HOW TO MAKE EXT VANISH

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Abstract. We describe a general construction of a module A from a given module B such that $\mathrm{Ext}(B, A) = 0$ and we apply it to answer several questions on splitters, cotorsion theories, and saturated rings.

1. Introduction

The main theorem, Theorem 2, gives a method for constructing from any module B, a related module A such that $\mathrm{Ext}(B, A) = 0$. The fact that the module A is the union of a chain $\{A_\alpha : \alpha < \lambda\}$ such that for all α, $A_{\alpha+1}/A_\alpha$ is isomorphic to B allows one to "control" the properties of A. We exploit this in order to settle some implicit and explicit questions about almost-free splitters (Theorem 7), almost cotorsion groups (Corollary 5), cotorsion theories (Theorem 10) and saturated rings (Theorem 13). We also give a sufficent condition for flat covers to exist (Corollary 11).

The inspiration for Theorem 2 was the specific construction used by Göbel and Shelah in [8] to solve a long-standing problem about "splitters", that is abelian groups A such that $\mathrm{Ext}(A, A) = 0$. In section 3 we show how their solution follows from our general theorem. Another result in [8] motivated Theorem 10; see the paragraph preceding the statement of the theorem.

Added in proof: After seeing a preprint of this paper, Enochs proved the flat cover conjecture for any ring by proving the hypothesis of our Corollary 11. Independently and at about the same time, Bican and El Bashir gave another proof of the flat cover conjecture, by a different method. (See the paper "All modules have flat covers" by L. Bican, R. El Bashir and E. Enochs, to appear in this journal.) Further applications of Theorem 10 and Enochs' method appear in our preprint "Precover classes induced by Ext".

2. The main construction

Until further notice, R is an arbitrary ring, and all modules are right R-modules. Given R-modules A and C, we use $\mathrm{Ext}(A, C)$ to denote $\mathrm{Ext}^1_R(A, C)$. The following lemma is standard (cf. [3, Prop. XII.1.14] or [2]).

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Lemma 1. *Suppose that $A = A_\mu$ is the union of a continuous chain of submodules, $A = \cup_{\alpha<\mu} A_\alpha$ such that $Ext(A_0, C) = 0$ and for all $\alpha+1 < \mu$, $Ext(A_{\alpha+1}/A_\alpha, C) = 0$. Then $Ext(A, C) = 0$.*

PROOF. The proof is by induction on $\beta \leq \mu$ that $\mathrm{Ext}(A_\beta, C) = 0$ (where $A_\mu = A$). Suppose first that $\beta = \gamma + 1$. The short exact sequence

$$0 \to A_\gamma \to A_\beta \to A_{\gamma+1}/A_\gamma \to 0$$

induces the exact sequence

$$0 = \mathrm{Ext}(A_{\gamma+1}/A_\gamma, C) \to \mathrm{Ext}(A_{\gamma+1}, C) \to \mathrm{Ext}(A_\gamma, C) = 0$$

so the middle term is zero. Now suppose that β is a limit ordinal. We must show that an arbitrary exact sequence

$$0 \to C \xrightarrow{\iota} N \xrightarrow{\pi} A_\beta \to 0$$

splits. To do this, we define by transfinite induction, for $\alpha < \beta$, a continuous increasing chain of homomorphisms $\rho_\alpha : A_\alpha \to N$ such that $\pi \circ \rho_\alpha =$ the identity on A_α, i.e., ρ_α is a splitting of $\pi \restriction \pi^{-1}[A_\alpha]$. Suppose that ρ_α has been defined for all $\alpha < \tau$. If τ is a limit ordinal, we let ρ_τ be $\cup_{\alpha<\tau}\rho_\alpha$. If $\tau = \gamma + 1 < \beta$ for some γ, let $\sigma : A_\tau \to N$ be some splitting of $\pi \restriction \pi^{-1}[A_\tau]$, which exists since $\mathrm{Ext}(A_\tau, C) = 0$ by induction. Since ρ_γ and $\sigma \restriction A_\gamma$ are both splittings of $\pi \restriction \pi^{-1}[A_\gamma]$, there is a homomorphism $\theta : A_\gamma \to C$ such that $\iota \circ \theta = \rho_\gamma - \sigma \restriction A_\gamma$. Since $\mathrm{Ext}(A_\tau/A_\gamma, C) = 0$, θ extends to a homomorphism $\theta' : A_\tau \to C$. If we define $\rho_\tau = \sigma + (\iota \circ \theta')$, then ρ_τ is a splitting of $\pi \restriction \pi^{-1}[A_\tau]$ which extends ρ_γ. $\square$

Theorem 2. *Suppose that κ and λ are cardinals such that $\kappa \geq |R|$ and $\lambda^\kappa = \lambda$. Suppose also that B and L are R-modules of cardinality $\leq \kappa$. Then there is a module A of cardinality λ such that A is the union of a continuous chain of submodules, $A = \cup_{\alpha<\lambda} A_\alpha$, such that*

(i) $A_0 = L$,

(ii) for all $\alpha < \lambda$, $A_{\alpha+1}/A_\alpha \cong B$ and

(iii) $Ext(B, A) = 0$.

Moreover, if $Ext(L, A) = 0$, then

(iv) $Ext(A, A) = 0$.

PROOF. Fix a short exact sequence

$$0 \to K \hookrightarrow F \to B \to 0$$

such that F is free of rank $\leq \kappa$, so K has cardinality $\leq \kappa$. Enumerate all set maps $\varphi : K \to \lambda$ as $\{\varphi_\alpha : \alpha < \lambda\}$ so that each map is repeated λ times. This is possible since $\lambda^\kappa = \lambda$. Write $\lambda = \cup_{\alpha<\lambda} A_\alpha$ as a continuous union of an increasing chain of sets such that $|A_0| = |L|$ and for all α the cardinality of $A_{\alpha+1} - A_\alpha$ is $|B| + |A_\alpha|$. Then for each map $\varphi : K \to \lambda$ there is an $\alpha < \lambda$ such that $\varphi = \varphi_\alpha$ and has range contained in A_α. (Note that, by König's Lemma, the cofinality of λ is $> \kappa$). We are going to define inductively an R-module structure on A_α. Let $A_0 \cong L$. Now suppose that the module structure on A_α has been defined. If φ_α is a homomorphism from K to A_α, let φ'_α be φ_α. Otherwise, let φ'_α be the zero homomorphism. Let the R-module structure on $A_{\alpha+1}$ be defined by the pushout

$$\begin{array}{ccc} F & \to & A_{\alpha+1} \\ \uparrow & & \uparrow \\ K & \xrightarrow{\varphi'_\alpha} & A_\alpha \end{array}$$

where the vertical arrows are inclusions. Then we have $A_{\alpha+1}/A_\alpha \cong F/K \cong B$. This describes the inductive step. If we set $A = \cup_{\alpha<\lambda} A_\alpha$ (as R-module), then by construction every homomorphism: $K \to A$ extends to a homomorphism : $F \to A$, so $\mathrm{Ext}(B, A) = 0$. Moreover, if $\mathrm{Ext}(L, A) = 0$, then Lemma 1 implies that $\mathrm{Ext}(A, A) = 0$ since $\mathrm{Ext}(A_{\alpha+1}/A_\alpha, A) = \mathrm{Ext}(B, A) = 0$ for all $\alpha < \lambda$. □

3. Applications to abelian groups

In this section R is a proper subring of $\mathbb{Q}$ (hence a p.i.d.) and all modules are R-modules; we will mainly consider those which are torsion-free as R-modules or, equivalently, as abelian groups ($\mathbb{Z}$-modules). Note that if A and C are R-modules, then $\mathrm{Ext}_R(A, C) = \mathrm{Ext}_{\mathbb{Z}}(A, C)$. If μ is a cardinal, finite or infinite, an R-module A is called *μ-free* if every submodule of rank $< \mu$ is free.

Lemma 3. *If A is constructed as in the proof of Theorem 2 where B and L are μ-free, then A is μ-free.*

PROOF. If $N \subseteq \cup_{\alpha<\lambda} A_\alpha$ has rank $< \mu$, then $N = \cup_{\alpha<\lambda}(N \cap A_\alpha)$ where for all $\alpha < \lambda$, $(N \cap A_{\alpha+1})/(N \cap A_\alpha)$ is isomorphic to a submodule of $A_{\alpha+1}/A_\alpha \cong B$ and hence free by hypothesis; this is enough to imply that N is free, by induction. □

Theorem 4. *For any μ-free abelian group D there is a μ-free A of cardinality $2^{|D|}$ such that $\mathrm{Ext}(D, A) = 0$.*

PROOF. Apply Theorem 2 with $B = D$, $L = 0$, $\kappa = |D|$, and $\lambda = 2^\kappa$ and use Lemma 3. □

An abelian group A is called *cotorsion* provided that $\mathrm{Ext}(G, A) = 0$ for every torsion-free abelian group G; a sufficient condition is that $\mathrm{Ext}(\mathbb{Q}, A) = 0$. The torsion-free cotorsion groups are exactly the groups of the form $\prod_{p \text{ prime}} \widehat{J_p^{(\alpha_p)}} \oplus \mathbb{Q}^{(\delta)}$ where J_p is the p-adic integers and $\widehat{\ }$ stands for the p-adic completion. (See [5, sec. 54] or [3, Chap. V].)

Hunter [13] proved that a sufficient condition for A to be cotorsion is that $\mathrm{Ext}(\mathbb{Z}^\kappa, A) = 0$ provided κ is such that $|A| \leq 2^\kappa$ and $\kappa < \kappa^\omega$. Göbel and Trlifaj [10] asked whether there is a single cardinal ρ such that $\mathrm{Ext}(\mathbb{Z}^\rho, A) = 0$ is a sufficient condition for A to be cotorsion. This is answered in the negative by the following.

Corollary 5. *For any cardinal $\kappa \geq \omega$, there is an abelian group A of cardinality 2^{2^κ} such that $\mathrm{Ext}(\mathbb{Z}^\kappa, A) = 0$ but A is not cotorsion.*

PROOF. Apply Theorem 4 with $D = \mathbb{Z}^\kappa$ and $\mu = \aleph_1$. Since A is $\aleph_1$-free it is not cotorsion. □

Note that $2^{2^\kappa} = (2^\kappa)^+$ in some models of ZFC, which shows that Hunter's bound on the size of A is as large as possible.

An abelian group A is called a *splitter* if $\mathrm{Ext}(A, A) = 0$. (The terminology is introduced in [16]; see also [8] for more on the history and interest of such groups.) The nucleus of A, $\mathrm{nuc}(A)$, is the subring of $\mathbb{Q}$ generated by $\{p^{-1} : p$ is a prime such that $pA = A\}$. If A is torsion-free, A is a splitter if it is free as a $\mathrm{nuc}(A)$-module or if it is cotorsion. Hausen [12] proved that countable torsion-free splitters were free over their nucleus; Göbel and Shelah [8] extended this to splitters of cardinality $< 2^{\aleph_0}$. (Note that torsion-free cotorsion groups have cardinality at least $2^{\aleph_0}$.)

Whether or not splitters of cardinality $\geq 2^{\aleph_0}$ had to be either nuc-free or cotorsion was open until Göbel and Shelah proved otherwise; the following is their result.

Theorem 6. *For any proper subring R of $\mathbb{Q}$, there is an abelian group A with nuc$(A) = R$ of cardinality $2^{\aleph_0}$ such that A is neither R-free nor cotorsion, but Ext$(A, A) = 0$.*

PROOF. Let B be an R-module of finite rank $r > 1$ ($r > 2$ if R is a discrete valuation ring) such that $\mathrm{nuc}(B) = R$ and B is r-free as R-module, but B is not R-free. Apply Theorem 2 with $L = 0$, $\kappa = \aleph_0$ and $\lambda = 2^{\aleph_0}$; let A be the resulting R-module, so that $\mathrm{Ext}(A, A) = 0$ by Theorem 2(iv). Since $B = A_1$ is a submodule of A, A is not R-free, and also $\mathrm{nuc}(A) = R$. Finally, A is not cotorsion because it is r-free by Lemma 3 and J_p contains a non-free module of rank $r - 1$ (cf. [6, Thm. 88.1]). □

In fact, Göbel and Shelah also showed [8, 4.11 and 5.5] that there exist arbitrarily large indecomposable splitters. In [9], Göbel and Shelah showed that there are no non-free $\aleph_1$-free splitters of cardinality $\aleph_1$(and hence it is consistent that there are none of cardinality $2^{\aleph_0}$) However, Corollary 8 following shows that there are, provably in ZFC, $\aleph_1$-free splitters of larger cardinality, and Corollary 9 (ii) shows that it is consistent that there are $\aleph_1$-free splitters of cardinality $\aleph_2$.

Theorem 7. *For any proper subring R of $\mathbb{Q}$, and every (regular) cardinal μ such that there is a μ-free abelian group of cardinality μ which is not free, there is an abelian group A with nuc$(A) = R$ of cardinality 2^{μ} such that A is μ-free as R-module, A is not R-free, and Ext$(A, A) = 0$.*

PROOF. Let B be an R-module of cardinality μ such that $\mathrm{nuc}(B) = R$ and B is not R-free, but every submodule of cardinality $< \mu$ is R-free. (Such a B exists under the hypothesis by [3, Cor. VII.3.13].) Apply Theorem 2 with $L = 0$, $\kappa = \mu$ and $\lambda = 2^{\mu}$; let A be the resulting R-module, so that $\mathrm{Ext}(B, A) = 0$. Then $\mathrm{Ext}(A, A) = 0$ by Lemma 1, A is not R-free since $B = A_1$ is a submodule of A, and A is μ-free by Lemma 3. □

The following corollaries (among others) then follow from known results (see for example [3]) about the hypothesis on μ in the theorem.

Corollary 8. *For every regular cardinal $\mu < \aleph_{\omega^2}$, there is a μ-free splitter of cardinality 2^{μ} which is not free. Moreover, for every regular cardinal μ less than the first cardinal fixed point there is a μ-free splitter which is not free.*

Corollary 9. *(i) It is consistent with ZFC that for every cardinal μ there is a μ-free splitter which is not free.*

(ii) It is consistent with ZFC that there is an $\aleph_1$-free splitter of cardinality $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ which is not free.

It appears to be open whether it is provable in ZFC that there is an $\aleph_1$-free splitter of cardinality $\aleph_2$ which is not free.

4. Applications to cotorsion theories

A *cotorsion theory* (or orthogonal theory of Ext) consists of a pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules such that $\mathcal{C} = \mathcal{F}^{\perp}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$ where for a class $\mathcal{S}$,

$$\mathcal{S}^{\perp} = \{M : M \text{ is an } R\text{-module and } \mathrm{Ext}(S, M) = 0 \text{ for all } S \in \mathcal{S}\}$$

and

$$^{\perp}\mathcal{S} = \{M : M \text{ is an } R\text{-module and } \mathrm{Ext}(M, S) = 0 \text{ for all } S \in \mathcal{S}\}$$

(See [15], [18].) A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to have *enough injectives* (resp., *enough projectives*) if for every module L there is a short exact sequence $0 \to L \to A \to F \to 0$ (resp., $0 \to A \to F \to L \to 0$) such that $A \in \mathcal{C}$ and $F \in \mathcal{F}$.

The cotorsion theory $(\mathcal{F}, \mathcal{C})$ is *cogenerated* by a class $\mathcal{Y}$ if $\mathcal{C} = \mathcal{Y}^{\perp}$.

Salce asked whether every cotorsion theory (of abelian groups) has enough projectives and injectives. Göbel and Shelah [8, Thm. 6.1] showed that this is the case if the cotorsion theory is cogenerated by a set of finite rank groups of a certain sort. Here we extend that result to every cotorsion theory (of modules) which is cogenerated by a set — as opposed to a proper class — of modules.

Theorem 10. *Every cotorsion theory which is cogenerated by a* set *of modules has enough projectives and enough injectives.*

PROOF. Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory, cogenerated by a set, S, of modules. Let B be the direct sum of the modules in S. Given a module L, choose κ such that $\kappa \geq |B| + |L| + |R|$; let $\lambda = 2^{\kappa}$. Let A be as in Theorem 2 for this B, L, κ and λ. Then A contains L and $A \in \mathcal{C}$ because $\mathrm{Ext}(B, A) = 0$. To prove that $F = A/L$ belongs to $\mathcal{F}$, it suffices to show that $\mathrm{Ext}(F, X) = 0$ whenever $\mathrm{Ext}(B, X) = 0$. However, $F = \cup_{\alpha<\lambda} F_\alpha$ where $F_\alpha = A_\alpha/L$, so $F_0 = 0$ and for each $\alpha < \lambda$, $F_{\alpha+1}/F_\alpha \cong A_{\alpha+1}/A_\alpha \cong B$. Hence, by Lemma 1, $\mathrm{Ext}(F, X) = 0$ when $\mathrm{Ext}(B, X) = 0$. This shows that $(\mathcal{F}, \mathcal{C})$ has enough injectives.

To show that $(\mathcal{F}, \mathcal{C})$ has enough projectives, we argue as in [15, Lemma 2.2]. Given a module L, choose a short exact sequence

$$0 \to K \to P \to L \to 0$$

where P is projective, and apply the first part of the proof to K to get the left-hand column in the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K & \to & P & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & U & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & F & = & F & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $A \in \mathcal{C}$ and $F \in \mathcal{F}$. Complete the diagram by letting U be a pushout. Since P and F belong to $\mathcal{F}$, so does U. Thus the second horizontal short exact sequence shows that there are enough projectives. $\square$

One cotorsion theory of interest is the *flat cotorsion theory* $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is the class of flat R-modules, and $\mathcal{C} = \mathcal{F}^{\perp}$ is the class of so-called cotorsion modules. (See for example [19, Def. 3.1.1]; this is a cotorsion theory by [19, Lemma 3.4.1].) The *flat cover conjecture* of E. Enochs [4] is equivalent to the conjecture that this cotorsion theory has enough projectives (for every ring R): see [19, Prop. 2.1.3 and

Thm. 2.2.8]. The flat cover conjecture has been proved only for certain classes of rings.[1] As a consequence of Theorem 10 we have:

Corollary 11. *If there is an R-module Q such that each R-module M is cotorsion if and only if $Ext(Q, M) = 0$ (i.e., if the flat cotorsion theory is cogenerated by a set), then the flat cover conjecture holds for R.*

In another terminology ([19]), Theorem 10 implies that for any class ${}^{\perp}\mathcal{C}$ such that $({}^{\perp}\mathcal{C})^{\perp} = Q^{\perp}$ for some module Q, every module has a ${}^{\perp}\mathcal{C}$-precover and a $Q^{\perp}$-preenvelope.

5. Applications to saturated rings

In this section, R is a non-semisimple (= non-completely reducible) ring and all modules are right R-modules.

A module M is called a *Whitehead test module for injectivity* – or an *i-test module* – provided that for every module N, $\mathrm{Ext}(M, N) = 0$ implies N is injective. Note that each i-test module is non-projective, and that there is always a proper class of i-test modules [18].

Let κ be a cardinal. R is said to be a *κ-saturated ring* if all non-projective $\leq \kappa$-generated modules are i-test. Further, R is *fully saturated* if all non-projective modules are i-test. Clearly, κ-saturated implies λ-saturated for all $\lambda \leq \kappa$. Moreover, by [18, Theorem 4.9], R is κ-saturated (fully saturated) iff either $R = T$ or $R = S \boxplus T$ where S is semisimple, T is indecomposable and κ-saturated (fully saturated), and $\boxplus$ denotes the ring direct sum. So the structure theory of saturated rings reduces to describing the indecomposable ones.

Saturated rings were investigated in [17] and [18]. In [18, Theorem 6.6], the following assertion was proved to be consistent with ZFC:

> *Let $\kappa > |R|$, $cf(\kappa) = \omega$ and R be an indecomposable κ^+-saturated ring. Then either*
> *(i) R is isomorphic to a full matrix ring over a local right artinian ring, or*
> *(ii) R is Morita equivalent to the ring of all upper triangular 2×2 matrices over a skew-field.*

We will apply Theorem 2 to show that a stronger result is actually a theorem of ZFC. First, we need a definition.

Definition 12. *Let K be a skew-field and $0 < n, p < \omega$. Denote by $GT(n, p, K)$ the subring of the full matrix ring $M_{(n+p)\times(n+p)}(K)$ consisting of all matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where $A \in M_{n\times n}(K)$, $B \in M_{n\times p}(K)$, $C \in M_{p\times p}(K)$, and 0 is the zero matrix of type $p \times n$ over K.*

Theorem 13. *Let R be an indecomposable $\aleph_0$-saturated ring. Then either*
(i) R isomorphic to a full matrix ring over a local quasi-Frobenius ring, or
(ii) R is isomorphic to $GT(n, p, K)$ for a skew-field K and for some $0 < n, p < \omega$.

Before proving Theorem 13, we prove two lemmas. In the terminology of [18, Definition 4.10], the first lemma says that 2-saturated rings of type Ia are quasi-Frobenius:

[1] *Added in proof:* The conjecture is now proved; see the Introduction.

Lemma 14. *Assume R is a full matrix ring over a local right artinian ring such that R is 2-saturated. Then R is a quasi-Frobenius ring.*

PROOF. Assume R is not quasi-Frobenius, hence not right self-injective. Let S be the (unique up to isomorphism) simple module, which, without loss of generality, is a right ideal of R. Then $\mathrm{Ext}(S, R) \neq 0$ since R is 1-saturated. Let

$$0 \to R \hookrightarrow N \to S \to 0$$

be a non-split exact sequence of modules. Then R is an essential submodule of N and N is ≤ 2-generated. Since the composition length of N is one more than the composition length of R, we infer that N is not free (= not projective). Put $B = N^{(\omega)}$. Let $\kappa = |B|$, $\lambda = 2^\kappa$ and let $A = \cup_{\alpha<\lambda} A_\alpha$ be as in Theorem 2 (with $L = 0$). Then $\mathrm{Ext}(N, A) = 0$, so A is injective. For each $\alpha < \lambda$, define a free submodule F_α of $A_{\alpha+1}$ such that $F_\alpha \cap A_\alpha = 0$, and $(A_\alpha \oplus F_\alpha)/A_\alpha$ is the kernel of the canonical projection of $A_{\alpha+1}/A_\alpha \cong B$ onto $S^{(\omega)}$. This is possible since the kernel is a free module (of rank ω). Moreover, the kernel is essential in $A_{\alpha+1}/A_\alpha$, so $A_\alpha \oplus F_\alpha$ is essential in $A_{\alpha+1}$. For each $\alpha \leq \lambda$, let $G_\alpha = \oplus_{\beta<\alpha} F_\beta$. By induction on $0 < \alpha \leq \lambda$, it follows that G_α is an essential submodule of A_α (where we put $A_\lambda = A$). So A is an injective hull of the module G_λ. Let A' be a submodule of A such that A' is an injective hull of $A_1 = B$. By induction on $0 < \alpha \leq \lambda$, we prove that $\mathrm{Soc}(A_\alpha/A_1) \subseteq (G_\alpha + A_1)/A_1$. This is clear for $\alpha = 1$ and for α a limit ordinal. If $0 < \alpha < \lambda$, then $(A_\alpha \oplus F_\alpha)/A_1$ is essential in $A_{\alpha+1}/A_1$, so $\mathrm{Soc}(A_{\alpha+1}/A_1) = \mathrm{Soc}((A_\alpha \oplus F_\alpha)/A_1) \subseteq (G_\alpha \oplus F_\alpha)/A_1 = G_{\alpha+1}/A_1$ by induction premise. If $A_1 \neq A'$, then there is a module D with $A_1 \subset D \subseteq A'$ and $D/A_1 \cong S$. It follows that $D/A_1 \subseteq (G_\lambda + A_1)/A_1$, so $D \subseteq A_1 \oplus \bigoplus_{0<\alpha<\lambda} F_\alpha$ and $D = A_1 \oplus (D \cap \bigoplus_{0<\alpha<\lambda} F_\alpha)$. Then A_1 is not essential in D, a contradiction. This proves that $A' = A_1$, that is, B and N are injective. The injective hull E of S is ≤ 2-generated since it is a summand of N and not projective since N is a direct sum of copies of E. Since R is right artinian, we have $\mathrm{Ext}(E, E^{(\sigma)}) = 0$ for all cardinals σ. By [18, Lemma 6.8], there are a cardinal ρ and a module C such that $\mathrm{Ext}(E, C) = 0$ and there is an exact sequence

$$0 \to R \to C \to E^{(\rho)} \to 0.$$

Since $\mathrm{Ext}(E, C) = 0$, E is not projective and R is 2-saturated, C is injective. Applying $\mathrm{Ext}^i(R/S, -)$, we get

$$0 = \mathrm{Ext}(R/S, E^{(\rho)}) \to \mathrm{Ext}^2(R/S, R) \to \mathrm{Ext}^2(R/S, C) = 0.$$

This gives $\mathrm{Ext}^2(R/S, R) = 0$. Applying $\mathrm{Ext}^i(-, R)$ to the exact sequence $0 \to S \to R \to R/S \to 0$, we get

$$0 = \mathrm{Ext}(R, R) \to \mathrm{Ext}(S, R) \to \mathrm{Ext}^2(R/S, R) = 0$$

so $\mathrm{Ext}(S, R) = 0$, a contradiction. $\square$

Lemma 15. *Suppose that R is a regular ring which is simple but not semisimple. Then for any non-zero R-module N, $N^{(\omega)}$ is not injective.*

PROOF. There is a right ideal I of R which is not finitely-generated; without loss of generality I is countably generated. Then $I = \oplus_{n\in\omega} e_n R$ where the e_n are orthogonal idempotents [11, Proposition 2.13]. We assert that $Ne_n \neq 0$. Indeed, if it were zero, then $Ne_nR = 0$, but $R\,e_nR = R$ since R is simple, so N would be zero. Now, for each $n < \omega$, pick $a_n \in N$ such that $a_ne_n \neq 0$ and define $f : I \to \oplus_{n\in\omega} N_n$

(where $N_n = N$) such that $f(e_n) = a_n e_n \in N_n$. This is possible since $e_n s = 0$ implies $f(e_n)s = a_n e_n s = 0$. Clearly f can't be extended to a homomorphism on R. □

PROOF. *(of Theorem 13)* If R has at least two non-isomorphic simple modules, then (ii) holds by [18, Theorems 5.6 and 5.16]. Assume that all simple modules are isomorphic (to a module S). We will prove that (i) holds. By Lemma 14, it suffices to show that R is a full matrix ring over a local artinian ring. By [18, Theorem 6.1], it suffices to prove that R cannot be a simple, von Neumann regular ring such that every right ideal of R is countably generated. Suppose, to the contrary that it is; then R is right hereditary, hence not right self-injective by [14, Corollary 2.23], since R is not semisimple. So $\mathrm{Ext}(S, R) \neq 0$, since R is 2-saturated. Let

$$0 \to R \hookrightarrow N \to S \to 0$$

be a non-split exact sequence of modules. Then R is an essential submodule in N, so $\mathrm{Soc}(N) = \mathrm{Soc}(R) = 0$. Moreover, N is $\leq$ 2-generated and non-projective [11, Theorem 1.11]. Let $B = N^{(\omega)}$. Then B is countably generated and non-projective. We will prove that B is not i-test, thus contradicting the assumption of R being $\aleph_0$-saturated. Let $\kappa = |B|$ $(= |R|)$, $\lambda = 2^\kappa$ and let A be as in Theorem 2 (with $L = 0$). Then $\mathrm{Ext}(B, A) = 0$ and it suffices to prove that A is not injective. Aiming for a contradiction, suppose that A is injective. Then A contains a copy of $I(B)$, the injective envelope of B, since $A_1 \cong B$. Therefore $A/A_1 \cong (I(B)/B) \oplus C$ for some R-module C. We prove that S is not isomorphic to a submodule of A/A_1. Indeed, by Theorem 2, $A/B = \cup_{1 \leq \alpha < \lambda} A_\alpha / A_1$, so since S is simple, if there is an embedding θ of S into A/A_1, there is a least $\beta < \lambda$ such that the range of θ is contained in A_β/A_1. Clearly, $\beta = \gamma + 1$ for some γ (since S is cyclic) and then θ induces an embedding of S into $(A_{\gamma+1}/A_1)/(A_\gamma/A_1) \cong A_{\gamma+1}/A_\gamma \cong B$. However, this is impossible, since $\mathrm{Soc}(B) = 0$. On the other hand, by Lemma 15, B is not injective, so $I(B)/B$ is non-zero. Since R is $\aleph_0$-saturated, $\mathrm{Ext}(S, B) \neq 0$, so $\mathrm{Hom}(S, I(B)/B) \neq 0$. So S embeds in $I(B)/B$, and in A/A_1, a contradiction. □

Note that the assumption of $\aleph_0$-saturated is essential in Theorem 13: there exist noetherian domains that are n-saturated for all $n < \omega$, but not right artinian [18, Examples 4.6 and 4.7].

It remains open whether every indecomposable $\aleph_0$-saturated (or fully saturated) ring R which satisfies condition (i) in Theorem 13 is actually an artinian valuation ring (cf. [18, Theorem 4.5]); if so, we would have a complete characterization of $\aleph_0$-saturated (or fully saturated) rings. We do have such a characterization for hereditary rings:

Corollary 16. *Let R be a non-semisimple ring. Then the following are equivalent:*

(i) R is a right hereditary $\aleph_0$-saturated ring;

(ii) R is a right hereditary fully saturated ring;

(iii) $R = T$ or $R = S \boxplus T$, where S is a semisimple ring and T is isomorphic to $GT(n, p, K)$ for a skew-field K and for some $0 < n, p < \omega$.

PROOF. By Theorem 13 and [18, Corollary 6.2]. □

Since condition (iii) is left-right symmetric, it is also equivalent to the left-sided versions of (i) and (ii).

6. A DUALIZATION

Our main construction (Theorem 2) relies on several facts from homological algebra (such as Lemma 1), category theory (such as the pushout construction) and set theory (the simple fact that for each $\kappa \geq \omega$ there is λ with $\lambda^\kappa = \lambda$). It is the dualization of the latter fact that fails: $\kappa^\lambda > \lambda$ for all λ, so the proof of Theorem 2 cannot be dualized. A natural question arises whether there is still a way to prove a dual of Theorem 2, or of some of its particular instances.

In Proposition 18 below, we will prove that the homological and category-theoretic facts can be dualized. First, note that Lemma 1 is true in a slightly more general setting:

Lemma 17. *Let $(A_\alpha \mid \alpha \leq \mu)$ be a sequence of modules and $(f_{\alpha\beta} \mid \alpha \leq \beta \leq \mu)$ be a sequence of monomorphisms such that $\{(A_\alpha, f_{\alpha\beta}) \mid \alpha \leq \beta \leq \mu\}$ is a direct system which is continuous (in the sense that $A_0 = 0$ and $A_\alpha = \varinjlim_{\gamma<\alpha} A_\gamma$ for all limit ordinals $\alpha \leq \mu$).*

Let C be a module such that Ext$(A_{\alpha+1}/f_{\alpha,\alpha+1}(A_\alpha), C) = 0$ for all $\alpha + 1 \leq \mu$. Then Ext$(A_\mu, C) = 0$.

PROOF. By induction, as in Lemma 1. □

Now, Lemma 17 has the following dual:

Proposition 18. *Let $(A_\alpha \mid \alpha \leq \mu)$ be a sequence of modules and $(g_{\beta\alpha} \mid \alpha \leq \beta \leq \mu)$ be a sequence of epimorphisms such that $\{(A_\alpha, g_{\beta\alpha}) \mid \alpha \leq \beta \leq \mu\}$ is an inverse system which is continuous (in the sense that $A_0 = 0$ and $A_\alpha = \varprojlim_{\gamma<\alpha} A_\gamma$ for all limit ordinals $\alpha \leq \mu$). For each $\alpha < \mu$, let $K_\alpha =$Ker$(g_{\alpha+1,\alpha})$.*

Assume that C is a module such that Ext$(C, K_\alpha) = 0$ for all $\alpha < \mu$. Then Ext$(C, A_\mu) = 0$.

PROOF. By induction on $\alpha \leq \mu$, we prove that $\mathrm{Ext}(C, A_\alpha) = 0$. Let $\alpha < \nu$. The short exact sequence

$$0 \to K_\alpha \hookrightarrow A_{\alpha+1} \stackrel{g_{\alpha+1,\alpha}}{\to} A_\alpha \to 0$$

induces the exact sequence

$$0 = \mathrm{Ext}(C, K_\alpha) \to \mathrm{Ext}(C, A_{\alpha+1}) \to \mathrm{Ext}(C, A_\alpha) = 0$$

so the middle term is zero.

Suppose that α is a limit ordinal, so $A_\alpha = \varprojlim_{\beta<\alpha} A_\beta$. For each $\beta < \alpha$, denote by π_β the projection of A_α to A_β. Since all the inverse system maps are surjective, so is π_β.

Let $C \cong F/K$, where F is a free module. Denote by ϵ the inclusion of K into F. It remains to be shown that any homomorphism $\varphi \in \mathrm{Hom}(K, A_\alpha)$ can be extended to some $\phi \in \mathrm{Hom}(F, A_\alpha)$ so that $\varphi = \phi\epsilon$.

Take $\varphi \in \mathrm{Hom}(K, A_\alpha)$. By induction on $\beta < \alpha$, define $h_\beta \in \mathrm{Hom}(F, A_\beta)$ such that $h_\beta\epsilon = \pi_\beta\varphi$ and $g_{\beta\gamma}h_\beta = h_\gamma$ for all $\gamma \leq \beta$. For $\beta = 0$, put $h_0 = 0$. If $\beta < \alpha$ is a limit ordinal, then h_β is defined as the inverse limit of $(h_\gamma \mid \gamma < \beta)$. Let $\beta < \alpha$. By induction premise, $\mathrm{Ext}(C, A_{\beta+1}) = 0$, so there exists $k_{\beta+1}$ such that $k_{\beta+1}\epsilon = \pi_{\beta+1}\varphi$. Put $\delta = h_\beta - g_{\beta+1,\beta}k_{\beta+1}$. Then $\delta\epsilon = 0$, so δ induces a homomorphism $\overline{\delta} \in \mathrm{Hom}(C, A_\beta)$. Since $\mathrm{Ext}(C, K_\beta) = 0$, there is $\Delta \in \mathrm{Hom}(F, A_{\beta+1})$ such that $\Delta\epsilon = 0$ and $\overline{\delta} = g_{\beta+1,\beta}\overline{\Delta}$, so $\delta = g_{\beta+1,\beta}\Delta$. Then $h_{\beta+1} = k_{\beta+1} + \Delta$ satisfies $h_{\beta+1}\epsilon = \pi_{\beta+1}\varphi$ and $g_{\beta+1,\beta}h_{\beta+1} = h_\beta$, hence $h_{\beta+1}g_{\beta+1,\gamma} = h_\gamma$ for all $\gamma \leq \beta + 1$.

Finally, by the inverse limit property, there is $\phi \in \mathrm{Hom}(F, A_\alpha)$ such that $\pi_\beta \phi = h_\beta$ for all $\beta < \alpha$. Then $\pi_\beta \phi \epsilon = \pi_\beta \varphi$ for all $\beta < \alpha$, so $\phi \epsilon = \varphi$. $\square$

Proposition 18 can be used to construct splitters. For example, let $R = \mathbb{Z}$. Fix a prime p and consider the inverse system of canonical projections $g_{mn} : \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^n}$, $n \leq m < \omega$. So $A_m = \mathbb{Z}_{p^m}$ and $K_m \cong \mathbb{Z}_p$ for all $m < \omega$. Take $C = \mathbb{J}_p = \varprojlim_{m<\omega} \mathbb{Z}_{p^m}$, the group of all p-adic integers. Since $\mathrm{Ext}(\mathbb{J}_p, \mathbb{Z}_p) = 0$, Proposition 18 implies (the well-known fact) that $\mathbb{J}_p$ is a splitter.

Note that the vanishing of $\mathrm{Ext}(\mathbb{J}_p, \mathbb{Z}_p)$ cannot be proved by a pull-back argument dual to the one in the proof of Theorem 2. Indeed, there is a remarkable difference from the direct limit case: in the proof of Theorem 2, each homomorphism φ from K to A factors through the direct system, that is, there is $\alpha < \lambda$ such that φ maps into A_α - because $\lambda = |A|$ is "big enough". But in our example, though $|\mathbb{J}_p| = 2^\omega$ is "big enough", there are many homomorphisms from $\mathbb{J}_p$ to $\mathbb{Z}_{p^\infty}/\mathbb{Z}_p$ that do not factor through the inverse system - simply because those which factor must have a finite image.

However, if there is a dual theorem to Theorem 2 for R-modules, then a dual argument to that in Theorem 10 would give a proof of the flat cover conjecture for R-modules, since the class of flat modules is always generated by a set. In fact, what we need is a dual of Theorem 2 for the particular right R-module $B = \mathrm{Hom}_{\mathbb{Z}}(B', \mathbb{Q}/\mathbb{Z})$ where B' is the direct sum of all cyclic left R-modules: $B' = \oplus\{R/I : I$ a left ideal of $R\}$.

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