COVERS INDUCED BY EXT

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Abstract. We prove a generalization of the Flat Cover Conjecture by showing for any ring R that (1) each (right R-) module has a Ker $\mathrm{Ext}(-,\mathcal{C})$-cover, for any class of pure-injective modules $\mathcal{C}$, and that (2) each module has a Ker $\mathrm{Tor}(-,\mathcal{B})$-cover, for any class of left R-modules $\mathcal{B}$.

For Dedekind domains, we describe Ker $\mathrm{Ext}(-,\mathcal{C})$ explicitly for any class of cotorsion modules $\mathcal{C}$; in particular, we prove that (1) holds, and that Ker $\mathrm{Ext}(-,\mathcal{C})$ is a cotilting torsion-free class. For right hereditary rings, we prove the consistency of the existence of special Ker $\mathrm{Ext}(-,\mathcal{G})$-precovers for any set of modules $\mathcal{G}$.

1. Introduction

A classical result of Eckmann-Schopf says that if $\mathcal{I}$ is the class of all injective (right R-) modules, then each module has an $\mathcal{I}$-envelope. Bass proved that if $\mathcal{P}$ is the class of all projective modules, then each module has a $\mathcal{P}$-cover iff R is a right perfect ring. Bass' result is often interpreted as a lack of duality for modules over non-right perfect rings.

Call a module M a *dual module* provided that there are a ring S, an R,S-bimodule N, and an injective cogenerator, Q, of Mod-S such that $M \cong \mathrm{Hom}_S(N,Q)$ (as right R-modules). There are two important instances of dual modules: if $S = \mathbb{Z}$ and $Q = \mathbb{Q}/\mathbb{Z}$, then the dual module is called the *character module* of the left R-module N; if R is a k-algebra over a field k and $S = Q = k$, then any module M which is finite dimensional over k is a dual module.

A well-known result (cf. [15]) says that the dual module M is injective iff N is a flat left R-module. So a natural candidate for dualizing the Eckmann-Schopf result (to arbitrary rings) is obtained by replacing $\mathcal{P}$ by $\mathcal{F}$, the class of all flat modules. This led Edgar Enochs [11] to formulate the *Flat Cover Conjecture (FCC): "every module over every ring has an $\mathcal{F}$-cover"*. Only recently the conjecture has been proved, independently by Enochs and El Bashir [4].

Enochs' proof proceeds by showing that the hypothesis of Corollary 11 of the authors' [10] is true for any ring R. The heart of his argument is a proof that there is a cardinal κ (depending only on R) such that every flat R-module A is the union of an increasing continuous sequence $(A_\alpha \mid \alpha < \sigma)$ of pure submodules (for some σ depending on A) such that for all $\alpha + 1 < \sigma$, $\mathrm{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$ and $A_{\alpha+1}/A_\alpha$ is

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flat. The hypothesis of [10, Cor. 11] then follows as in the proof of Corollary 10 below.

This motivates the following definition

Definition 1. For any right R-module A and any cardinal κ, a *κ-refinement of A* (of length σ) is an increasing sequence $(A_\alpha \mid \alpha \leq \sigma)$ of pure submodules of A such that $A_0 = 0$, $A_\sigma = A$, $A_\alpha = \cup_{\beta<\alpha} A_\beta$ for all limit ordinals $\alpha \leq \sigma$, and $\text{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$ for all $\alpha + 1 \leq \sigma$.

In homological terms, the FCC says that every module has an $\mathcal{A}$-cover, where $\mathcal{A}$ is the kernel of the contravariant Ext functor $\text{Ext}(-, \mathcal{C})$ and $\mathcal{C}$ is the class of *all* dual modules (or, respectively, $\mathcal{A}$ is the kernel of the covariant Tor functor $\text{Tor}(-, \mathcal{B})$ and $\mathcal{B}$ is the class of *all* left R-modules. Precise definitions are given in the next section.)

Using κ-refinements, we will generalize the FCC by replacing $\mathcal{C}$ by *any* class of *pure-injective* modules (resp., replacing $\mathcal{B}$ by *any* class of left R-modules. See Corollaries 10 and 11).

In Theorem 16, we prove that $\mathcal{C}$ can be *any* class of *cotorsion* modules when R is a Dedekind domain; in that case, we also give a full description of the kernel. Assuming Gödel's Axiom of Constructibility (V = L), we prove existence of special Ker $\text{Ext}(-, \mathcal{G})$-precovers for *any set* of modules $\mathcal{G}$ provided that R is a right hereditary ring (Theorem 14).

2. Preliminaries

For a ring R, denote by Mod-R the category of all right R-modules. We will use "module" to mean right R-module. Also, Hom, Ext and Tor will stand for Hom_R, Ext^1_R and Tor^R_1, respectively.

Definition 2. Let $\mathcal{C} \subseteq \text{Mod-}R$ and let $\mathcal{B}$ be a class of left R-modules. We define

$$^\perp\mathcal{C} = \text{Ker Ext}(-, \mathcal{C}) = \{D \mid \text{Ext}(D, C) = 0 \text{ for all } C \in \mathcal{C}\},$$

and similarly

$$\mathcal{C}^\perp = \text{Ker Ext}(\mathcal{C}, -) = \{D \mid \text{Ext}(C, D) = 0 \text{ for all } C \in \mathcal{C}\},$$

$$\text{Ker Tor}(-, \mathcal{B}) = \{A \mid \text{Tor}(A, B) = 0 \text{ for all } B \in \mathcal{B}\}.$$

For a module C, we will write $^\perp C$ instead of $^\perp\{C\}$.

We start by recalling a lemma relating κ-refinements to the vanishing of Ext:

Lemma 3. *Let C be a module. Suppose that $A = A_\mu$ is the union of a continuous chain of submodules, $A = \cup_{\alpha<\mu} A_\alpha$ such that $A_0 \in {}^\perp C$ and for all $\alpha + 1 < \mu$, $A_{\alpha+1}/A_\alpha \in {}^\perp C$. Then $A \in {}^\perp C$.*

PROOF. Well-known (see [7, Thm. 1.2], [12, Lemma IV.2.1], or [10, Lemma 1]). □

We will often use the following notions and facts concerning precovers and covers:

Definition 4. Let $\mathcal{A} \subseteq \text{Mod-}R$ and $M \in \text{Mod-}R$.

A homomorphism $\phi \in \text{Hom}(A, M)$ with $A \in \mathcal{A}$ is called an *$\mathcal{A}$-precover* of M if the induced map $\text{Hom}(A', A) \to \text{Hom}(A', M)$ is surjective for all $A' \in \mathcal{A}$. An $\mathcal{A}$-precover $\phi \in \text{Hom}(A, M)$ is an *$\mathcal{A}$-cover* provided that each $\psi \in \text{Hom}(A, A)$ satisfying $\phi = \phi\psi$ is an automorphism of A.[1]

A precover ϕ is called *special* provided that $\text{Ker}(\phi) \in \mathcal{A}^{\perp}$ and ϕ is surjective.

A *(special) $\mathcal{A}$-preenvelope* and an *$\mathcal{A}$-envelope* are defined dually; see [22, §1.2].

If $\phi : A \to M$ is surjective, $A \in \mathcal{A}$ and $\text{Ker}(\phi) \in \mathcal{A}^{\perp}$, then ϕ is a special $\mathcal{A}$-precover of M (see [22, 2.1.3]). Furthermore, by [22, 2.1.1 and 2.2.12], we have:

Theorem 5. *Let $\mathcal{A} \subseteq$ Mod-R be a class containing all projective modules and closed under direct limits and extensions. Assume that a module M has an $\mathcal{A}$-precover. Then M has an $\mathcal{A}$-cover, the $\mathcal{A}$-cover is special, and it is uniquely determined up to isomorphism.*

A submodule A of a module B is a *pure* submodule ($A \subseteq_* B$, for short) if for each finitely presented module F, the functor $\text{Hom}(F, -)$ preserves exactness of the sequence $0 \to A \to B \to B/A \to 0$. (See, for example, [17, pp. 53f] or [14, pp. 94ff].) We will need the following properties of pure submodules:

Lemma 6. *Let R be a ring and let $\kappa \geq$ card$(R) + \aleph_0$.*

(i) Let M be a module and X be a subset of M with card$(X) \leq \kappa$. Then there is a pure submodule $N \subseteq_* M$ such that $X \subseteq N$ and card$(N) \leq \kappa$.

(ii) Assume $C \subseteq_* A$ and $B/C \subseteq_* A/C$. Then $B \subseteq_* A$.

(iii) Assume $A_0 \subseteq \dots \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq \dots$ is a chain of pure submodules of M. Then $\cup_\alpha A_\alpha$ is a pure submodule of M.

PROOF. Well-known (see [14, Theorem 6.4]). □

For convenience we state a consequence of [10, Theorem 10] in the terminology of this paper.

Theorem 7. *If $\mathcal{C}$ is a class of modules such that $({}^{\perp}\mathcal{C})^{\perp} = Q^{\perp}$ for some module Q, then every module has a special ${}^{\perp}\mathcal{C}$-precover.*

PROOF. This follows from [10, Theorem 10] because $({}^{\perp}\mathcal{C}, ({}^{\perp}\mathcal{C})^{\perp})$ is a cotorsion theory and to say that it has enough projectives is to say that every module has a special ${}^{\perp}\mathcal{C}$-precover. □

3. Covers induced by Ext and Tor

Modules that are injective with respect to pure embeddings are called *pure-injective* [14, §7]. For example, any dual module is pure injective.

Theorem 8. *Let R be a ring and $\mathcal{C}$ be a class of pure-injective modules. Let $\kappa =$ card$(R) + \aleph_0$. Then the following conditions are equivalent for any module A:*

(i) $A \in {}^{\perp}\mathcal{C}$;

(ii) there is a cardinal λ such that A has a κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$ with $A_{\alpha+1}/A_\alpha \in {}^{\perp}\mathcal{C}$ for all $\alpha < \lambda$.

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[1] Here, we follow the terminology of Enochs and Xu [22]. The corresponding terminology of Auslander, Reiten and Smalø (e.g. in [2]) is that of a right approximation and a minimal right approximation.

PROOF. (i) implies (ii): Let $\kappa = \operatorname{card}(R) + \aleph_0$. If $\operatorname{card}(A) \leq \kappa$, we can let $\lambda = 1$, $A_0 = 0$, and $A_1 = A$. So we can assume that $\operatorname{card}(A) > \kappa$. Let $\lambda = \operatorname{card}(A)$. Then $A \cong F/K$ where $F = R^{(\lambda)}$ is a free module. We enumerate the elements of F in a λ-sequence: $F = \{x_\alpha \mid \alpha < \lambda\}$. By induction on α, we will define a sequence $(A_\alpha \mid \alpha \leq \lambda)$ so that for all $\alpha \leq \lambda$, A_α is pure in A and belongs to ${}^\perp\mathcal{C}$. Since each $C \in \mathcal{C}$ is pure-injective, it will follow from the long exact sequence induced by

$$0 \to A_\alpha \to A_{\alpha+1} \to A_{\alpha+1}/A_\alpha \to 0$$

that $A_{\alpha+1}/A_\alpha \in {}^\perp\mathcal{C}$ for all $\alpha < \lambda$.

A_α will be constructed so that it equals $(R^{(I_\alpha)} + K)/K$ for some $I_\alpha \subseteq \lambda$ such that $R^{(I_\alpha)} \cap K$ is pure in K. Let $A_0 = 0$. Assume A_β has been defined for all $\beta < \sigma$. Suppose first that $\sigma = \alpha + 1$. By induction on $n < \omega$ we will define an increasing chain $F_0 \subseteq F_1 \subseteq \dots$ and then put $A_{\alpha+1} = \cup_{n<\omega}(F_n + K)/K$. We require that $\operatorname{card}(F_{n+1}/F_n) \leq \kappa$ for all $n < \omega$, and furthermore: for n odd, $(F_n + K)/K$ is pure in F/K; for n even, $F_n = R^{(J_n)}$ for some $J_n \supseteq J_{n-2} \supseteq \cdots \supseteq J_0$ and $F_n \supseteq K'_n$ where $F_{n-1} \cap K \subseteq K'_n \subseteq_* K$.

First, put $F_0 = R^{(I_\alpha)}$ and let $J_0 = I_\alpha$ and $K'_0 = R^{(I_\alpha)} \cap K$. Assume F_{n-1} has been constructed and n is odd. By part (i) of Lemma 6 there is a pure submodule $(F_n + K)/(F_{n-2} + K) \subseteq_* F/(F_{n-2} + K)$ of cardinality $\leq \kappa$ containing $(x_\alpha R + F_{n-1} + K)/(F_{n-2} + K)$. Moreover, we can choose F_n so that $\operatorname{card}(F_n/F_{n-1}) \leq \kappa$. By part (ii) of Lemma 6, $(F_n + K)/K$ is pure in F/K.

Assume $n > 0$ is even. We first define K'_n: by part (i) of Lemma 6, we find a pure submodule $K'_n/K'_{n-2} \subseteq_* K/K'_{n-2}$ of cardinality $\leq \kappa$ containing $(F_{n-1} \cap K)/K'_{n-2}$. This is possible since $K'_{n-2} \supseteq F_{n-3} \cap K$ and $(F_{n-1} \cap K)/(F_{n-3} \cap K)$ embeds in F_{n-1}/F_{n-3}, so it has cardinality $\leq \kappa$. By part (ii) of Lemma 6, we have $K'_n \subseteq_* K$.

We can choose $J_n \subseteq \lambda$ such that $\operatorname{card}(J_n - J_{n-2}) \leq \kappa$ and $F_{n-1} + K'_n \subseteq R^{(J_n)} = F_n$. This is possible since $\operatorname{card}((F_{n-1} + K'_n)/F_{n-2}) \leq \kappa$; indeed, we have the exact sequence

$$0 \to F_{n-1}/F_{n-2} \to (F_{n-1} + K'_n)/F_{n-2} \to (F_{n-1} + K'_n)/F_{n-1} \to 0$$

and $(F_{n-1} + K'_n)/F_{n-1} \cong K'_n/(F_{n-1} \cap K)$ has cardinality $\leq \kappa$ because it is a homomorphic image of K'_n/K'_{n-2}.

Now, define $A_{\alpha+1} = \cup_{n<\omega}(F_n + K)/K$ and $I_{\alpha+1} = \cup_{n<\omega} J_{2n}$. By part (iii) of Lemma 6, $A_{\alpha+1} \subseteq_* A$. Clearly, $\operatorname{card}(A_{\alpha+1}/A_\alpha) \leq \kappa$.

We have $A_{\alpha+1} \cong F'/K'$, where $F' = \cup_{n<\omega} F_{2n}$ and $K' = F' \cap K$. Also, $F' = R^{(I_{\alpha+1})}$ is free, and $K' = \cup_{n<\omega} K'_{2n}$ is pure in K by construction and part (iii) of Lemma 6.

Let $C \in \mathcal{C}$. In order to prove that $\operatorname{Ext}(A_{\alpha+1}, C) = 0$, we have to extend any $f \in \operatorname{Hom}(K', C)$ to an element of $\operatorname{Hom}(F', C)$. First, f extends to K, since $K' \subseteq_* K$ and C is pure-injective. By the assumption (i), we can extend further to F, and then restrict to F'.

Finally, if $\sigma \leq \lambda$ is a limit ordinal, let $A_\sigma = \cup_{\beta<\sigma} A_\beta$; that A_σ has the desired properties follows from Lemma 3 and part (iii) of Lemma 6.

(ii) implies (i): This is clear by Lemma 3. □

Lemma 9. *If $\mathcal{A} \subseteq$ Mod-R is equal to ${}^\perp\mathcal{C}$ for a class $\mathcal{C}$ of pure-injective modules, then every module M which has an $\mathcal{A}$-precover has an $\mathcal{A}$-cover.*

PROOF. This follows from Theorem 5 and the following observation of Angeleri-Mantese-Tonolo-Trlifaj: Assume P is a pure-injective module. Then ${}^\perp P$ is closed under homomorphic images of pure epimorphisms. The canonical map of a direct sum onto a direct limit is well-known to be a pure epimorphism (cf. [21, 33.9(2)]). So ${}^{\perp}P$ is closed under direct limits. □

Corollary 10. *Let R be a ring and $\mathcal{C}$ be a class of pure-injective modules. Then every module has a ${}^{\perp}\mathcal{C}$-cover.*

PROOF. Let $\kappa = \mathrm{card}(R) + \aleph_0$. Denote by H the direct sum of a representative set of the class $\{A \mid \mathrm{card}(A) \leq \kappa \,\&\, \mathrm{Ext}(A, \mathcal{C}) = 0\}$. Clearly, $({}^{\perp}\mathcal{C})^{\perp} \subseteq H^{\perp}$. Conversely, take $D \in H^{\perp}$. Let $A \in {}^{\perp}\mathcal{C}$; by Theorem 8, A has a κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$. By choice of H, $\mathrm{Ext}(A_{\alpha+1}/A_\alpha, D) = 0$ for all $\alpha < \lambda$ and hence, by Lemma 3, $\mathrm{Ext}(A, D) = 0$. So $D \in ({}^{\perp}\mathcal{C})^{\perp}$. This proves that $({}^{\perp}\mathcal{C})^{\perp} = H^{\perp}$. By Theorem 7, every module has a special ${}^{\perp}\mathcal{C}$-precover. An application of Lemma 9 finishes the proof. □

If $\mathcal{C}$ is the class of *all* pure-injective modules then ${}^{\perp}\mathcal{C}$ is the class of all flat modules, so Corollary 10 implies the FCC. However, in general, ${}^{\perp}\mathcal{C}$ will be larger than the class of flat modules.

Theorem 8 and Corollary 10 remain true for any notion of "pure" that satisfies properties (i) – (iii) in Lemma 6. For example, this happens for the RD-purity [12, II.§3]; hence we get analogous results for the particular case when $\mathcal{C}$ is a class of RD-injective modules.

There is an analogue of Theorem 8 for the bifunctor Tor:

Corollary 11. *Let R be a ring and $\mathcal{B}$ be any class of left R-modules. Let $\kappa = \mathrm{card}(R) + \aleph_0$. The following conditions are equivalent for any module A:*

(i) $A \in \mathrm{Ker\ Tor}(-, \mathcal{B})$,

(ii) there is a cardinal λ such that A has a κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$ such that $A_{\alpha+1}/A_\alpha \in \mathrm{Ker\ Tor}(-, \mathcal{B})$ for all $\alpha < \lambda$.

PROOF. For each $B \in \mathcal{B}$, let $C(B) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$ be the character module of B. Put $\mathcal{C} = \{C(B) \mid B \in \mathcal{N}\}$. Then $\mathcal{C}$ is a class of pure-injective modules. By [6, Proposition VI.5.1], ${}^{\perp}\mathcal{C} = \mathrm{Ker\ Tor}(-, \mathcal{B})$, so the assertion follows from Theorem 8. □

Theorem 12. *Let R be a ring.*

(i) Let $\mathcal{B}$ be a class of left R-modules. Then every module has a Ker Tor$(-, \mathcal{B})$-cover.

(ii) Let $\mathcal{D}$ be a class of dual modules. Then every module has a ${}^{\perp}\mathcal{D}$-cover.

PROOF. (i) As above, we have $\mathcal{A} = \mathrm{Ker\ Tor}(-, \mathcal{B}) = {}^{\perp}\mathcal{C}$ where $\mathcal{C}$ is a class of pure-injective modules. Then every module has an $\mathcal{A}$-cover by Corollary 10.

(ii) Since any dual module is pure-injective, every module has a ${}^{\perp}\mathcal{D}$-cover by Corollary 10. □

Taking $\mathcal{B}$ to be the class of all left R-modules, we obtain the FCC again, this time as a consequence of Theorem 12(i).

Corollary 13. *(i) Let k be a field and R be a k-algebra. Let $\mathcal{M}$ be a class of k-finite dimensional modules. Then every module has a ${}^{\perp}\mathcal{M}$-cover.*

(ii) Assume that R is a right pure-semisimple ring. Let $\mathcal{M}$ be any class of modules. Then every module has a ${}^{\perp}\mathcal{M}$-cover.

PROOF. (i) Since any finite dimensional module is dual (in the k-vector space duality), the assertion follows from Theorem 12(ii).

(ii) Since every R-module is pure-injective (see [14, Theorem 8.4]) this follows from Corollary 10. □

4. Hereditary rings

By [10, Theorem 10], every module has a special $M^{\perp}$-preenvelope, for any module M. When M is pure-injective, Theorem 8 (for the class $\mathcal{C} = \{M\}$) yields the dual assertion that every module has a special ${}^{\perp}M$-precover. It is an open problem (even for $R = \mathbb{Z}$) whether for every M (or even for $M = \mathbb{Z}$), every module has a special ${}^{\perp}M$-precover. However, we can prove a consistency result in the case when R is a right hereditary ring:

Theorem 14. *Assume $V = L$. Let R be a right hereditary ring and $\mathcal{G}$ be a* set *of modules. Let $\kappa = \prod_{G \in \mathcal{G}} card(G) + card(R) + \aleph_0$.*

(i) Let A be a module of cardinality $\rho > \kappa$ such that $A \in {}^{\perp}\mathcal{G}$. Then there is a κ-refinement $(A_\alpha \mid \alpha \leq \rho)$ such that $A_{\alpha+1}/A_\alpha \in {}^{\perp}\mathcal{G}$ for all $\alpha < \rho$.

(ii) Every module has a special ${}^{\perp}\mathcal{G}$-precover.

PROOF. Replacing $\mathcal{G}$ by $\prod_{G \in \mathcal{G}} G$, we can, without loss of generality, assume that $\mathcal{G} = \{G\}$ for a single module G. Part (i) is then a consequence of Theorem 5.5(2), page 50, of [8], which is proved there for the ring $\mathbb{Z}$, but which has the same proof for any hereditary ring R. The sequence $(A_\nu \mid \nu \leq \mathrm{cf}(\rho))$ given there has quotients $A_{\nu+1}/A_\nu$ which are of cardinality $< \rho$, but by induction on $\rho \geq \kappa^+$, we can refine this sequence by inserting between A_ν and $A_{\nu+1}$, whenever $\mathrm{card}(A_{\nu+1}/A_\nu) > \kappa$, a sequence $(A_{\nu,\tau} \mid \tau \leq \rho_\nu)$ such that $\rho_\nu = \mathrm{card}(A_{\nu+1}/A_\nu)$, $A_{\nu,0} = A_\nu$, $A_{\nu,\rho_\nu} = A_{\nu+1}$ and for all $\tau < \rho_\nu$, $\mathrm{card}(A_{\nu,\tau+1}/A_{\nu,\tau}) \leq \kappa$. Moreover, one can check that each member of the refined sequence $(A_{\nu,\tau} \mid \nu \leq \mathrm{cf}(\rho), \tau \leq \rho_\nu)$ has fewer than ρ predecessors, and hence the whole sequence has length ρ. (See also [3, Thm. 3.1] where the result is proved for $\mathcal{G} = \{R\}$.)

Part (ii) follows as in Corollary 10. □

Remark 15. (1) The proof does not extend to proper classes of modules because of the dependence of κ on the cardinality of $\mathcal{G}$. This contrasts with Theorem 8, where $\kappa = \mathrm{card}(R) + \aleph_0$ does not depend on $\mathcal{C}$. Also, Theorem 14 (ii) cannot be improved to claim the existence of ${}^{\perp}\mathcal{G}$-covers. Indeed, if R is right hereditary, but not right perfect, and $\mathcal{G} = \{F\}$ where F is the free module of rank $2^{\mathrm{card}(R)}$, then (under V = L) ${}^{\perp}\mathcal{G}$ is the class of all projective modules, cf. [20, Theorem 3.13(ii)]. Since R is not right perfect, there exist modules without ${}^{\perp}\mathcal{G}$-covers, by the classical result of Bass.

(2) In order to be able to conclude that every module has a special ${}^{\perp}\mathcal{G}$-precover, it is not necessary that the length of the refined sequence be a cardinal, $\rho = \mathrm{card}(A)$, rather than just an ordinal. We do not know if it is provable in ZFC (say for $R = \mathbb{Z}$) that for every G there is a κ such that every A satisfying $\mathrm{Ext}(A, G) = 0$ has a κ-refinement (of some length σ) whose factors (i.e. $A_{\alpha+1}/A_\alpha$) are in ${}^{\perp}G$.

(3) For the case of $G = \mathbb{Z} = R$, in any model of ZFC in which there are non-free Whitehead groups, there exists $A \in {}^{\perp}\mathbb{Z}$ such that there is no $\aleph_0$-refinement of A whose factors are in ${}^{\perp}\mathbb{Z}$: take A to be a non-free Whitehead group and use the fact

that countable Whitehead groups are free. Furthermore, for any explicitly given cardinal κ (e.g. κ is $\aleph_{586}$ or $\aleph_{\omega_1+\omega 3+29}$), there is no theorem of ZFC which says that every $A \in {}^{\perp}\mathbb{Z}$ has a κ-refinement whose factors are in ${}^{\perp}\mathbb{Z}$; this is because there is a model of ZFC in which there are non-free Whitehead groups but every Whitehead group of size $\leq \kappa$ is free (see [9, 2.8]).

(4) There is a model of ZFC + GCH such that for any non-cotorsion $\mathbb{Z}$-module G, and any κ, there is an A such that $\operatorname{Ext}(A, G) = 0$ but there is no κ-refinement of A of length $= \operatorname{card}(A)$ whose factors are in ${}^{\perp}G$. We use a model of the uniformization principle designated UP in [20, p. 1526]. As in there, or in [19], given κ, we can construct a $\mathbb{Z}$-module A of some cardinality $\lambda > \kappa$ such that $\operatorname{Ext}(A, G) = 0$ and A has a λ-filtration $\cup_{\nu<\lambda} A'_\nu$ such that for a stationary set of ν, $A'_{\nu+1}/A'_\nu \cong \mathbb{Q}$. A standard argument then shows that for any κ-refinement $(A_\alpha \mid \alpha \leq \lambda)$, there is an $\alpha < \beta < \lambda$ such that $\mathbb{Q}$ is a submodule of A_β/A_α, and hence $\operatorname{Ext}(A_\beta/A_\alpha, G) \neq 0$ since G is not cotorsion.

In contrast to Remark 15(4) we have the following theorem for cotorsion modules over Dedekind domains.

Recall that a module C is *cotorsion* if $\operatorname{Ext}(F, C) = 0$ for every flat module F (cf. [22, p. 52]). For example, any pure-injective module is cotorsion. If R is a Dedekind domain, then C is cotorsion iff $\operatorname{Ext}(Q(R), C) = 0$ where $Q(R)$ is the quotient field of R.

For a module M, denote by $\operatorname{Cog}(M)$ the class of all modules cogenerated by M (i.e., the class of all submodules of products of copies of M). A module M is *cotilting* if ${}^{\perp}M = \operatorname{Cog}(M)$. $\operatorname{Cog}(M)$ is then a torsion-free class, called the *cotilting torsion-free class*, cf. [5, §1].

Theorem 16. *Let R be a Dedekind domain and Spec(R) be the spectrum of R. Let $\mathcal{C}$ be a class of cotorsion modules.*

(i) There is a set $S_{\mathcal{C}} \subseteq \mathit{Spec}(R)$ such that

$$ {}^{\perp}\mathcal{C} = \{A \in \mathit{Mod}\text{-}R \mid \forall P \in S_{\mathcal{C}} : R/P \not\subseteq A\}. $$

In fact, $S_{\mathcal{C}} = \{P \in \mathit{Spec}(R) \mid \exists C \in \mathcal{C} : R/P \notin {}^{\perp}C\}$.

(ii) There is a class $\mathcal{P}$ of pure-injective modules such that ${}^{\perp}\mathcal{C} = {}^{\perp}\mathcal{P}$. This is a consequence of any one of the following facts for an arbitrary cotorsion module C:
(a) ${}^{\perp}C = {}^{\perp}\prod\{\hat{R}_P \mid P \in S_C\}$ where $\hat{R}_P = \mathit{Hom}(E(R/P), E(R/P))$;
(b) ${}^{\perp}C = {}^{\perp}\mathit{PE}(C)$ where $\mathit{PE}(C)$ is the pure-injective envelope of C;
(c) ${}^{\perp}C = {}^{\perp}F$ where F is the flat cover of C; moreover, F is pure-injective.

(iii) ${}^{\perp}\mathcal{C}$ is a cotilting torsion-free class and every module has a ${}^{\perp}\mathcal{C}$-cover.

PROOF. (i) Let A be a module. Denote by $T(A)$ the torsion part of A. Since every element of $\mathcal{C}$ is cotorsion and $A/T(A)$ is flat, we have $A \in {}^{\perp}\mathcal{C}$ iff $T(A) \in {}^{\perp}\mathcal{C}$. We also have $\operatorname{Soc}(E(T(A)) \trianglelefteq T(A) \trianglelefteq E(T(A))$ and $\operatorname{Soc}(E(T(A)) = \operatorname{Soc}(T(A)) \cong \oplus_{0 \neq P \in Spec(R)} (R/P)^{(\alpha_P)}$ for some cardinals α_P. By Matlis' theory [16] (see also [17, Theorem 18.4]) we have $E(R/P) = \cup_{n<\omega} P^{-n}(R/P)$, so $E(R/P)$ has an (infinite) composition series with factors isomorphic to R/P, for every $0 \neq P \in \operatorname{Spec}(R)$. By Lemma 3 we get that $T(A) \in {}^{\perp}\mathcal{C}$ iff $\operatorname{Soc}(T(A)) \in {}^{\perp}\mathcal{C}$ iff $R/P \in {}^{\perp}\mathcal{C}$ for all $0 \neq P \in \operatorname{Spec}(R)$ such that R/P is a submodule of A. Note that $R/P \in {}^{\perp}\mathcal{C}$ iff $P \notin S_{\mathcal{C}}$. It follows that $A \in {}^{\perp}\mathcal{C}$ iff R/P is not a submodule of A for all $P \in S_{\mathcal{C}}$.

(ii) (a) Let $0 \neq P \in \operatorname{Spec}(R)$. By part (i), ${}^{\perp}\hat{R}_P = \{A \mid \forall Q \in S_{\hat{R}_P} : R/Q \not\subseteq A\}$, where $S_{\hat{R}_P} = \{Q \in \operatorname{Spec}(R) \mid R/Q \notin {}^{\perp}\hat{R}_P\}$.

By Matlis' theory, if $q \in R \setminus P$, then $q.$ is an automorphism of $E(R/P)$, and hence of $\hat{R}_P$. Since $\hat{R}_P = \mathrm{Hom}(E(R/P), \oplus_{Q \in Spec(R)} E(R/Q))$, $\hat{R}_P$ is pure-injective and flat, but not injective. Since $q.$ is a monomorphism of $E(\hat{R}_P)$ we infer that the torsion module $M_P = E(\hat{R}_P)/\hat{R}_P$ is q-torsion-free. We also have $\mathrm{Ext}(R/Q, \hat{R}_P) = \mathrm{Hom}(R/Q, M_P)$ for all $0 \neq Q \in \mathrm{Spec}(R)$. It follows that $\mathrm{Ext}(R/Q, \hat{R}_P) = 0$ for all $Q \neq P$. Since $\mathrm{Soc}(M_P) \neq 0$ and $\mathrm{Soc}(M_P)$ is a direct sum of copies of R/P, we get $\mathrm{Ext}(R/P, \hat{R}_P) \neq 0$.

This proves that $S_{\hat{R}_P} = \{P\}$. If $J = \prod\{\hat{R}_P \mid P \in S_C\}$, then J is pure-injective and $S_J = S_C$, so ${}^\perp C = {}^\perp J$ by part (i).

(b) By part (i) it suffices to show that for all P in $\mathrm{Spec}(R)$, $R/P \in {}^\perp C$ if and only if $R/P \in {}^\perp \mathrm{PE}(C)$. But C is elementarily equivalent to $\mathrm{PE}(C)$ ([18]; see also [14, Thm 7.51]). Once we show that there is a first-order sentence θ_P in the language of R-modules such that for any module M, $\mathrm{Ext}(R/P, M) = 0$ if and only if $M \models \theta_P$, we are done. Now P is generated by two elements, say p_1, p_2, and is finitely presented; say the relations are generated by $\{\sum_{i=1}^2 r_{ij} p_i = 0 \mid j = 1, \ldots, m\}$. Also, $\mathrm{Ext}(R/P, M) = 0$ if and only if every homomorphism from P to M extends to a homormorphism from R to M. Therefore $\mathrm{Ext}(R/P, M) = 0$ if and only if

$$M \models \forall x_1 \forall x_2 [(\bigwedge_{j=1}^{m} \sum_{i=1}^{2} r_{ij} x_i = 0) \Rightarrow (\exists y (\wedge_{i=1}^2 p_i y = x_i))].$$

(c) Since C is cotorsion, F is flat and cotorsion, hence pure-injective [22, Lemma 3.2.3]. For each $P \in \mathrm{Spec}(R)$, denote by R_P the localization of R at P, by P_P the (unique) maximal ideal of R_P, and by $k(P)$ the residue field R_P/P_P. By [22, Theorem 4.1.15], $F \cong \prod_{P \in Spec(R)} T_P$, where T_P is the completion of a free R_P-module of rank π_P in the P_P-adic topology. The cardinals π_P ($P \in \mathrm{Spec}(R)$), are uniquely determined by C, and called the *0-th dual Bass numbers* of C, [22, §5.2].

Xu's formula for computing dual Bass numbers [22, Theorem 5.2.2] gives $\pi_P = \dim_{k(P)} k(P) \otimes_{R_P} \mathrm{Hom}(R_P, C)$. In particular, $\pi_P = 0$ iff $k(P) \otimes_{R_P} \mathrm{Hom}(R_P, C) = 0$ iff $\mathrm{Im}(\nu_P \otimes_{R_P} 1) = \mathrm{Hom}(R_P, C)$, where ν_P is the embedding of P_P into R_P. The latter is equivalent to $P_P.\mathrm{Hom}(R_P, C) = \mathrm{Hom}(R_P, C)$.

Since R_P is a noetherian valuation domain, the ideal P_P is principal, $P_P = s.R_P$ for some $s \in P_P$. So $\pi_P = 0$ iff $s.\mathrm{Hom}(R_P, C) = \mathrm{Hom}(R_P, C)$.

On the other hand, if $0 \neq P \in \mathrm{Spec}(R)$, then $R/P \cong k(P)$ as R-modules, so $R/P \in {}^\perp C$ iff $\mathrm{Hom}(\nu_P, C)$ is surjective. The latter is equivalent to $s.\mathrm{Hom}(R_P, C) = \mathrm{Hom}(R_P, C)$, and hence to $\pi_P = 0$. It follows that $S_C = \{P \in \mathrm{Spec}(R) \mid P \neq 0 \,\&\, \pi_P \neq 0\}$.

By [22, Lemma 4.1.5], $T_P \cong \mathrm{Hom}(E(R/P), E(R/P)^{(\pi_P)})$, so $q.$ is an automorphism of T_P for each $q \in R \setminus P$, and as in part (a), we get $S_{T_P} = \{P\}$ whenever $P \neq 0$ and $\pi_P \neq 0$. Since $S_{T_0} = \emptyset$, we infer that $S_F = S_C$, so ${}^\perp C = {}^\perp F$ by part (i).

(iii) By part (ii) and Corollary 10, every module has a special ${}^\perp\mathcal{C}$-cover. Since ${}^\perp\mathcal{C}$ is closed under submodules and products, [1, Theorem 2.5] gives that ${}^\perp\mathcal{C}$ is a cotilting torsion-free class. $\square$

In [13, §2], cotilting torsion-free classes of abelian groups were characterized. We have the following for modules over Dedekind domains:

Corollary 17. *Let R be a Dedekind domain and $\mathcal{T}$ be a class of modules. Then the following conditions are equivalent:*

(i) $\mathcal{T}$ is a cotilting torsion-free class such that $\mathcal{T}$ is closed under direct limits.

(ii) There is a set of non-zero prime ideals, $\mathcal{P}$, such that

$$\mathcal{T} = \{A \in \textit{Mod-R} \mid \forall P \in \mathcal{P} : R/P \nsubseteq A\}.$$

PROOF. (i) implies (ii): We have $\mathcal{T} = {}^{\perp}C$ for a cotilting module C. Since $\mathcal{T}$ is closed under direct limits and contains all projective modules, C is cotorsion. By part (i) of Theorem 16, we can take $\mathcal{P} = S_C$.

(ii) implies (i): By the proof of part (ii) of Theorem 16, we have $S_{\hat{R}_P} = \{P\}$ for each non-zero prime ideal P. So $\mathcal{T} = {}^{\perp}\prod\{\hat{R}_P \mid P \in \mathcal{P}\}$, and $\mathcal{T}$ is a cotilting torsion-free class closed under direct limits by (the proof of) part (iii). □

5. Open Problems

(1) *Characterize the rings R such that for each $M \in$ Mod-R, every module has a special ${}^{\perp}M$-precover.* By Theorem 14, this is the case for any right hereditary ring R assuming Gödel's Axiom of Constructibility (V = L). Also, this is true in ZFC in the case when R is right pure-semisimple, by Corollary 13(ii).

(2) Denote by $\mathcal{W}$ the class of all Whitehead groups, [9]. *Does every abelian group have a special $\mathcal{W}$-precover (in ZFC)?* This is a particular case of (1) for $R = M = \mathbb{Z}$. Under V = L, every Whitehead group is free, so the answer is positive.

(3) *Can Theorem 16 be extended to wider classes of rings (such as Prüfer domains or commutative Noetherian rings of finite Krull dimension)?* In particular, *for which rings is it the case that for every class $\mathcal{C}$ of cotorsion modules, every module has a special ${}^{\perp}\mathcal{C}$-precover?*

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