SET-THEORETIC METHODS IN MODULE THEORY

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(corrected version)

R is an associative ring with identity.
"module" means left R-module.

Filtrations

Definitions.

1. A filtration of a module A is a continuous chain of submodules $\{A_\alpha : \alpha < \sigma\}$ of A whose union is A such that $A_0 = 0$ and $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ for all limit ordinals $\beta < \sigma$.
2. Let $\mathcal{C}$ be a class of modules.
A module A is said to be $\mathcal{C}$-filtered if it has a filtration as above s.t. , $A_{\alpha+1}/A_\alpha \in \mathcal{C}$ for all $\alpha + 1 < \sigma$.

Examples.

1. Every module A has a filtration $\{A_\alpha : \alpha < \sigma\}$ where $\sigma =$ the size of a minimal generating set and each A_α is $< \sigma$-generated.
2. Every free module is $\{R\}$-filtered. (And conversely.)

Projective modules

Theorem. (Kaplansky)

A module is projective if and only if it is $\mathcal{C}$-filtered, where $\mathcal{C}$ is the set of countably-generated projective modules.

Recall: P is projective iff it is a direct summand of a free module iff for every epimorphism $f : M \longrightarrow N$ and every homomorphism $g : P \longrightarrow N$, there is $h : P \longrightarrow M$ such that $g = f \circ h$.

Definitions

A class $\mathcal{A}$ of modules is κ-deconstructible if every module in $\mathcal{A}$ is $\mathcal{C}$-filtered, where $\mathcal{C}$ is the set of $\leq \kappa$-generated elements of $\mathcal{A}$.
$\mathcal{A}$ is deconstructible (or bounded) if it is κ-deconstructible for some κ.

Singular Compactness

Shelah's Singular Compactness Theorem (v.1)

Let λ be a singular cardinal and M an R-module which is $\leq \lambda$-generated.
Let R be a p.i.d.
Assume that for every regular cardinal $\kappa < \lambda$, every $< \kappa$-generated submodule of M is free.
Then M is free.

(λ is *singular* if the cofinality of λ is $< \lambda$ iff there is a strictly increasing sequence $\{\mu_\nu : \nu < \tau\}$ of cardinals $< \lambda$ and length $\tau < \lambda$ whose supremum is λ.)

Singular Compactness, version 2

Shelah's Singular Compactness Theorem (v.2)

Let λ be a singular cardinal and M an R-module which is $\leq \lambda$-generated.
Let R be an hereditary ring
Assume that for every regular cardinal $\kappa < \lambda$, every $< \kappa$-generated submodule of M is projective.
Then M is projective.

(λ is *singular* if the cofinality of λ is $< \lambda$ iff there is a strictly increasing sequence $\{\mu_\nu : \nu < \tau\}$ of cardinals $< \lambda$ and length $\tau < \lambda$ whose supremum is λ.)

(R is *hereditary* if every submodule of a projective module is projective iff every left ideal is projective.)

Singular Compactness, version 3

Shelah's Singular Compactness Theorem (v.3)

Let λ be a singular cardinal and M an R-module which is $\leq \lambda$-generated.
Let R be any ring
Assume that for every regular cardinal $\kappa < \lambda$, "enough" $< \kappa$-generated submodules of M are projective.
Then M is projective.

" enough": There is a set S_κ of $< \kappa$-generated projective submodules of M such that every subset of M of cardinality $< \kappa$ is contained in a member of S_κ; and S_κ is closed under unions of well-ordered chains of length $< \kappa$.

Singular Compactness, version 4

Let $\mathcal{C}$ be a set of countably-presented modules.

Shelah's Singular Compactness Theorem (v.4)

Let λ be a singular cardinal and M an R-module which is $\leq \lambda$-generated.
Let R be any ring
Assume that for every regular cardinal $\kappa < \lambda$, "enough" $< \kappa$-generated submodules of M are $\mathcal{C}$-filtered.
Then M is $\mathcal{C}$-filtered.

" enough": There is a set S_κ of $< \kappa$-generated $\mathcal{C}$-filtered submodules of M such that every subset of M of cardinality $< \kappa$ is contained in a member of S_κ; and S_κ is closed under unions of well-ordered chains of length $< \kappa$.

(Recall: A is $\mathcal{C}$-filtered if it has a filtration s.t. , $A_{\alpha+1}/A_\alpha \in \mathcal{C}$ for all α.)

Singular Compactness, version 5

Let $\mathcal{C}$ be a set of $\leq \mu$-presented modules.

Shelah's Singular Compactness Theorem (v.5)

Let λ be a singular cardinal $> \mu$ and M an R-module which is $\leq \lambda$-generated.
Let R be any ring
Assume that for every regular cardinal $\kappa < \lambda$ and $> \mu$, "enough" $< \kappa$-generated submodules of M are $\mathcal{C}$-filtered.
Then M is $\mathcal{C}$-filtered.

" enough": There is a set S_κ of $< \kappa$-generated $\mathcal{C}$-filtered submodules of M such that every subset of M of cardinality $< \kappa$ is contained in a member of S_κ; and S_κ is closed under unions of well-ordered chains of length $< \kappa$.

(Recall: A is $\mathcal{C}$-filtered if it has a filtration s.t. , $A_{\alpha+1}/A_\alpha \in \mathcal{C}$ for all α.)

Classes defined by Ext

$\mathcal{S}$ a class of modules

Definitions

$^{\perp}\mathcal{S} = \{N \mid \mathrm{Ext}^1(N, M) = 0 \text{ for all } M \in \mathcal{S}\}$

$\mathcal{S}^{\perp} = \{N \mid \mathrm{Ext}^1(M, N) = 0 \text{ for all } M \in \mathcal{S}\}$

Recall:
$\mathrm{Ext}^1(A, B) = 0$ iff every short exact sequence

$$0 \to B \to M \to A \to 0$$

splits,
i.e., up to isomorphism the only one is

$$0 \to B \to B \oplus A \to A \to 0$$

Examples/Definitions.

- $\{Projective\ modules\} = {}^{\perp}\{all\ modules\}$
- $\{Injective\ modules\} = \{all\ modules\}^{\perp}$
- ${}^{\perp}\{R\}$ = the class of Whitehead modules
- (R an ID)
  ${}^{\perp}\{\text{torsion modules}\}$ = the class of Baer modules

Deconstructibility

Let $\mathcal{A} = {}^{\perp}\mathcal{B}$.
Fact: If A is the union of a filtration $\{A_\alpha : \alpha < \sigma\}$ such that $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ for all $\alpha < \sigma$, then $A \in \mathcal{A}$.

The key question

Can we reduce knowledge of members of $\mathcal{A}$ to knowledge of its "small" members?
i.e., is $\mathcal{A}$ bounded?
i.e., is there κ such that every $A \in \mathcal{A}$ is κ-deconstructible?
i.e., is every every $A \in \mathcal{A}$ the union of a continuous chain of submodules $\{A_\alpha : \alpha < \sigma\}$ such that each $A_{\alpha+1}/A_\alpha$ is $\leq \kappa$-generated and belongs to $\mathcal{A}$?

The regular case

Let κ be a regular cardinal.

The *regular* theorem. (v.1 Shelah)

Assume $V = L$. Let R be a hereditary ring and let $\mathcal{A} = {}^{\perp}\{N\}$ and $\kappa > |R| + |N| + \aleph_0$.
If A is $\leq \kappa$-generated and has a filtration $\{A_\alpha : \alpha < \kappa\}$ of $< \kappa$-generated submodules belonging to $\mathcal{A}$, then
there is a subfiltration $\{A_{f(\alpha)} : \sigma < \kappa\}$ such that $A_{f(\alpha+1)}/A_{f(\alpha)} \in \mathcal{A}$ for all $\alpha < \kappa$. (Here $f : \kappa \to \kappa$ is a continuous increasing function.)

Application: Whitehead groups

Theorem. (Shelah, et. al.)

Assume V = L. If R is a hereditary ring, then for any R-module N, ${}^{\perp}\{N\}$ is $|R| + |N| + \aleph_0$-deconstructible.

Consequences

(i) (1973) Assuming V = L, every Whitehead group ($\mathbb{Z}$-module) is free.
(ii) Assuming V = L, if R is a p.i.d. of cardinality $\leq \aleph_1$ which is not a complete discrete valuation ring, then every Whitehead R-module is free.

Remarks.
(i) and (ii) are not provable in ZFC.
For some p.i.d's of cardinality $\geq \aleph_2$, the conclusion of (ii) is provably false (in ZFC).
(iii) Saroch - Trlifaj: can replace "R is hereditary" by "$\mathcal{A}$ is closed under pure submodules."

Regular case, version 2 (in ZFC)

$\mathcal{A} = {}^{\perp}\mathcal{B}$ where $\mathcal{B}$ is closed under direct sums.

The *regular* theorem. (v.2 E-Fuchs-Shelah)

Assume $\mathcal{B}$ is closed under arbitrary direct sums. If A is $\leq \kappa$-generated and has a filtration $\{A_\alpha : \alpha < \kappa\}$ of $< \kappa$-generated submodules belonging to $\mathcal{A}$ and s.t. proj. dim.$(A + \alpha/A_{\alpha+1}) \leq 1$, then
there is a subfiltration $\{A_{f(\alpha)} : \sigma < \kappa\}$ such that $A_{f(\alpha+1)}/A_{f(\alpha)} \in \mathcal{A}$ for all $\alpha < \kappa$.

A word about the proof

A subset S of κ is called stationary if it has non-empty intersection with the range of every continuous increasing $f : \kappa \to \kappa$.

It suffices to prove that

$$S \overset{def}{=} \{\alpha < \kappa : \exists \mu_\alpha > \alpha \text{ s.t. } A_{\mu_\alpha}/A_\alpha \notin \mathcal{A}\}$$

is *not* stationary, for then any f missing S will do.

Aiming for a contradiction, we assume S is stationary and show $\mathrm{Ext}^1(A, B) \neq 0$ for some $B \in \mathcal{B}$ ($B = N$ in v. 1), i.e. $A \notin {}^\perp\mathcal{B}$.

Version 2: W.l.o.g. if $\alpha \in S$, $\mu_\alpha = \alpha + 1$, i.e. there is $B_\alpha \in \mathcal{B}$ such that $\mathrm{Ext}^1(A_{\alpha+1}/A_\alpha, B_\alpha) \neq 0$.
We construct a non-splitting short exact sequence

$$0 \to \oplus_{\beta \in S} B_\beta \to M \to A \to 0$$

as the union of a chain

$$0 \to \oplus_{\beta < \nu} B_\beta \to M_\nu \to A_\nu \to 0.$$

If there were a splitting $g : A \to M$ of

$$0 \to \oplus_{\beta \in S} B_\beta \to M \to A \to 0$$

there would be $\alpha \in S$ such that $g \restriction A_\alpha : A_\alpha \to M_\alpha$.
But we choose

$$0 \to \oplus_{\beta < \alpha+1} B_\beta \to M_{\alpha+1} \to A_{\alpha+1} \to 0$$

to prevent any such from extending.

In Version 1, we must construct a chain

$$0 \to N \to M_\nu \to A_\nu \to 0$$

and we must use the $\diamondsuit$ prediction principles to predict and kill a particular splitting : $A_\alpha \to M_\alpha$ for each $\alpha \in S$.

Application: Baer modules

Baer (1936): countable Baer $\mathbb{Z}$-modules are free.
Griffith (1969): all Baer $\mathbb{Z}$-modules are free.
Kaplansky (1962): Baer modules over arbitrary IDs are flat and of proj. dim ≤ 1. Are they all projective?

Consequence (1990)

(R an ID) The class of Baer modules is $\aleph_0$-deconstructible. Hence, if every countably generated Baer R-module is projective, then every Baer R-module is projective.

(Recall: A is Baer if $A \in {}^{\perp}\{\text{torsion modules}\}$.)

Theorem. (Angeleri Hugel-Bazzoni-Herbera 2005)

Every countably generated Baer module over an arbitrary ID is projective. Hence, every Baer module is projective.

Regular case, version 3

$\mathcal{A} = {}^{\perp}\mathcal{B}$ where $\mathcal{B}$ is closed under direct sums.

The *regular* theorem. (v.3 Šťovíček-Trlifaj)

Assume $\mathcal{B}$ is closed under arbitrary direct sums. If A is $\leq \kappa$-generated and has a filtration $\{A_\alpha : \alpha < \kappa\}$ of $< \kappa$-generated submodules belonging to $\mathcal{A}$, then
there is a subfiltration $\{A_{f(\alpha)} : \sigma < \kappa\}$ such that $A_{f(\alpha+1)}/A_{f(\alpha)} \in \mathcal{A}$ for all $\alpha < \kappa$.

Cotorsion pairs

Definition. (Salce 1979)

A cotorsion pair is a pair of classes of modules $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$

Examples

- (Projectives, R-modules)
- (R-modules, Injectives)
- $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$
  $({}^{\perp}\mathcal{S}, ({}^{\perp}\mathcal{S})^{\perp})$
- *Case of* $\mathcal{S} = \{$Pure-injective modules$\}$:
  $\{$*Flat modules*$\} = {}^{\perp}\mathcal{S}$
  $\{$*Flat modules*$\}^{\perp} = \{$*(Enoch) Cotorsion modules*$\}$

  (Flat, Cotorsion) is a cotorsion pair.

Tilting cotorsion pairs

Definition/Theorem

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *n-tilting* if

- Every element of $\mathcal{A}$ is of proj. dim. $\leq n$;
- $\text{Ext}^i(A, B) = 0$ for all $i \geq 2$ and all $A \in \mathcal{A}, B \in \mathcal{B}$; and
- $\mathcal{B}$ is closed under direct sums.

Example/Theorem

Let $\mathcal{S}$ be a set of modules (of proj. dim $\leq n$) such that each member M of $\mathcal{S}$ has a projective resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$$

where each P_ℓ is **finitely-generated**.
Let $\mathcal{B} = \mathcal{S}^{\perp_\infty} \stackrel{def}{=} \{N \mid \text{Ext}^i(M, N) = 0 \text{ for all } M \in \mathcal{S} \text{ and all } i \geq 1\}$.
Then $({}^\perp\mathcal{B}, \mathcal{B})$ is an n-tilting cotorsion pair of finite type.

There is an analogous definition for countable type.

Key question

Is every tilting cotorsion pair of finite type?

Theorems

1. (Bazzoni-E-Trlifaj 2003) All 1-tilting pairs are of countable type.
2. (Bazzoni-Herbera 2005) All 1-tilting pairs are of finite type.
3. (Šťovíček-Trlifaj 2005) All n-tilting cotorsion pairs are of countable type.
4. (Bazzoni-Šťovíček 2005) All n-tilting cotorsion pairs are of finite type.

Complete cotorsion pairs

Definition. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete if it has enough projectives and injectives, i.e., for every module M, there is a short exact sequence

$$0 \to B \to A \to M \to 0$$

such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$; or, equivalently (Salce), for every module M, there is a short exact sequence

$$0 \to M \to B \to A \to 0$$

such that $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Theorem. (E-Trlifaj)

The cotorsion pair $(\mathcal{A}, \mathcal{B})$ is complete if it is *generated by a set*, i.e., $\mathcal{B} = \{M\}^{\perp}$ for some module M.

Application: Flat covers

$(\mathcal{A}, \mathcal{B})$ is generated by a set iff $\mathcal{A}$ is deconstructible.
(Enochs for *if*; Šťovíček-Trlifaj for *only if*)

Theorem (Enochs 2000)

The (Flat, Cotorsion) pair is deconstructible,
hence complete, so flat covers exist for modules over any ring.

$R = \mathbb{Z}$

Theorems.

1. (E-Trlifaj) $(\mathcal{A}, \mathcal{B})$ is deconstructible, and hence complete, if every member of $\mathcal{B}$ is cotorsion.

2. (E-Shelah-Trlifaj) It is consistent with ZFC + GCH that for every N which is not cotorsion, the cotorsion pair $({}^{\perp}\{N\}, ({}^{\perp}\{N\})^{\perp})$ is not generated by a set, hence not deconstructible.

3. (E-Shelah) It is consistent with ZFC + GCH that $({}^{\perp}\{\mathbb{Z}\}, ({}^{\perp}\{\mathbb{Z}\})^{\perp})$ is not complete.