p_{\mathrm{b}}^1 = (u_{\mathrm{b}}^1, 0)$ as $\xi \to -\infty$. By looking at the linearization of (3.5) about p_{b}^1 we deduce that the convergence of $\phi_{\mathrm{b}}(\xi)$ to p_{b}^1 is exponential at a rate $\frac{1}{2}\sqrt{2}$. Combining this with Theorem 4.3 (ii), $\nu \geq 2\sqrt{2}$ and $\check{c} - \check{c}_0 = O(\varepsilon)$ we estimate

$$w'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) = \frac{\varepsilon}{\check{c}}\left(u_{a,\varepsilon}(-L_\varepsilon) - \gamma w_{a,\varepsilon}(-L_\varepsilon)\right) = \frac{\varepsilon}{\check{c}_0}\left(u_{\mathrm{b}}^1 - \gamma w_{\mathrm{b}}^1\right) + O\left(\varepsilon^2|\log\varepsilon|\right).$$

In addition, the derivative $\phi_{\rm b}'(\xi)$ converges exponentially to 0 at a rate $\frac{1}{2}\sqrt{2}$ as $\xi \to -\infty$. Finally, recall that $\check{c}_0(a) = \sqrt{2}(\frac{1}{2} - a)$. Using all the previous observations, we estimate

$$M_{\rm b,2} = \left\langle \begin{pmatrix} e^{\check{c}_0 L_\varepsilon} v_{\rm b}'(-L_\varepsilon) \\ -e^{\check{c}_0 L_\varepsilon} u_{\rm b}'(-L_\varepsilon) \\ \displaystyle\int_\infty^{-L_\varepsilon} e^{-\check{c}_0 \hat{\xi}} u_{\rm b}'(\hat{\xi}) d\hat{\xi} \end{pmatrix}, \begin{pmatrix} u_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \\ v_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \\ w_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle = -\frac{\varepsilon}{\check{c}_0}\left(u_{\rm b}^1 - \gamma w_{\rm b}^1\right) \int_{-\infty}^{\infty} u_{\rm b}'(\xi) e^{-\check{c}_0 \xi} d\xi + O\left(\varepsilon^2 |\log \varepsilon|, \varepsilon^{\sqrt{2} a_0 \nu}\right).$$

Without loss of generality we may assume $\nu \geq \sqrt{2}/a_0$. Thus, we take $\nu \geq \max\{2\sqrt{2}, \sqrt{2}/a_0, 2/\mu\} > 0$ (see (6.5)). With this choice of ν the approximation result follows. Since we have $0 < \gamma < 4$, the line $w = \gamma^{-1} u$ intersects the cubic $w = f(u)$ only at $u = 0$. So, it holds $u_{\rm b}^1 - \gamma w_{\rm b}^1 > 0$. Moreover, we have $u_{\rm b}'(\xi) = v_{\rm b}(\xi) < 0$ for all $x \in \mathbb{R}$. Combing these two items, it follows $M_{\rm b,2} > \varepsilon/k_0$. □

Recall that the solution $\varphi_{\rm b,ad}(\xi)$, defined in (6.20), to the adjoint system (6.19) converges exponentially to 0 as $\xi \to -\infty$ with rate $\sqrt{2}a$. Thus, in the nonhyperbolic regime $0 < a \ll 1$, the first two coordinates of Ψ_*, defined in (6.61), are no longer of higher-order, as was the case in the hyperbolic regime. Therefore, in addition to the product $w_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \int_{-\infty}^{L_\varepsilon} u_{\rm b}'(\xi) e^{-\check{c}_0 \xi} d\xi$, we also have to bound the inner product

$$\left\langle \begin{pmatrix} \varphi_{\rm b,ad}(-L_\varepsilon) \\ 0 \end{pmatrix}, \phi_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \right\rangle, \tag{6.86}$$

from below away from 0. Recall from §4.1 that the pulse solution $\phi_{a,\varepsilon}(\xi)$ is at $\xi = Z_{a,\varepsilon} - L_\varepsilon$ in the neighborhood $\mathcal{U}_F$ of the fold point $(u^*, 0, w^*)$, where $u^* = \frac{1}{3}\left(a + 1 + \sqrt{a^2 - a + 1}\right)$ and $w^* = f(u^*)$. In $\mathcal{U}_F$ there exists a coordinate transform $\Phi_\varepsilon \colon \mathcal{U}_F \to \mathbb{R}^3$ bringing system (3.1) into the canonical form (4.1). In system (4.1) the dynamics on the two-dimensional invariant manifold $z = 0$ is decoupled from the dynamics along the straightened out strong unstable fibers in the z-direction. The flow on the invariant manifold $z = 0$ can be estimated; see Propositions 4.1 and 4.2. Therefore, our approach is to transfer to local coordinates by applying Φ_ε to the inner product (6.86). The estimates on the dynamics of (4.1) leads to bounds on $\phi_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon)$ in the local coordinates. In addition, the other term $(\phi_{\rm b,ad}'(-L_\varepsilon), 0)$ in the inner product (6.86) can be determined to leading order in the local coordinates, since the linear action of Φ_ε is explicit. Furthermore, if we have $\varepsilon > K_0 a^3$, then the leading order of $\phi_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon)$ can also be determined in local coordinates using the estimates on the x-derivative given in Proposition 4.1 (ii). The procedure described above leads to the following result.

Proposition 6.16. *For each sufficiently small $a_0 > 0$, there exists $\varepsilon_0 > 0$ and $K_0, k_0 > 1$, such that for each $(a, \varepsilon) \in (0, a_0) \times (0, \varepsilon_0)$ the quantity $M_{\rm b,2}$ in Theorem 6.11 satisfies $M_{\rm b,2} > \varepsilon/k_0$. If we have in addition $\varepsilon > K_0 a^3$, then $M_{\rm b,2}$ is bounded as $\varepsilon^{2/3}/k_0 < M_{\rm b,2} < \varepsilon^{2/3} k_0$ and can be approximated a-uniformly by*

$$M_{\rm b,2} = \frac{a^2}{4\sqrt{2}} - \frac{(18 - 4\gamma)^{2/3}}{9\sqrt{2}} \Theta^{-1}\left(\frac{-3a}{2(18 - 4\gamma)^{1/3} \varepsilon^{1/3}}\right) \varepsilon^{2/3} + O(\varepsilon |\log \varepsilon|),$$

where Θ is defined in (4.7).

Proof. We start by estimating the lower term in the inner product $M_{\rm b,2}$. Similarly as in the proof of Proposition 6.15, we estimate a-uniformly

$$w_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) = \frac{\varepsilon}{\check{c}_0}\left(u_{\rm b}^1 - \gamma w_{\rm b}^1\right) + O\left(\varepsilon^{5/3} |\log \varepsilon|\right),$$

using Theorem 4.3 (ii). The ε-independent quantity $u_{\rm b}^1 - \gamma w_{\rm b}^1 > 0$ is approximated by $\frac{2}{3} - \frac{4}{27}\gamma + O(a)$ and is bounded away from 0, since $u_{\rm b}^1 = \frac{2}{3}(1 + a)$, $w_{\rm b}^1 = f(u_{\rm b}^1)$ and $0 < \gamma < 4$. In addition, $u_{\rm b}'(\xi)$ is strictly negative, independent of ε and a and converges to 0 at an exponential rate $\frac{1}{2}\sqrt{2}$ as $\xi \to \pm\infty$; see (3.6). Therefore, we estimate

$$\tilde{k}_0 \varepsilon < \left\langle \int_\infty^{-L_\varepsilon} e^{-\check{c}_0 \hat{\xi}} u_{\rm b}'(\hat{\xi}) d\hat{\xi}, w_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \right\rangle < \varepsilon |\log \varepsilon| / \tilde{k}_0 \tag{6.87}$$

for some $\tilde{k}_0 > 0$ independent of a and ε.

We continue by estimating the upper terms in the inner product $M_{b,2}$. The linearization about the fixed point $(u_b^1, 0)$ of (3.5) has eigenvalues $\frac{1}{2}\sqrt{2}$ and $-\sqrt{2}a$ and corresponding eigenvectors $v_+ = (1, \frac{1}{2}\sqrt{2})$ and $v_- = (1, -\sqrt{2}a)$, respectively. By [24, Theorem 1] $\phi_b'(\xi)e^{-\xi/\sqrt{2}}$ converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to an eigenvector $\alpha_+ v_+$ as $\xi \to -\infty$ for some $\alpha_+ \in \mathbb{R} \setminus \{0\}$. Using the explicit formula (3.6) for $\phi_b(\xi)$, we deduce $\alpha_+ = -\frac{1}{2}\sqrt{2}e^{\xi_{b,0}/\sqrt{2}}$, where $\xi_{b,0} \in \mathbb{R}$ denotes the initial translation. Without loss of generality we take $\xi_{b,0} = 0$ so that $\alpha_+ = -\frac{1}{2}\sqrt{2}$; see Remark 3.1. Thus, we approximate a-uniformly

$$e^{\check{c}_0 L_\varepsilon} \begin{pmatrix} v_b'(-L_\varepsilon) \\ -u_b'(-L_\varepsilon) \end{pmatrix} = \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon} \begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} + O\left(\varepsilon^2\right), \tag{6.88}$$

using $\nu \geq 2\sqrt{2}$. For the remaining computations, we transform into local coordinates in the neighborhood $\mathcal{U}_F$ of the fold point $(u^*, 0, w^*)$; see §4.1. Recall from the proof of Theorem 4.3 that $\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)$ is contained in the fold neighborhood $\mathcal{U}_F$ for $a_0, \varepsilon_0 > 0$ sufficiently small. We apply the coordinate transform $\Phi_\varepsilon \colon \mathcal{U}_F \to \mathbb{R}^3$ bringing system (3.1) into the canonical form (4.1). Recall from §4.1 that Φ_ε is C^r-smooth in a and ε in a neighborhood of $(a, \varepsilon) = 0$. Moreover, Φ_ε can be decomposed about $(u^*, 0, w^*)$ into a linear and a nonlinear part

$$\Phi_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathcal{N}\left[\begin{pmatrix} u \\ v \\ w \end{pmatrix} - \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix}\right] + \tilde{\Phi}_\varepsilon \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \mathcal{N} = \partial\Phi_\varepsilon \begin{pmatrix} u^* \\ 0 \\ w^* \end{pmatrix} = \begin{pmatrix} -\beta_1 & \dfrac{\beta_1}{\check{c}} & \dfrac{\beta_1}{\check{c}^2} \\ 0 & 0 & \dfrac{\beta_2}{\check{c}} \\ 0 & \dfrac{1}{\check{c}} & \dfrac{1}{\check{c}^2} \end{pmatrix}, \tag{6.89}$$

where

$$\begin{aligned} \beta_1 &= \left(a^2 - a + 1\right)^{1/3} (u^* - \gamma w^*)^{-1/3} > 0, \\ \beta_2 &= \check{c}\left(a^2 - a + 1\right)^{1/6} (u^* - \gamma w^*)^{-2/3} > 0, \end{aligned}$$

uniformly in a and ε. The nonlinearity $\tilde{\Phi}_\varepsilon$ satisfies $\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = \partial\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = 0$ and $\partial\tilde{\Phi}_\varepsilon$ is bounded a- and ε-uniformly. Differentiating $(x_{a,\varepsilon}(\xi), y_{a,\varepsilon}(\xi), z_{a,\varepsilon}(\xi)) = \Phi_\varepsilon(\phi_{a,\varepsilon}(\xi))$ yields

$$\begin{pmatrix} x_{a,\varepsilon}'(\xi) \\ y_{a,\varepsilon}'(\xi) \\ z_{a,\varepsilon}'(\xi) \end{pmatrix} = \left[\mathcal{N} + \partial\tilde{\Phi}(\phi_{a,\varepsilon}(\xi))\right] \begin{pmatrix} u_{a,\varepsilon}'(\xi) \\ v_{a,\varepsilon}'(\xi) \\ w_{a,\varepsilon}'(\xi) \end{pmatrix}.$$

Recall that $(\phi_b(\xi), w_b^1)$ converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to $(u_b^1, 0, w_b^1)$. Thus, by Theorem 4.3 (ii) and $\nu \geq 2\sqrt{2}$ we have

$$\|\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) - (u_b^1, 0, w_b^1)\| \leq C\varepsilon^{2/3}|\log\varepsilon|, \tag{6.90}$$

where $C > 0$ denotes a constant independent of a and ε. Recall that $u_b^1 = \frac{2}{3}(1+a)$, $u^* = \frac{1}{3}\left(a + 1 + \sqrt{a^2 - a + 1}\right)$, $w_b^1 = f(u_b^1)$, $w^* = f(u^*)$ and $f'(u^*) = 0$. Therefore, we estimate

$$\left|u^* - \tfrac{2}{3}\right|, \left|w^* - \tfrac{4}{27}\right| \leq Ca \quad \left|u_b^1 - u^* - \tfrac{1}{2}a\right|, \left|w_b^1 - w^*\right| \leq Ca^2. \tag{6.91}$$

Combining estimates (6.90) and (6.91) with $\partial\tilde{\Phi}_\varepsilon(u^*, 0, w^*) = 0$, we estimate

$$\|\partial\tilde{\Phi}_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon))\| \leq C\left(\varepsilon^{2/3}|\log\varepsilon| + a\right). \tag{6.92}$$

Using (6.88) and

$$\left(\mathcal{N}^{-1}\right)^* = \begin{pmatrix} -\dfrac{1}{\beta_1} & 0 & 0 \\ 0 & -\dfrac{1}{\beta_2} & \dfrac{\check{c}}{\beta_2} \\ 1 & \check{c} & 0 \end{pmatrix},$$

we approximate a-uniformly

$$
\begin{aligned}
e^{\check{c}_0 L_\varepsilon}\left\langle\!\!\left\langle \begin{pmatrix} v_{\rm b}'(-L_\varepsilon) \\ -u_{\rm b}'(-L_\varepsilon)\end{pmatrix}, \begin{pmatrix} u_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ v_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\end{pmatrix}\right\rangle\!\!\right\rangle &= \left\langle \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon}\begin{pmatrix} -1 \\ \sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} u_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ v_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ w_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\end{pmatrix}\right\rangle + O\left(\varepsilon^2\right) \\
&= \left\langle \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon}\left(\mathcal{N}^{-1}\right)^*\begin{pmatrix} -1 \\ \sqrt{2} \\ 0 \end{pmatrix}, \mathcal{N}\begin{pmatrix} u_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ v_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ w_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\end{pmatrix}\right\rangle + O\left(\varepsilon^2\right) \\
&= \left\langle \frac{1}{2} e^{-\sqrt{2}aL_\varepsilon}\begin{pmatrix} \frac{1}{\beta_1} \\ -\frac{\sqrt{2}}{\beta_2} \\ \sqrt{2}\check{c}-1 \end{pmatrix}, (I+\Delta)\begin{pmatrix} x_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ y_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ z_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\end{pmatrix}\right\rangle + O\left(\varepsilon^2\right),
\end{aligned} \tag{6.93}
$$

where $\Delta := -\partial\tilde{\Phi}_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon))\left(\mathcal{N}+\partial\tilde{\Phi}_\varepsilon(\phi_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon))\right)^{-1}$. First, by (6.92) it holds $\|\Delta\| \leq C\left(\varepsilon^{2/3}|\log\varepsilon| + a\right)$. Second, from the equations (4.1) one observes that $|y_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)| < C\varepsilon$. Third, by Theorem 4.3 the pulse $\phi_{a,\varepsilon}(\xi)$ exits the fold neighborhood at $\xi = Z_{a,\varepsilon} - \xi_{\rm b}$, where $\xi_{\rm b} = O(1)$. The dynamics in the z-component in (4.1) decays exponentially in backward time with rate greater than $\check{c}/2$ by taking the neighborhood $\mathcal{U}_F$ smaller if necessary. Note that $\check{c}$ is bounded from below away from 0 by an a-independent constant. Thus, we may assume that the a-independent constant ν satisfies $\nu \geq 2(\check{c})^{-1}$, i.e. we take $\nu \geq \max\{2\sqrt{2}, 2(\check{c})^{-1}, 2/\mu\} > 0$ (see (6.5)). With this choice of ν, we estimate $|z_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon)| \leq C\varepsilon$. So, using the equation for z' in (4.1), one observes that $|z_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)| \leq C\varepsilon$. Combining the previous three observations with (6.93), we approximate a-uniformly

$$
\begin{aligned}
e^{(\sqrt{2}a+\check{c}_0)L_\varepsilon}\left\langle\!\!\left\langle \begin{pmatrix} v_{\rm b}'(-L_\varepsilon) \\ -u_{\rm b}'(-L_\varepsilon)\end{pmatrix}, \begin{pmatrix} u_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) \\ v_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\end{pmatrix}\right\rangle\!\!\right\rangle &= \frac{1}{2}x_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\left\langle \begin{pmatrix} \frac{1}{\beta_1} \\ -\frac{\sqrt{2}}{\beta_2} \\ \sqrt{2}\check{c}-1 \end{pmatrix}, (I+\Delta)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\rangle + O(\varepsilon), \\
&= \frac{1}{2\beta_1}x_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon)\left(1+O\left(\varepsilon^{2/3}|\log\varepsilon|+a\right)\right) + O(\varepsilon).
\end{aligned} \tag{6.94}
$$

From Propositions 4.1 and 4.2 it follows that for any $k^\dagger > 0$ there exists $\varepsilon_0, a_0 > 0$ such that for $(a,\varepsilon) \in (0,a_0)\times(0,\varepsilon_0)$ it holds $x_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) > k^\dagger\varepsilon$. Moreover, $\beta_1 > 0$ is bounded by an a-independent constant. Thus, by taking $k^\dagger > 0$ sufficiently large, we estimate

$$
M_{{\rm b},2} > e^{-\sqrt{2}aL_\varepsilon}\frac{k^\dagger\varepsilon}{4\beta_1} + \tilde{k}_0\varepsilon, \tag{6.95}
$$

using (6.87) and (6.94). This proves the first assertion.

Suppose we are in the regime $\varepsilon > K_0 a^3$ for some $K_0 > 0$, so that $a = O\left(\varepsilon^{1/3}\right)$. On the one hand, using (6.89) and (6.91) we approximate the x-coordinate $x_{\rm b}$ of $\Phi_\varepsilon(u_{\rm b}^1, 0, w_{\rm b}^1)$ by

$$
x_{\rm b} = -\beta_1\left(u_{\rm b}^1 - u^*\right) + \frac{\beta_1}{\check{c}^2}\left(w_b^1 - w^*\right) + O\left(a^2\right) = -\frac{\beta_1 a}{2} + O(a^2).
$$

On the other hand, since $\partial\Phi_\varepsilon$ is bounded a- and ε-uniformly, we have by (6.90) that $|x_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon) - x_{\rm b}| \leq C\varepsilon^{2/3}|\log\varepsilon|$. Hence, using $K_0 a^3 < \varepsilon$, we estimate

$$
\left|x_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon) + \tfrac{1}{2}\beta_1 a\right| \leq C\left(\varepsilon^{2/3}|\log\varepsilon| + a^2\right) \leq C\varepsilon^{2/3}|\log\varepsilon|. \tag{6.96}
$$

Therefore, Propositions 4.1 and 4.2 yield, provided $K_0 > 0$ is chosen sufficiently large (with lower bound independent of a and ε),

$$
x_{a,\varepsilon}'(Z_{a,\varepsilon}-L_\varepsilon) = \theta_0\left(x_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon)^2 - \Theta^{-1}\left(x_{a,\varepsilon}(Z_{a,\varepsilon}-L_\varepsilon)\varepsilon^{-1/3}\right)\varepsilon^{2/3}\right) + O(\varepsilon). \tag{6.97}
$$

First, by (6.91) it holds

$$\theta_0 = \frac{1}{\check{c}}(a^2 - a + 1)^{1/6}\,(u^* - \gamma w^*)^{1/3} = \frac{\sqrt{2}}{3}\,(18 - 4\gamma)^{1/3} + O(a),$$
$$\beta_1 = \left(a^2 - a + 1\right)^{1/3}(u^* - \gamma w^*)^{-1/3} = 3\,(18 - 4\gamma)^{-1/3} + O(a).$$

Second, in the regime $K_0 a^3 < \varepsilon$ we have

$$\left|e^{-\sqrt{2}aL_\varepsilon} - 1\right| \leq C\varepsilon^{1/3}|\log\varepsilon|.$$

Third, by combining (6.96) and (6.97), we observe $x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) = O(\varepsilon^{2/3})$. We substitute (6.96) and (6.97) into (6.94) and approximate $M_{\mathrm{b},2}$ with the aid of the previous three observations and identity (6.87) by

$$\begin{aligned}
M_{\mathrm{b},2} &= \frac{1}{2\beta_1}x'_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon) + O\left(\varepsilon|\log\varepsilon|\right)\\
&= \frac{\theta_0}{2\beta_1}\left(x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)^2 - \Theta^{-1}\left(x_{a,\varepsilon}(Z_{a,\varepsilon} - L_\varepsilon)\varepsilon^{-1/3}\right)\varepsilon^{2/3}\right) + O\left(\varepsilon|\log\varepsilon|\right)\\
&= \frac{a^2}{4\sqrt{2}} - \frac{(18-4\gamma)^{2/3}}{9\sqrt{2}}\Theta^{-1}\left(\frac{-3a}{2\,(18-4\gamma)^{1/3}\,\varepsilon^{1/3}}\right)\varepsilon^{2/3} + O\left(\varepsilon|\log\varepsilon|\right).
\end{aligned}$$

This is the desired leading order approximation of $M_{\mathrm{b},2}$. In the regime $K_0 a^3 < \varepsilon$, for $K_0 > 1$ sufficiently large, the bound $\varepsilon^{2/3}/k_0 < M_{\mathrm{b},2} < \varepsilon^{2/3}k_0$ follows from this approximation, using that Θ^{-1} is smooth and $\Theta^{-1}(0) < 0$. □

Remark 6.17. By Theorem 6.11 the second eigenvalue λ_1 of (6.6) is to leading order approximated by the quotient $M_{\mathrm{b},2}M_{\mathrm{b},1}^{-1}$. We give a geometric interpretation of the quantities $M_{\mathrm{b},1}$ and $M_{\mathrm{b},2}$ in both the hyperbolic and nonhyperbolic regimes.

For the interpretation of the quantity $M_{\mathrm{b},1}$ we append the Nagumo eigenvalue problem to the Nagumo existence problem (3.5) along the back

$$\begin{aligned}
u_\xi &= v,\\
v_\xi &= \check{c}_0 v - f(u) + w_{\mathrm{b}}^1,\\
\tilde{u}_\xi &= \tilde{v},\\
\tilde{v}_\xi &= \check{c}_0\tilde{v} - f'(u)\tilde{u} + \lambda\tilde{u}.
\end{aligned} \tag{6.98}$$

Note that $(\phi_{\mathrm{b}}(\xi), \phi'_{\mathrm{b}}(\xi))$ is a heteroclinic solution to (6.98) for $\lambda = 0$ connecting the equilibria $(p_{\mathrm{b}}^1, 0)$ and $(p_{\mathrm{b}}^0, 0)$. The space of bounded solutions to the adjoint equation of the linearization of (6.98) at $\lambda = 0$ about $(\phi_{\mathrm{b}}(\xi), \phi'_{\mathrm{b}}(\xi))$ is spanned by $(\psi_{\mathrm{ad},1}(\xi), 0)$ and $(\psi_{\mathrm{ad},2}(\xi), \psi_{\mathrm{ad},1}(\xi))$, where $\psi_{\mathrm{ad},1}(\xi) = (v'_{\mathrm{b}}(\xi), -u'_{\mathrm{b}}(\xi))e^{-\check{c}_0\xi}$. The Melnikov integral

$$M_{\mathrm{b},1} = \int_{-\infty}^{\infty}\left(u'_{\mathrm{b}}(\xi)\right)^2 e^{-\check{c}_0\xi}d\xi,$$

measures how the intersection between the stable manifold $\mathcal{W}^s(p_{\mathrm{b}}^0, 0)$ and unstable manifold $\mathcal{W}^u(p_{\mathrm{b}}^1, 0)$ breaks at $(\phi_{\mathrm{b}}(0), \phi'_{\mathrm{b}}(0))$ in the direction of $(\psi_{\mathrm{ad},2}(0), \psi_{\mathrm{ad},1}(0))$ as we vary λ. Note that the quantity M_{f}, defined in (6.75), has a similar interpretation.

In the hyperbolic regime $M_{\mathrm{b},2}$ is to leading order given by (6.85). The positive sign of the quantity $u_{\mathrm{b}}^1 - \gamma w_{\mathrm{b}}^1$ in (6.85) corresponds to the fact that solutions on the right slow manifold move in the direction of positive w. For the geometric interpretation of the integral

$$\int_{-\infty}^{\infty} u'_{\mathrm{b}}(\xi)e^{-\check{c}_0\xi}d\xi, \tag{6.99}$$

in (6.85) we observe that the dynamics in the layers of the fast problem (3.3) are given by the Nagumo systems

$$\begin{aligned}
u_\xi &= v,\\
v_\xi &= \check{c}_0 v - f(u) + w.
\end{aligned} \tag{6.100}$$

For $w = w_{\rm b}^1$ system (6.100) admits the heteroclinic solution $\phi_{\rm b}(\xi)$ connecting the equilibria $p_{\rm b}^1$ and $p_{\rm b}^0$. The space of bounded solutions to the adjoint problem of the linearization of (6.100) at $w = w_{\rm b}^1$ about $\phi_{\rm b}(\xi)$ is spanned by $\psi_{{\rm ad},1}(\xi)$. One readily observes that (6.99) is a Melnikov integral measuring how the intersection between the stable manifold $\mathcal{W}^s(p_{\rm b}^0)$ and unstable manifold $\mathcal{W}^u(p_{\rm b}^1)$ breaks at $\phi_{\rm b}(0)$ in the direction of $\psi_{{\rm ad},1}(0)$ as we vary w in (6.100), i.e. as we move through the fast fibers in the layer problem (3.3).

In the nonhyperbolic regime $M_{{\rm b},2}$ is estimated by (6.95). As can be observed from the proof of Proposition 6.16, the sign of $M_{{\rm b},2}$ is dominated by the inner product

$$\left\langle \begin{pmatrix} v_{\rm b}'(-L_\varepsilon) \\ -u_{\rm b}'(-L_\varepsilon) \end{pmatrix}, \begin{pmatrix} u_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \\ v_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon) \end{pmatrix} \right\rangle$$

of the adjoint of the singular back solution and the derivative of the pulse solution near the fold point. This inner product determines the orientation of the pulse solution as it passes over the fold before jumping off in the strong unstable direction along the singular back solution. In essence, upon passing up and over the fold, the solution jumps off along a strong unstable fiber to the left. In the fold coordinates, the sign of this inner product amounts to the sign of the derivative $x_{a,\varepsilon}'(Z_{a,\varepsilon} - L_\varepsilon)$ of the x-coordinate of the pulse solution in the local coordinates around the fold (4.1). The sign of this derivative is determined by the direction of the Riccati flow in the blow-up charts near the fold; see system (4.8).

6.6 The region R_2

The goal of the section is to prove that the region $R_2(\delta, M)$ contains no eigenvalues of (6.6) for any $M > 0$ and each $\delta > 0$ sufficiently small. As described in §6.2.1 our approach is to show that problem (6.6) admits exponential dichotomies on each of the intervals $I_{\rm f}, I_r, I_{\rm b}$ and I_ℓ, which together form a partition of the whole real line $\mathbb{R}$. The exponential dichotomies on I_r and I_ℓ are yet established in Proposition 6.5. The exponential dichotomies on $I_{\rm f}$ and $I_{\rm b}$ are generated from exponential dichotomies of a reduced eigenvalue problem via roughness results. Our plan is to compare the projections of the aforementioned exponential dichotomies at the endpoints of the intervals. The obtained estimates yield that any exponentially localized solution to (6.6) must be trivial for $\lambda \in R_2$.

6.6.1 A reduced eigenvalue problem

We establish for ξ in $I_{\rm f}$ or $I_{\rm b}$ a reduced eigenvalue problem by setting ε to 0 in system (6.6), while approximating $\phi_{a,\varepsilon}(\xi)$ with (a translate of) the front $\phi_{\rm f}(\xi)$ or the back $\phi_{\rm b}(\xi)$, respectively. However, we do keep the λ-dependence in contrast to the reduction done in the region R_1. Thus, the reduced eigenvalue problem reads

$$\psi_\xi = A_j(\xi,\lambda)\psi, \quad A_j(\xi,\lambda) = A_j(\xi,\lambda;a) := \begin{pmatrix} -\eta & 1 & 0 \\ \lambda - f'(u_j(\xi)) & \check{c}_0 - \eta & 1 \\ 0 & 0 & -\frac{\lambda}{\check{c}_0} - \eta \end{pmatrix}, \quad j = {\rm f}, {\rm b}, \tag{6.101}$$

where $u_j(\xi)$ denotes the u-component of $\phi_j(\xi)$, λ is in R_2 and a is in $[0, \frac{1}{2} - \kappa]$. By its triangular structure, system (6.101) leaves the subspace $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$ invariant. The dynamics of (6.101) on that space is given by

$$\varphi_\xi = C_j(\xi,\lambda)\varphi, \quad C_j(\xi,\lambda) = C_j(\xi,\lambda;a) := \begin{pmatrix} -\eta & 1 \\ \lambda - f'(u_j(\xi)) & \check{c}_0 - \eta \end{pmatrix}, \quad j = {\rm f}, {\rm b}. \tag{6.102}$$

We remark that problem (6.102) corresponds to the weighted eigenvalue problem of the Nagumo systems $u_t = u_{xx} + f(u)$ and $u_t = u_{xx} + f(u) - w_{\rm b}^1$ about the traveling-wave solutions $u_{\rm f}(x + \check{c}_0 t)$ and $u_{\rm b}(x + \check{c}_0 t)$, respectively.

We show that systems (6.101) and (6.102) admit exponential dichotomies on both half-lines. The translated derivative $e^{-\eta\xi}\phi_j'(\xi)$ is an exponentially localized solution to (6.102) at $\lambda = 0$, which admits no zeros. Therefore, by Sturm-Liouville theory, $\lambda = 0$ is the eigenvalue of largest real part of (6.102). So, problems (6.102) admit no exponentially localized solutions for $\lambda \in R_2(\delta, M)$ by taking $\delta > 0$ sufficiently small. This fact allows us to paste the exponential dichotomies on both half-lines of systems (6.102) and (6.101) to a single exponential dichotomy on $\mathbb{R}$. This is the content of the following result.

Proposition 6.18. *Let $\kappa, M > 0$. For each $\delta > 0$ sufficiently small, $a \in [0, \frac{1}{2} - \kappa]$ and $\lambda \in R_2(\delta, M)$ system* (6.101) *admits exponential dichotomies on $\mathbb{R}$ with λ- and a-independent constants $C, \frac{\mu}{2} > 0$, where $\mu > 0$ is as in Lemma 6.3.*

Proof. By Lemma 6.3, provided $\delta > 0$ is sufficiently small, the asymptotic matrices $C_{j,\pm\infty}(\lambda) = C_{j,\pm\infty}(\lambda; a) := \lim_{\xi\to\pm\infty} C_j(\xi, \lambda)$ of (6.102) have for $a \in [0, 1/2 - \kappa]$ and $\lambda \in R_2(\delta, M)$ a uniform spectral gap larger than $\mu > 0$. Hence, it follows from [29, Lemmata 1.1 and 1.2] that system (6.102) admits for $(\lambda, a) \in R_2 \times [0, 1/2 - \kappa]$ exponential dichotomies on both half-lines with constants $C, \mu > 0$ and projections $\Pi^{u,s}_{j,\pm}(\xi, \lambda) = \Pi^{u,s}_{j,\pm}(\xi, \lambda; a)$, $j = \mathrm{f}, \mathrm{b}$. We emphasize that the constant $C > 0$ is independent of λ and a, because $R_2 \times [0, 1/2 - \kappa]$ is compact.

By Sturm-Liouville theory (see e.g. [20, Theorem 2.3.3]) system (6.102) has precisely one eigenvalue $\lambda = 0$ on $\mathrm{Re}(\lambda) \geq -\delta$ (taking $\delta > 0$ smaller if necessary). Therefore, system (6.102) admits no bounded solutions for $\lambda \in R_2$. Hence, we can paste the exponential dichotomies as in [5, p. 16-19] by defining $\Pi^s_j(0, \lambda)$ to be the projection onto $R(\Pi^s_{j,+}(0, \lambda))$ along $R(\Pi^u_{j,-}(0; \lambda))$. Thus, system (6.102) admits for $(\lambda, a) \in R_2 \times [0, 1/2 - \kappa]$ an exponential dichotomy on $\mathbb{R}$ with λ- and a-independent constants $C, \mu > 0$ and projections $\Pi^{u,s}_j(\xi, \lambda) = \Pi^{u,s}_j(\xi, \lambda; a)$, $j = \mathrm{f}, \mathrm{b}$.

By the triangular structure of system (6.101) the exponential dichotomy on $\mathbb{R}$ of the subsystem (6.102) can be transferred to the full system (6.101) using a variation of constants formula; see also the proof of Corollary 6.7. The exponential dichotomy on $\mathbb{R}$ of system (6.101) has constants $C, \min\{\mu, \eta - \frac{\delta}{c_0}\} > 0$, where $C > 0$ is independent of a and λ. The result follows by taking $\delta > 0$ sufficiently small using that $\mu \leq \eta$ by Lemma 6.3. □

6.6.2 Absence of point spectrum in R_2

With the aid of the following lemma we show that the region R_2 contains no eigenvalues of (6.6).

Lemma 6.19 ([14, Lemma 6.10])**.** *Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ with $a < b$ and $A \in C([a, b], \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose the equation*

$$\varphi_x = A(x)\varphi, \tag{6.103}$$

has an exponential dichotomy on $[a, b]$ with constants $C, m > 0$ and projections $P^{u,s}_1(x)$. Denote by $T(x, y)$ the evolution of (6.103)*. Let P_2 be a projection such that $\|P^s_1(b) - P_2\| \leq \delta_0$ for some $\delta_0 > 0$ and let $v \in \mathbb{C}^n$ a vector such that $\|P^s_1(a)v\| \leq k\|P^u_1(a)v\|$ for some $k \geq 0$. If we have $\delta_0(1 + kC^2 e^{-2m(b-a)}) < 1$, then it holds*

$$\|P_2 T(b, a)v\| \leq \frac{\delta_0 + kC^2 e^{-2m(b-a)}\,(1 + \delta_0)}{1 - \delta_0\,(1 + kC^2 e^{-2m(b-a)})}\|(1 - P_2)T(b, a)v\|.$$

Proposition 6.20. *Let $M > 0$ be as in Proposition 6.1. There exists $\delta, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ system* (6.6) *admits no nontrivial exponentially localized solution for $\lambda \in R_2(\delta, M)$.*

Proof. We start by establishing exponential dichotomies of system (6.6) on the intervals $I_\mathrm{f} = (-\infty, L_\varepsilon]$ and $I_\mathrm{b} = [Z_{a,\varepsilon} - L_\varepsilon, Z_{a,\varepsilon} + L_\varepsilon]$. Let $\lambda \in R_2(\delta, M)$. We regard the eigenvalue problem (6.6) as an ε-perturbation of system (6.101). Indeed, by Theorem 4.3 (i)-(ii), for each sufficiently small $a_0 > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ we estimate the difference between the coefficient matrices of both systems along the front and the back by

$$\begin{aligned} \|A(\xi, \lambda) - A_\mathrm{f}(\xi, \lambda)\| &\leq C\varepsilon|\log\varepsilon|, \quad \xi \in (-\infty, L_\varepsilon], \\ \|A(Z_{a,\varepsilon} + \xi, \lambda) - A_\mathrm{b}(\xi, \lambda)\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \quad \xi \in [-L_\varepsilon, L_\varepsilon], \end{aligned} \tag{6.104}$$

where $\rho(a) = \frac{2}{3}$ for $a < a_0$ and $\rho(a) = 1$ for $a \geq a_0$ and C is independent of λ, a and ε. By Proposition 6.18 system (6.101) has an exponential dichotomy on $\mathbb{R}$ with λ- and a-independent constants $C, \frac{\mu}{2} > 0$ and projections $Q^{u,s}_j(\xi, \lambda) = Q^{u,s}_j(\xi, \lambda; a)$

for $j = \mathrm{f},\mathrm{b}$. Denote by $P_j^{u,s}(\lambda) = P_j^{u,s}(\lambda; a)$ the spectral projection onto the (un)stable eigenspace of the asymptotic matrices $A_{j,\pm\infty}(\lambda) = A_{j,\pm\infty}(\lambda; a)$ of system (6.101). As in the proof of Proposition 6.9 we obtain the estimate

$$\|Q_j^{u,s}(\pm\xi,\lambda) - P_{j,\pm}^{u,s}(\lambda)\| \leq C\left(e^{-\frac{1}{2}\sqrt{2}\xi} + e^{-\frac{\mu}{2}\xi}\right), \quad j = \mathrm{f},\mathrm{b}, \tag{6.105}$$

for $\xi \geq 0$. By estimate (6.104) roughness [4, Theorem 2] yields exponential dichotomies on $I_\mathrm{f} = (-\infty, L_\varepsilon]$ and $I_\mathrm{b} = [Z_{a,\varepsilon}-L_\varepsilon, Z_{a,\varepsilon}+L_\varepsilon]$ for system (6.6) with λ- and a-independent constants $C, \frac{\mu}{2} > 0$ and projections $\mathcal{Q}_j^{u,s}(\xi,\lambda) = \mathcal{Q}_j^{u,s}(\xi,\lambda; a,\varepsilon)$, which satisfy

$$\begin{aligned} \|\mathcal{Q}_\mathrm{f}^{u,s}(\xi,\lambda) - Q_\mathrm{f}^{u,s}(\xi,\lambda)\| &\leq C\varepsilon|\log\varepsilon|, \\ \|\mathcal{Q}_\mathrm{b}^{u,s}(Z_{a,\varepsilon}+\xi,\lambda) - Q_\mathrm{b}^{u,s}(\xi,\lambda)\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \end{aligned} \tag{6.106}$$

for $|\xi| \leq L_\varepsilon$.

On the other hand, system (6.6) admits by Proposition 6.5 exponential dichotomies on $I_r = [L_\varepsilon, Z_{a,\varepsilon} - L_\varepsilon]$ and on $I_\ell = [Z_{a,\varepsilon} + L_\varepsilon, \infty)$ with constants $C, \mu > 0$ and projections $\mathcal{Q}_{r,\ell}^{u,s}(\xi,\lambda) = \mathcal{Q}_{r,\ell}^{u,s}(\xi,\lambda; a,\varepsilon)$. The projections satisfy at the endpoints

$$\begin{aligned} \left\|[\mathcal{Q}_r^s - \mathcal{P}](L_\varepsilon,\lambda)\right\| &\leq C\varepsilon|\log\varepsilon|, \\ \left\|[\mathcal{Q}_r^s - \mathcal{P}](Z_{a,\varepsilon} - L_\varepsilon,\lambda)\right\|, \left\|[\mathcal{Q}_\ell^s - \mathcal{P}](Z_{a,\varepsilon} + L_\varepsilon,\lambda)\right\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|, \end{aligned} \tag{6.107}$$

where $\mathcal{P}(\xi,\lambda) = \mathcal{P}(\xi,\lambda; a,\varepsilon)$ denote the spectral projections onto the stable eigenspace of $A(\xi,\lambda)$.

Having established exponential dichotomies for (6.6) on the intervals $I_\mathrm{f}, I_r, I_\mathrm{b}$ and I_ℓ, our next step is to compare the associated projections at the endpoints of the intervals. Recall that $A_j(\xi,\lambda)$ converges at an exponential rate $\frac{1}{2}\sqrt{2}$ to the asymptotic matrix $A_{j,\pm\infty}(\lambda)$ as $\xi \to \pm\infty$ for $j = \mathrm{f},\mathrm{b}$. Combining this with (6.104) and $\nu \geq 2\sqrt{2}$ we estimate

$$\begin{aligned} \|A(L_\varepsilon,\lambda) - A_{\mathrm{f},\infty}(\lambda)\| &\leq C\varepsilon|\log\varepsilon|, \\ \|A(Z_{a,\varepsilon} \pm L_\varepsilon,\lambda) - A_{\mathrm{b},\pm\infty}(\lambda)\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|. \end{aligned}$$

By continuity the same bound holds for the spectral projections associated with these matrices. Combining this fact with $\nu \geq \max\{2\sqrt{2}, 2/\mu\}$, (6.105), (6.106) and (6.107) we obtain

$$\begin{aligned} \left\|[\mathcal{Q}_r^{u,s} - \mathcal{Q}_\mathrm{f}^{u,s}](L_\varepsilon,\lambda)\right\| &\leq C\varepsilon|\log\varepsilon|, \\ \left\|[\mathcal{Q}_\ell^{u,s} - \mathcal{Q}_\mathrm{b}^{u,s}](Z_{a,\varepsilon} + L_\varepsilon,\lambda)\right\|, \left\|[\mathcal{Q}_r^{u,s} - \mathcal{Q}_\mathrm{b}^{u,s}](Z_{a,\varepsilon} - L_\varepsilon,\lambda)\right\| &\leq C\varepsilon^{\rho(a)}|\log\varepsilon|. \end{aligned} \tag{6.108}$$

The last step is an application of Lemma 6.19. Let $\psi(\xi)$ be an exponentially localized solution to (6.6) at some $\lambda \in R_2$. This implies $\mathcal{Q}_\mathrm{f}^s(0,\lambda)\psi(0) = 0$. An application of Lemma 6.19 yields

$$\|\mathcal{Q}_r^s(L_\varepsilon,\lambda)\psi(L_\varepsilon)\| \leq C\varepsilon|\log\varepsilon|\|\mathcal{Q}_r^u(L_\varepsilon,\lambda)\psi(L_\varepsilon)\|, \tag{6.109}$$

using (6.108) and $\nu \geq 2/\mu$. We proceed in a similar fashion by applying Lemma 6.19 to the inequality (6.109) and using (6.108) to obtain a similar inequality at the endpoint $Z_{a,\varepsilon} - L_\varepsilon$. Applying the Lemma once again, we eventually obtain

$$\|\mathcal{Q}_\ell^s(Z_{a,\varepsilon} + L_\varepsilon,\lambda)\psi(Z_{a,\varepsilon} + L_\varepsilon)\| \leq C\varepsilon^{\rho(a)}|\log\varepsilon|\|\mathcal{Q}_\ell^u(Z_{a,\varepsilon} + L_\varepsilon,\lambda)\psi(Z_{a,\varepsilon} + L_\varepsilon)\| = 0,$$

where the latter equality is due to the fact that $\psi(\xi)$ is exponentially localized. Thus, ψ is the trivial solution to (6.6). □

7 Proofs of main stability results

We studied the essential spectrum in §5 and the point spectrum in §6 of the linearization $\mathcal{L}_{a,\varepsilon}$. In this section we complete the proofs of the main stability results: Theorem 2.2 and Theorem 2.4.

Proof of Theorem 2.2. In the regime $\varepsilon < Ka^2$, the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ is contained in the half-plane $\mathrm{Re}(\lambda) \leq -\min\{\varepsilon\gamma, a\} = -\varepsilon\gamma$ by Proposition 5.1. Consider the regions R_1, R_2 and R_3 defined in §6.2.1. By Propositions 6.1, 6.4 and 6.20 there is no point spectrum of $\mathcal{L}_{a,\varepsilon}$ in the regions R_2 and R_3 to the right hand side of the essential spectrum. By Proposition 6.4, Theorem 6.11 and Proposition 6.14 the point spectrum in R_1 to the right hand side of the essential spectrum consists of the simple translational eigenvalue $\lambda_0 = 0$ and at most one other real eigenvalue λ_1 approximated by $-M_{\mathrm{b},2}M_{\mathrm{b},1}^{-1}$, where $M_{\mathrm{b},1} > 0$ is independent of ε and bounded by an a-independent constant. Subsequently, we use Propositions 6.15 and 6.16 to estimate $M_{\mathrm{b},2}$. We conclude that there exists a constant $b_0 > 0$ such that $\lambda_1 < -\varepsilon b_0$. □

Proof of Theorem 2.4. It follows by Proposition 6.15 that the potential eigenvalue $\lambda_1 < 0$ of $\mathcal{L}_{a,\varepsilon}$ is approximated (a-uniformly) by $\lambda_1 = -M_1\varepsilon + O\left(|\varepsilon \log \varepsilon|^2\right)$ in the hyperbolic regime, where M_1 is given by

$$\begin{aligned} M_1 = M_1(a) &:= \frac{\left(\gamma w_{\mathrm{b}}^1 - u_{\mathrm{b}}^1\right)\int_{-\infty}^{\infty} u_{\mathrm{b}}'(\xi)e^{-\check{c}_0\xi}d\xi}{\check{c}_0\int_{-\infty}^{\infty}\left(u_{\mathrm{b}}'(\xi)\right)^2 e^{-\check{c}_0\xi}d\xi} \\ &= \frac{18(a+1) - \gamma\left(4a^3 - 6a^2 - 6a + 4\right)}{9a\,(1-a)\,(1-2a)} \\ &> 0, \end{aligned} \tag{7.1}$$

where we used the explicit expressions for the front and the back given in (3.6) and substituted $u_{\mathrm{b}}^1 = \frac{2}{3}(1+a)$, $w_{\mathrm{b}}^1 = f(u_{\mathrm{b}}^1)$ and $\check{c}_0 = \sqrt{2}(\frac{1}{2} - a)$.

By Theorem 5.1 the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ intersects the real axis only at points $\lambda \leq -\varepsilon(\gamma + a^{-1})$ in the hyperbolic regime. So, if $M_1 < \gamma + a^{-1}$ is satisfied, then λ_1 lies to the right hand side of the essential spectrum. In that case, λ_1 is by Proposition 6.4 contained in the point spectrum of $\mathcal{L}_{a,\varepsilon}$. This proves the first assertion.

By Theorem 6.11 and Proposition 6.16, there exists $K_0, k_0 > 1$, independent of a and ε, such that, if $\varepsilon > K_0 a^3$, then

$$\lambda_1 = -\frac{M_{\mathrm{b},2}}{M_{\mathrm{b},1}} + O\left(\left|\varepsilon^{2/3}\log\varepsilon\right|^2\right),$$

satisfies $1/k_0\varepsilon^{2/3} < \lambda_1 < k_0\varepsilon^{2/3}$. By Theorem 5.1 the essential spectrum of $\mathcal{L}_{a,\varepsilon}$ intersects the real axis only at points $\lambda \leq -\min\{\varepsilon(\gamma + a^{-1}), \frac{1}{2}a + \frac{1}{2}\varepsilon\gamma\}$. Thus, in the regime $K_0 a^3 < \varepsilon < Ka^2$ the essential spectrum intersects the real axis at points $\lambda < -K_0^{1/3}\varepsilon^{2/3}$. Taking $K_0 > 1$ larger if necessary, it follows that λ_1 lies to the right hand side of the essential spectrum and λ_1 is by Proposition 6.4 an eigenvalue of $\mathcal{L}_{a,\varepsilon}$.

With the aid of (3.6) we calculate

$$M_{\mathrm{b},1} = \int_{-\infty}^{\infty}\left(u_{\mathrm{b}}'(\xi)\right)^2 e^{-\check{c}_0\xi}d\xi = \frac{1}{3\sqrt{2}} + O(a),$$

taking the initial translation $\xi_{\mathrm{b},0}$ of $u_{\mathrm{b}}(\xi)$ equal to 0; see Remark 3.1. Moreover, if $K_0 a^3 < \varepsilon^{1+\alpha}$ for some $\alpha > 0$, then we compute with the aid of Proposition 6.16

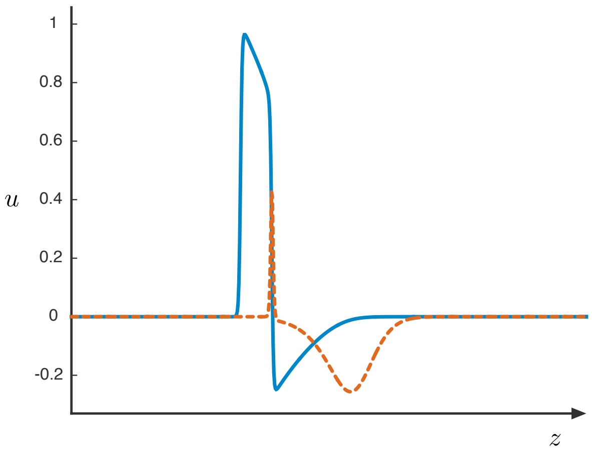
$$M_{\mathrm{b},2} = -\frac{(18-4\gamma)^{2/3}}{9\sqrt{2}}\Theta^{-1}(0)\varepsilon^{2/3} + O\left(\varepsilon^{(2+\alpha)/3}, \varepsilon|\log\varepsilon|\right),$$

uniformly in a and α, where Θ is defined in (4.7). With these leading order computations of $M_{\mathrm{b},1}$ and $M_{\mathrm{b},2}$ the approximation (2.4) of λ_1 follows in the regime $K_0 a^3 < \varepsilon^{1+\alpha}$, $\varepsilon < Ka^2$. □

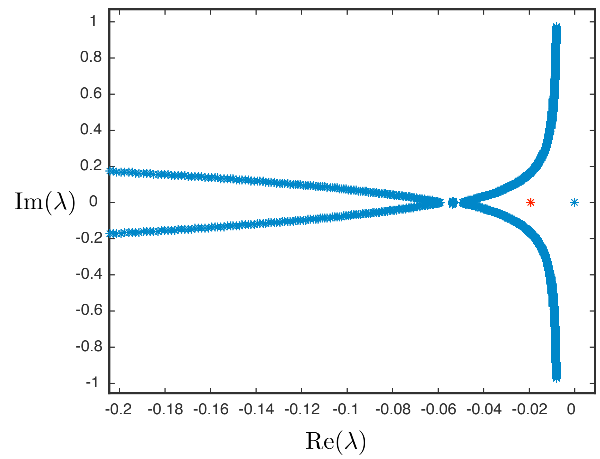
8 Numerics

In this section, we discuss numerical results pertaining to Theorem 2.4. As the theorem is primarily a spectral stability result, in the numerical analysis below we solve the traveling-wave ODE

$$\begin{aligned} 0 &= u_{\xi\xi} - cu_\xi + f(u) - w, \\ 0 &= -cw_\xi + \varepsilon(u - \gamma w). \end{aligned} \tag{8.1}$$



(a) Shown is the u-component of the pulse solution (blue) obtained numerically for $(c, a, \varepsilon, \gamma) = (0.5480, 0.0997, 0.0021, 3.5)$. Also plotted is the u-component of the eigenfunction (dashed red) corresponding to the eigenvalue $\lambda_1 = -0.0194$.



(b) Shown is the spectrum of the operator $\mathcal{L}_{a,\varepsilon}$ associated with the pulse in Figure 6a. Note that the eigenvalue $\lambda_1 = -0.0194$ (shown in red) lies to the right of the essential spectrum.

Figure 6

for stationary solutions of (2.2) along with the eigenvalue problem (2.3) to obtain information on the behavior of the potential second eigenvalue λ_1 of $\mathcal{L}_{a,\varepsilon}$; in particular, we focus on the location of λ_1 with respect to the essential spectrum and its asymptotic behavior as $\varepsilon \to 0$.

8.1 Position of λ_1 with respect to the essential spectrum

In the nonhyperbolic regime $K_0 a^3 < \varepsilon$ it is always the case that λ_1 lies to the right of the essential spectrum and is in fact an eigenvalue of $\mathcal{L}_{a,\varepsilon}$ by Theorem 2.4 (ii). In the hyperbolic regime there is a condition in Theorem 2.4 (i) which ensures that λ_1 lies to the right of the essential spectrum and is an eigenvalue of $\mathcal{L}_{a,\varepsilon}$. We comment on this condition. Note that for parameter values $(a, \gamma) = (0.0997, 3.5)$ the condition is satisfied

$$M_1 = 12.498 < 13.530 = \gamma + a^{-1}.$$

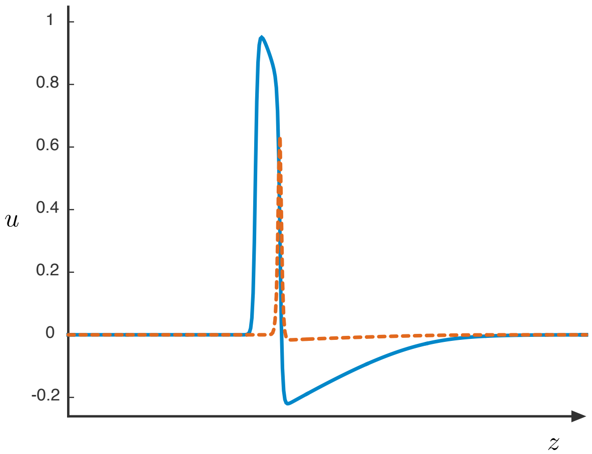
Here M_1 is calculated with the aid of formula (7.1). If $(u(x - ct), w(x - ct))$ is a traveling-wave solution to (2.1) with wave speed c, then $(u(\xi), w(\xi))$ solves (8.1), or equivalently, $(u(\xi), u'(\xi), w(\xi))$ satisfies the traveling wave ODE

$$\begin{aligned} u_\xi &= v, \\ v_\xi &= cv - f(u) + w, \\ w_\xi &= \frac{\varepsilon}{c}(u - \gamma w). \end{aligned} \tag{8.2}$$

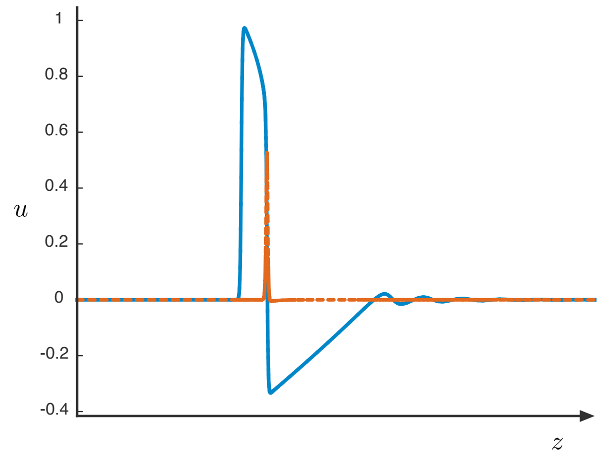
In Matlab, we solve (8.2) numerically for the parameter values $(a, \varepsilon, \gamma) = (0.0997, 0.0021, 3.5)$ where we obtain the monotone pulse solution shown in Figure 6a with wave speed $c = 0.5480$; we also solve the eigenvalue problem (2.3) and obtain a solution with eigenvalue $\lambda_1 = -0.0194$; the corresponding eigenfunction of $\mathcal{L}_{a,\varepsilon}$ is plotted along with the pulse in Figure 6a. The spectrum associated with the pulse is plotted in Figure 6b. Note that the eigenvalue $\lambda_1 = -0.0194$ appears indeed to the right of the essential spectrum.

8.2 Asymptotics of λ_1 as $\varepsilon \to 0$

We now turn to the asymptotics of the eigenvalue λ_1 of $\mathcal{L}_{a,\varepsilon}$ as $\varepsilon \to 0$. To study this, we continue traveling-pulse solutions to (2.1) numerically along different curves in the parameters c, a and ε in order to illustrate the behavior of the eigenvalue



(a) Shown is the u-component of the monotone pulse solution (blue) obtained numerically for $(c, a, \varepsilon, \gamma) = (0.4446, 0.1671, 0.0021, 0.5)$. Also plotted is the u-component of the weighted eigenfunction (dashed red) corresponding to the eigenvalue $\lambda_1 = -0.0408$.



(b) Shown is the u-component of the oscillatory pulse solution (blue) obtained numerically for $(c, a, \varepsilon, \gamma) = (0.6864, 0.0059, 0.0021, 0.5)$. Also plotted is the u-component of the weighted eigenfunction (dashed red) corresponding to the eigenvalue $\lambda_1 = -0.0374$.

Figure 7

λ_1 in the hyperbolic and nonhyperbolic regimes treated in Theorem 2.4. In order to ensure that we obtain the correct value for λ_1, we use a small exponential weight $\eta > 0$ to shift the essential spectrum away from the imaginary axis, i.e. we look for solutions to the eigenvalue problem (2.3) bounded in the weighted norm $\|\psi\|_{-\eta} = \sup_{\xi \in \mathbb{R}} \|\psi(\xi) e^{-\eta \xi}\|$. This amounts to replacing (2.3) with the shifted version

$$\psi_\xi = (A_0(\xi, \lambda) - \eta)\,\psi. \tag{8.3}$$

This procedure is justified and explained in detail in §6.2.2. In short, if $[0, 1/2 - \kappa]$ is the allowed range for a in the existence result Theorem 2.1, then for the choice $\eta = \frac{1}{2}\sqrt{2}\kappa$, λ_1 lies to the right of the shifted essential spectrum and is always an eigenvalue of the shifted problem (8.3). In the following, we fix $\eta = 0.1$. Thus, we restrict to a-values in $[0, 0.3586]$.

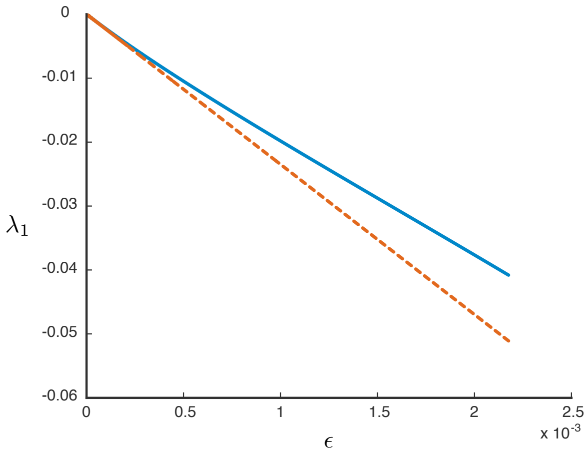
8.2.1 Hyperbolic regime

We first consider the hyperbolic regime: according to Theorem 2.4 (i), for sufficiently small $\varepsilon > 0$, the eigenvalue λ_1 of (8.3) is approximated by

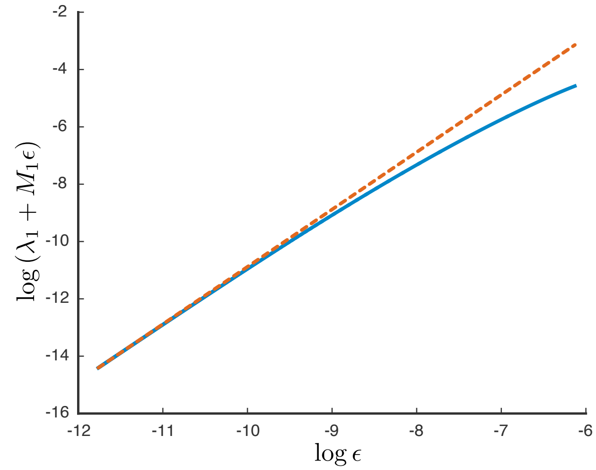
$$\lambda_1 = -M_1 \varepsilon + \mathcal{O}\left(|\varepsilon \log \varepsilon|^2\right), \tag{8.4}$$

where $M_1 > 0$ is given by (7.1). Using Matlab, we solve (8.2) numerically for the parameter values $(a, \varepsilon, \gamma) = (0.1671, 0.0021, 0.5)$ where we obtain the monotone pulse solution shown in Figure 7a with wave speed $c = 0.4446$. In addition, we solve the eigenvalue problem (8.3) and obtain a solution with eigenvalue $\lambda_1 = -0.0408$; the corresponding weighted eigenfunction of (8.3) is plotted along with the pulse in Figure 7a. To see whether (8.4) gives a good prediction for the location of the eigenvalue λ_1 in the hyperbolic regime, we fix the parameter a and using the continuation software package AUTO, we append the weighted eigenvalue problem (8.3) to the existence problem (8.2) and continue in the parameters (c, ε) letting $\varepsilon \to 0$ to determine the asymptotics of the eigenvalue λ_1. We regard c here as a free parameter, because the value of $c = \check{c}(a, \varepsilon)$ for which (2.1) admits a traveling-pulse solution depends on a and ε by Theorem 2.1. Thus, instead of prescribing $c = \check{c}(a, \varepsilon)$ we require AUTO to continue along a 1-dimensional curve in the (c, ε)-plane of homoclinic solutions to 0 of (8.2).

The results of the continuation process are plotted in Figure 8. In Figure 8a, the continuation of the eigenvalue λ_1 is plotted against ε along with the first order approximation $\lambda_1 \approx -M_1 \varepsilon$ for the eigenvalue λ_1 from Theorem 2.4 (i). There is good



(a) Plotted is the curve (blue) obtained for the continuation of the eigenvalue λ_1 as $\varepsilon \to 0$ in the monotone pulse case. Here we have fixed $a = 0.1671$ and the wave speed c varies along the continuation. For comparison, we also plot the first order approximation (dashed red) $\lambda \approx -M_1\varepsilon$ for the eigenvalue λ_1 from Theorem 2.4 (i).



(b) Shown is a log-log plot of the differences (blue) of the two curves in Figure 8a, that is, we plot $\log(\lambda_1 + M_1\varepsilon)$ vs. $\log\varepsilon$ where the values for λ_1 were obtained using the numerical continuation. Also plotted (dashed red) is a straight line of slope 2.

Figure 8

agreement as $\varepsilon \to 0$. In addition, in Figure 8b, a log-log plot of the difference of the two curves in Figure 8a is plotted along with a straight line of slope 2. Asymptotically, there is good agreement between these two curves, which suggests that the difference between the numerically computed values for λ_1 and the approximation $\lambda_1 \approx -M_1\varepsilon$ is indeed higher order.

8.2.2 Nonhyperbolic regime

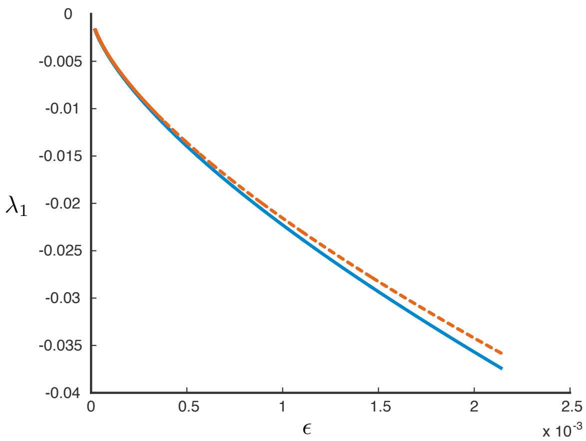
We next consider the nonhyperbolic regime. Take $K^* > 1/4$. By Theorem 2.1 and Remark 3.4, provided $a, \varepsilon > 0$ are sufficiently small with $K^* a^2 < \varepsilon$, the tail of the pulse solution is oscillatory. Hence for sufficiently small $\varepsilon > 0$, in the region of oscillatory pulses, one expects by Theorem 2.4 (ii) that the eigenvalue λ_1 of (8.3) becomes asymptotically $\mathcal{O}\left(\varepsilon^{2/3}\right)$. Using Matlab, we solve (8.2) numerically for the parameter values $(a, \varepsilon, \gamma) = (0.0059, 0.0021, 0.5)$ and obtain the oscillatory pulse solution shown in Figure 7b with wave speed $c = 0.6864$. We also solve the eigenvalue problem (8.3) and obtain a solution with eigenvalue $\lambda_1 = -0.0374$ and corresponding weighted eigenfunction which is plotted along with the pulse in Figure 7b. To determine the asymptotics of the eigenvalue λ_1 in the oscillatory regime, we now continue this solution letting $\varepsilon \to 0$ along the curve $\varepsilon = 61.9026a^2$ so that it holds $\varepsilon > K^* a^2$ along this curve. Note that we regard c again as a free parameter for the same reasons as in §8.2.1.

We compare the results of the continuation process with the results of Theorem 2.4. Along the curve $\varepsilon = 61.9026a^2$, for sufficiently small $a, \varepsilon > 0$, by Theorem 2.4 (ii) the eigenvalue is given by

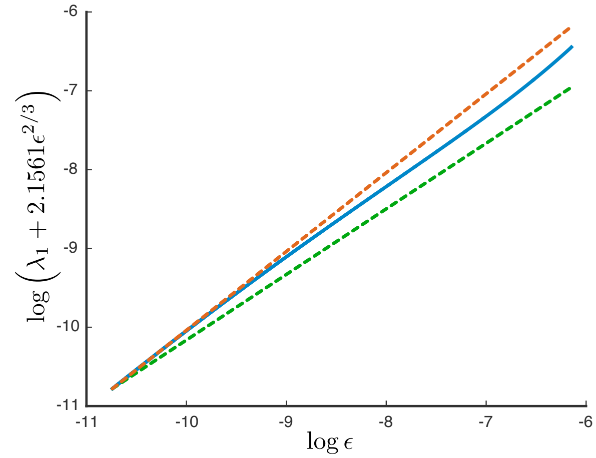
$$\lambda_1 = -\tfrac{1}{3}(18 - 4\gamma)^{2/3}\zeta_0\varepsilon^{2/3} + \mathcal{O}\left(\varepsilon^{5/6}\right) \approx -2.1561\varepsilon^{2/3}, \tag{8.5}$$

where we used that $\zeta_0 \approx 1.0187$.

The results of the continuation process are shown in Figure 9; in Figure 9a, the continuation of the eigenvalue λ_1 is plotted against ε in blue along with the first order approximation (8.5) in red. In Figure 9b, a log-log plot of the difference of the two curves in Figure 9 is plotted along with straight lines of slope 1 and 5/6. Asymptotically, the log of the difference lies between these two lines, which suggests that the difference between the numerically computed values for λ_1 and the approximation is indeed higher order.



(a) Plotted is the curve obtained for the continuation of the eigenvalue λ_1 as $\varepsilon \to 0$ in the oscillatory pulse case. Here we continue along the curve $\varepsilon = 61.9026a^2$, and the wave speed $c = \check{c}(a, \varepsilon)$ varies along the continuation. For comparison, we also plot the first order approximation (dashed red) $\lambda \approx -2.1561\varepsilon^{2/3}$ for the eigenvalue λ_1 from Theorem 2.4 (ii).



(b) Shown is a log-log plot of the differences (blue) of the two curves in Figure 9a, that is, we plot $\log\left(\lambda_1 + 2.1561\varepsilon^{2/3}\right)$ vs. $\log \varepsilon$ where the values for λ_1 were obtained using the numerical continuation. Also plotted are a straight line (dashed red) of slope 1 and a straight line (dashed green) of slope 5/6.

Figure 9

9 Discussion and outlook

In this paper, we proved the spectral and nonlinear stability of fast pulses with oscillatory tails that exist in the FitzHugh-Nagumo system

$$u_t = u_{xx} + u(u-a)(1-u) - w,$$
$$w_t = \varepsilon(u - \gamma w),$$

in the regime where $0 < a, \varepsilon \ll 1$. We showed that the linearization of this PDE about a fast pulse has precisely two eigenvalues near the origin when considered in an appropriate weighted function space. One of these eigenvalues λ_0 is situated at the origin due to translational invariance, and we proved that the second nontrivial eigenvalue λ_1 is real and strictly negative, thus yielding stability. Our proof also recovers the known result that fast pulses with monotone tails, which exist for fixed $0 < a < \frac{1}{2}$, are stable. Comparing the case of monotone versus oscillatory tails, there are some challenges present in the oscillatory case due to the nonhyperbolicity of the slow manifolds at the two fold points where the Nagumo front and back jump off to the other branches of the slow manifold. Our results show that these challenges are not just technical but rather result in qualitatively different behaviors. First, the fold at the equilibrium rest state facilitates the onset of the oscillations in the tails of the pulses. Second, the symmetry present due to the cubic nonlinearity means that the back has to jump off the other fold point. Due to the interaction of the back with this second fold point, the scaling of the critical eigenvalue λ_1 in the oscillatory case is given by $\varepsilon^{2/3}$, in contrast to the monotone case where it scales with ε. Moreover, the criterion that needs to be checked to ascertain the sign of λ_1 is different in these two cases.

Our proof of spectral stability is based on Lyapunov-Schmidt reduction, and, more specifically, on the approach taken in [16] to prove the stability of fast pulses with monotone tails for the discrete FitzHugh-Nagumo system. We begin with the linearization of the FitzHugh-Nagumo equation about the fast pulse and write the associated eigenvalue problem as

$$\psi_\xi = A(\xi, \lambda)\psi, \tag{9.1}$$

where $A(\xi, \lambda) \to \hat{A}(\lambda)$ as $|\xi| \to \infty$. The ξ-dependence in the matrix $A(\xi, \lambda)$ reflects the passage of the fast pulse along the front, through the right branch of the slow manifold, the jump-off at the upper-right knee along the back, and down the left branch of the slow manifold. Key to our approach is the fact that the spectrum of the matrix $A(\xi, \lambda)$ near the slow manifolds has a consistent splitting into one unstable and two center-stable eigenvalues, and that an exponential weight moves the center eigenvalue into the left half-plane. Eigenfunctions therefore correspond to solutions that decay exponentially as $\xi \to -\infty$, while they may grow algebraically or even with a small exponential rate (corresponding to the center-stable matrix eigenvalues) as $\xi \to \infty$. The splitting along the slow manifolds guarantees the existence of exponential dichotomies along the slow manifolds and shows that they cannot contribute point eigenvalues. The splitting allows us also to decide whether the front and the back will contribute eigenvalues. For the FitzHugh-Nagumo system, both will contribute because their derivatives decay exponentially as $\xi \to -\infty$ so that they emerge along the unstable direction. In contrast, for the cases studied in [1, 14], the back decays algebraically as $\xi \to -\infty$ and therefore emerges from the center-stable direction instead of the unstable direction as required for eigenfunctions: hence, the back does not contribute an eigenvalue. Thus, for FitzHugh-Nagumo, both front and back will contribute an eigenvalue, and our approach consists of constructing, for each prospective eigenvalue λ in the complex plane, a piecewise continuous eigenfunction of the linearization, that is a piecewise continuous solution to (9.1), where we allow for precisely two jumps that occur in the middle of the front and the back. Finding eigenvalues then reduces to identifying values of λ for which these jumps vanish. Melnikov theory allows us to find expressions for these jumps that can then be solved.

We emphasize that this approach applies to the more general situation of a pulse that is constructed by concatenating several fronts and backs with parts of the slow manifolds: as long as there is a consistent splitting of eigenvalues, we can decide which fronts and backs contribute an eigenvalue, and then construct prospective eigenfunctions with as many jumps as expected eigenvalues, where the jumps occur near the fronts and backs that contribute. Equation (9.1) will have exponential dichotomies along the slow manifolds and along the fronts and backs that do not contribute eigenvalues, which allows for a reduction to a finite set of jumps with expansions that can be calculated using Melnikov theory.

Our method provides a piecewise continuous eigenfunction for any prospective eigenvalue λ. Thus, by finding the eigenvalues λ for which the finite set of jumps vanishes, we have therefore determined the corresponding eigenfunctions. In our analysis, this amounts to the observation that eigenfunctions are found by piecing together multiples of the derivatives of the Nagumo front $\beta_{\mathrm{f}}\phi_{\mathrm{f}}'$ and back $\beta_{\mathrm{b}}\phi_{\mathrm{b}}'$, where the ratio of the amplitudes $(\beta_{\mathrm{f}}, \beta_{\mathrm{b}})$ is determined by the corresponding eigenvalue (see Remark 6.13). As expected, the eigenfunction corresponding to the translational eigenvalue $\lambda_0 = 0$ is represented by $(\beta_{\mathrm{f}}, \beta_{\mathrm{b}}) = (1, 1)$. Moreover, assuming the second eigenvalue $\lambda_1 < 0$ lies to the right of the essential spectrum, the corresponding eigenfunction is centered at the back as we have $(\beta_{\mathrm{f}}, \beta_{\mathrm{b}}) = (0, 1)$. The implications for the dynamics of the pulse profile under small perturbations are as follows. If a perturbation is localized near the back of the pulse, then it excites only the eigenfunction corresponding to λ_1, and the back will move with exponential rate back to its original position relative to the front without interacting with the front. On the other hand, perturbations that affect also the front will cause a shift of the full profile. These two mechanisms provide a detailed description of the way in which solutions near the traveling pulse converge over time to an appropriate translate of the pulse.

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Conflict of Interest: The authors declare that they have no conflict of interest.

A Corner estimates

In this section we provide a proof of Theorem 4.5, based on a theorem in [6], regarding the nature of solutions upon entry to a neighborhood of a slow manifold.

Proof of Theorem 4.5. This proof is based on an argument in [6]. In the box

$$\mathcal{U}_E' := \{(U, V, W) : U, V \in [-\Delta, \Delta], W \in [-\Delta, W^* + \Delta]\},$$

for sufficiently small $\varepsilon > 0$, there exist constants $\alpha_{\pm}^{u/s} > 0$ such that

$$0 < \alpha_-^s < \Lambda(U, V, W; c, a, \varepsilon) < \alpha_+^s,$$
$$0 < \alpha_-^u < \Gamma(U, V, W; c, a, \varepsilon) < \alpha_+^u,$$

We first consider the V-coordinate. For any $\xi > \xi_1$, we have

$$|V(\xi)| \geq |V(\xi_1)|\, e^{\alpha_-^u(\xi - \xi_1)}.$$

Since $V(\xi_2) \in N_2$, we also have

$$|V(\xi_1)| \leq \Delta e^{-\alpha_-^u(\xi_2 - \xi_1)}.$$

We note that since the solution enters $\mathcal{U}_E'$ via N_1 and reaches N_2 at $\xi_2(\varepsilon)$, using the equation for W in (3.9), we have that $\xi_2(\varepsilon)$ satisfies $\xi_2(\varepsilon) \geq (C\varepsilon)^{-1}$. Therefore, using the upper bound on Γ we have that

$$|V(\xi)| \leq \Delta e^{-\alpha_-^u \xi_2 + \alpha_+^u \xi - (\alpha_+^u - \alpha_-^u)\xi_1} \leq C e^{-\frac{1}{C\varepsilon}},$$

for $\xi \in [\xi_1, \Xi(\varepsilon)]$.

The solution in the slow W-component may be written as

$$W(\xi) = W(\xi_1, \varepsilon) + \int_{\xi_1}^{\xi} \varepsilon(1 + H(U(s), V(s), W(s), c, a, \varepsilon)U(s)V(s))ds,$$

from which we infer that

$$|W(\xi) - W(\xi_1, \varepsilon)| \leq C\varepsilon(\xi - \xi_1) \leq C\varepsilon\Xi(\varepsilon), \quad \text{for } \xi \in [\xi_1, \Xi(\varepsilon)],$$

and hence

$$|W(\xi)| \leq C\varepsilon\Xi(\varepsilon) + |W(\xi_1, \varepsilon)|, \quad \text{for } \xi \in [\xi_1, \Xi(\varepsilon)].$$

Finally we consider the U-component. We have that the difference $(U(\xi) - U_0(\xi))$ satisfies

$$\begin{aligned} U' - U_0' &= -(\Lambda(U, V, W, c, a, \varepsilon)U - \Lambda(U_0, 0, 0, c, a, 0)U_0) \\ &= -\Lambda(U_0, 0, 0, c, a, 0)(U - U_0) + O\left(\varepsilon + |U - U_0| + |V| + |W|\right) U. \end{aligned}$$

with $U(\xi_1) - U_0(\xi_1) = \tilde{U}_0$ where $|\tilde{U}_0| \ll \Delta$. By possibly taking Δ smaller if necessary and using the fact that the rate of contraction in the U-component is stronger than α_-^s, we deduce that $(U(\xi) - U_0(\xi))$ satisfies a differential equation

$$X' = b_1(\xi)X + b_2(\xi), \quad X(\xi_1) = \tilde{U}_0,$$

where $b_1(\xi) < -\alpha_-^s/2 < 0$ and

$$|b_2(\xi)| \leq C\left(\varepsilon\Xi(\varepsilon) + |W(\xi_1, \varepsilon)|\right) e^{-\alpha_-^s \xi},$$

for $\xi \in [\xi_1, \Xi(\varepsilon)]$. Hence, it holds

$$|U(\xi) - U_0(\xi)| \leq C\left(\varepsilon\Xi(\varepsilon) + |\tilde{U}_0| + |W(\xi_1, \varepsilon)|\right),$$

for $\xi \in [\xi_1, \Xi(\varepsilon)]$, which completes the proof. □

B Exponential dichotomies and trichotomies

It is well-known that exponential separation is an important tool in studying spectral properties of traveling waves [31]. Below we provide the definitions of exponential dichotomies and trichotomies to familiarize the reader with our notation. For an extensive introduction we refer to [5, 29].

Definition. Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x, y)$ the evolution operator of

$$\varphi_x = A(x)\varphi. \tag{B.1}$$

Equation (B.1) has an *exponential dichotomy* on J with constants $K, \mu > 0$ and projections $P^s(x), P^u(x)\colon \mathbb{C}^n \to \mathbb{C}^n$, $x \in J$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) = 1$;
- $P^{u,s}(x)T(x, y) = T(x, y)P^{u,s}(y)$;
- $\|T(x, y)P^s(y)\|, \|T(y, x)P^u(x)\| \le Ke^{-\mu(x-y)}$ for $x \ge y$.

Equation (B.1) has an *exponential trichotomy* on J with constants $K, \mu, \nu > 0$ and projections $P^u(x), P^s(x), P^c(x)\colon \mathbb{C}^n \to \mathbb{C}^n$, $x \in J$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) + P^c(x) = 1$;
- $P^{u,s,c}(x)T(x, y) = T(x, y)P^{u,s,c}(y)$;
- $\|T(x, y)P^s(y)\|, \|T(y, x)P^u(x)\| \le Ke^{-\mu(x-y)}$ for $x \ge y$;
- $\|T(x, y)P^c(y)\| \le Ke^{\nu|x-y|}$.

Often we use the abbreviations $T^{u,s,c}(x, y) = T(x, y)P^{u,s,c}(y)$ leaving the associated projections of the dichotomy or trichotomy implicit.

References

[1] M. Beck, C. K. R. T. Jones, D. Schaeffer, and M. Wechselberger. Electrical waves in a one-dimensional model of cardiac tissue. *SIAM J. Appl. Dyn. Syst.*, 7(4):1558–1581, 2008.

[2] G. Carpenter. A geometric approach to singular perturbation problems with applications to nerve impulse equations. *J. Differential Equations*, 23(3):335–367, 1977.

[3] P. Carter and B. Sandstede. Fast pulses with oscillatory tails in the FitzHugh–Nagumo system. *SIAM J. Math. Anal.*, 47(5):3393–3441, 2015.

[4] W. A. Coppel. Dichotomies and reducibility. *J. Differential Equations*, 3:500–521, 1967.

[5] W. A. Coppel. *Dichotomies in stability theory*, volume 629 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1978.

[6] E. G. Eszter. *An Evans function analysis of the stability of periodic travelling wave solutions of the FitzHugh-Nagumo system*. PhD thesis, University of Massachusetts, 1999.

[7] J. W. Evans. Nerve axon equations. I. Linear approximations. *Indiana Univ. Math. J.*, 21:877–885, 1971/72.

[8] J. W. Evans. Nerve axon equations. III. Stability of the nerve impulse. *Indiana Univ. Math. J.*, 22:577–593, 1972/73.

[9] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, 31(1):53–98, 1979.

[10] R. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical journal*, 1(6):445, 1961.

[11] G. Flores. Stability analysis for the slow travelling pulse of the FitzHugh-Nagumo system. *SIAM J. Math. Anal.*, 22(2):392–399, 1991.

[12] S. P. Hastings. On the existence of homoclinic and periodic orbits for the FitzHugh-Nagumo equations. *Quarterly J. Mathematics*, 27(1):123–134, 1976.

[13] S P Hastings. Single and multiple pulse waves for the fitzhugh-nagumo. *SIAM Journal on Applied Mathematics*, 42(2):247–260, 1982.

[14] M. Holzer, A. Doelman, and T. J. Kaper. Existence and stability of traveling pulses in a reaction–diffusion-mechanics system. *J. Nonlinear Sci.*, 23(1):129–177, 2013.

[15] A. J. Homburg and B. Sandstede. Homoclinic and heteroclinic bifurcations in vector fields. *Handbook of dynamical systems*, 3:379–524, 2010.

[16] H. J. Hupkes and B. Sandstede. Stability of pulse solutions for the discrete FitzHugh-Nagumo system. *Trans. Amer. Math. Soc.*, 365(1):251–301, 2013.

[17] C. K. R. T. Jones. Stability of the travelling wave solution of the FitzHugh-Nagumo system. *Trans. Amer. Math. Soc.*, 286(2):431–469, 1984.

[18] C. K. R. T. Jones, T. J. Kaper, and N. Kopell. Tracking invariant manifolds up to exponentially small errors. *SIAM J. Mathematical Analysis*, 27(2):558–577, 1996.

[19] C. K. R. T. Jones, N. Kopell, and R. Langer. Construction of the FitzHugh-Nagumo pulse using differential forms. In *Patterns and dynamics in reactive media*, pages 101–115. Springer, 1991.

[20] T. Kapitula and K. Promislow. *Spectral and dynamical stability of nonlinear waves*, volume 185 of *Applied Mathematical Sciences*. Springer, New York, 2013. With a foreword by C. K. R. T. Jones.

[21] M. Krupa, B. Sandstede, and P. Szmolyan. Fast and slow waves in the FitzHugh–Nagumo equation. *Journal of Differential Equations*, 133(1):49–97, 1997.

[22] M Krupa and P Szmolyan. Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions. *SIAM J. Mathematical Analysis*, 33(2):286–314, 2001.

[23] M Krupa and P Szmolyan. Relaxation oscillation and canard explosion. *J. Differential Equations*, 174(2):312–368, 2001.

[24] N. Levinson. The asymptotic nature of solutions of linear systems of differential equations. *Duke Math. J.*, 15:111–126, 1948.

[25] X. B. Lin. Using Mel′nikov's method to solve Šilnikov's problems. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(3-4):295–325, 1990.

[26] E. Mishchenko. *Differential equations with small parameters and relaxation oscillations*, volume 13. Springer Science & Business Media, 2013.

[27] J. Nagumo, S. Arimoto, and S. Yoshizawa. An active pulse transmission line simulating nerve axon. *Proceedings of the IRE*, 50(10):2061–2070, 1962.

[28] K. J. Palmer. Exponential dichotomies and transversal homoclinic points. *J. Differential Equations*, 55(2):225–256, 1984.

[29] B. Sandstede. *Verzweigungstheorie homokliner Verdopplungen*. PhD thesis, University of Stuttgart, 1993.

[30] B. Sandstede. Stability of multiple-pulse solutions. *Trans. Amer. Math. Soc.*, 350(2):429–472, 1998.

[31] B. Sandstede. Stability of travelling waves. In *Handbook of dynamical systems*, volume 2, pages 983–1055. North-Holland, Amsterdam, 2002.

[32] B. Sandstede and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Phys. D*, 145(3-4):233–277, 2000.

[33] E. Yanagida. Stability of fast travelling pulse solutions of the FitzHugh-Nagumo equations. *J. Math. Biol.*, 22(1):81–104, 1985.