NEW RESULTS ON WHITEHEAD GROUPS

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(corrected version)

New results on Whitehead groups

plus some old results
and some new questions

Joint work with László Fuchs, Saharon Shelah and Jan Trlifaj

P. Eklof, L. Fuchs and S. Shelah, "Test groups for Whitehead groups", preprint.

The Whitehead class

Let $\mathcal{S}$ be a class of modules.

$${}^{\perp}\mathcal{S} = \{N \mid \mathrm{Ext}^1(N, M) = 0 \text{ for all } M \in \mathcal{S}\}$$

$$\mathcal{S}^{\perp} = \{N \mid \mathrm{Ext}^1(M, N) = 0 \text{ for all } M \in \mathcal{S}\}$$

Example

The class of Whitehead groups
$\mathcal{W} = {}^{\perp}\{\mathbb{Z}\}$

The Whitehead cotorsion pair

Definition. (Salce 1979)

A **cotorsion pair** is a pair of classes of modules $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$

The Whitehead cotorsion pair

$(\mathcal{W}, \mathcal{W}^{\perp}) = ({}^{\perp}\mathbb{Z}, ({}^{\perp}\mathbb{Z})^{\perp})$

Deconstructing cotorsion pairs

Definition.

The cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be κ-**deconstructible** if every $A \in \mathcal{A}$ can be written as the union of a continuous chain of submodules

$$A = \bigcup_{\alpha<\sigma} A_\alpha$$

such that $A_0 = 0$ and for all $\alpha < \sigma$, $A_{\alpha+1}/A_\alpha$ belongs to $\mathcal{A}$ and is $\leq \kappa$-generated.

Note. If $A = \bigcup_{\alpha<\sigma} A_\alpha$ s.t. $A_0 = 0$ and for all $\alpha < \sigma$, $A_{\alpha+1}/A_\alpha \in \mathcal{A}$, then $A \in \mathcal{A}$.

$(\mathcal{A}, \mathcal{B})$ is deconstructible if it is κ-deconstructible for some κ.

We will sometimes say "$\mathcal{A}$ is .(κ-)deconstructible."

Consequences of deconstructibility

1. If $(\mathcal{A}, \mathcal{B})$ is κ-deconstructible, and every member of $\mathcal{A}$ which is generated by $\leq \kappa$ elements is free (projective), then every member of $\mathcal{A}$ is free (projective).

Example. Baer modules
THEOREM. (Fuchs, et al.) *The cotorsion pair $(\mathcal{A}, \mathcal{B})$ is $\aleph_0$-deconstructible provided: $\mathcal{B}$ is closed under arbitrary direct sums and $\mathcal{A}$ consists of modules of proj dim ≤ 1.*

Corollary. *If every countably generated Baer R-module is projective, then every Baer R-module is projective.*

THEOREM. (Angeleri Hugel-Bazzoni-Herbera 2005) *Every countably generated Baer module over an arbitrary ID is projective.*

Consequence 2: completeness

2. If $(\mathcal{A}, \mathcal{B})$ is deconstructible, then it is *cogenerated* by a set, *that is, there is a* set $\mathcal{S}$ *such that* $\mathcal{S}^{\perp} = \mathcal{B}$*; equivalently, there is a module* P *such that* $\{P\}^{\perp} = \mathcal{B}$*.*

Hence: (E - Trlifaj 2001) $(\mathcal{A}, \mathcal{B})$ is *complete*, i.e., has enough projectives and injectives, i.e., special precovers exist.

Example $(R = \mathbb{Z})$

THEOREM. (E-Trlifaj) $(\mathcal{A}, \mathcal{B})$ *is deconstructible, and hence complete, if every member of* $\mathcal{B}$ *is cotorsion.*

THEOREM. (E-Shelah-Trlifaj 2004) *It is consistent with ZFC + GCH that for every* N *which is not cotorsion, the cotorsion pair* $({}^{\perp}\{N\}, ({}^{\perp}\{N\})^{\perp})$ *is not cogenerated by a set, hence not deconstructible.*

THEOREM. (E-Shelah 2003) *It is consistent with ZFC + GCH that* $(\mathcal{W}, \mathcal{W}^{\perp})$ *is not complete.*

Consequence 3: tensor products

3. If $(\mathcal{W}, \mathcal{W}^{\perp})$ is κ-deconstructible and the tensor product of two members of $\mathcal{W}$ of cardinality $\leq \kappa$ is again in $\mathcal{W}$, then the same is true for the tensor product of any two members of $\mathcal{W}$.

Proof. (E-Fuchs) Let A and B be W-groups, i.e. elements of $\mathcal{W}$. Without loss of generality, we can assume that A has cardinality $> \kappa$. Let $\{A_\nu : \nu < \sigma\}$ be a continuous chain as in the definition of κ-deconstructible.

Case: $|B| = \kappa$

Assuming $|B| = \kappa$, we prove $A \otimes B$ is a W-group.
$\{A_\nu \otimes B : \nu < \sigma\}$ is a continuous filtration of $A \otimes B$. Suffices to show that for all $\nu < \sigma$, $A_{\nu+1} \otimes B/A_\nu \otimes B$ is a W-group.
There is an exact sequence (Cartan-Eilenberg)

$$0 \to A_\nu \otimes B \to A_{\nu+1} \otimes B \to (A_{\nu+1}/A_\nu) \otimes B$$

and by hypothesis, the right-hand term is a W-group (of cardinality κ).
But then the quotient $A_{\nu+1} \otimes B/A_\nu \otimes B$ is a subgroup of a W-group and hence a W-group.

General case

Next suppose that $|B|\grave{>}\kappa$. Again we use the continuous filtration $\{A_\nu \otimes B : \nu < \sigma\}$and the exact sequence

$$0 \to A_\nu \otimes B \to A_{\nu+1} \otimes B \to (A_{\nu+1}/A_\nu) \otimes B$$

The right-hand term $(A_{\nu+1}/A_\nu) \otimes B$ is a W-group by the first case (and the symmetry of the tensor product) so we finish as before.

Models of set theory

Theorem

If $\diamondsuit_\lambda(E)$ holds for every regular cardinal $\lambda > \kappa$ and every stationary subset E of λ, then $\mathcal{W}$ is κ-deconstructible (and hence $(\mathcal{W}, \mathcal{W}^\perp)$ is cogenerated by a set and thus complete).

Examples.

(I) A model of V = L. Then every member of $\mathcal{W}$ is free (cf. Consequence 1).

(II) A model of GCH + Ax(S) + $\diamondsuit^*(\omega_1 \setminus S)$ plus
$\diamondsuit_\lambda(E)$ holds for every regular cardinal $\lambda > \aleph_1$ and every stationary subset E of λ.

Then there are non-free Whitehead groups. In fact, an $\aleph_1$-free group of cardinality $\aleph_1$ is Whitehead iff $\Gamma(A) \leq \tilde{S}$.

Claim

In Model (II) every tensor product of Whitehead groups (of arbitrary cardinality) is Whitehead.

Model (II)

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We will use

Consequence 3

If $(\mathcal{W}, \mathcal{W}^{\perp})$ is κ-deconstructible and the tensor product of two members of $\mathcal{W}$ of cardinality $\leq \kappa$ is again in $\mathcal{W}$, then the same is true for the tensor product of any two members of $\mathcal{W}$.

Proof of Claim

By Consequence 3, it suffices to prove that the tensor product of two Whitehead groups of cardinality $\aleph_1$ is a Whitehead group.
In this model, a group A of cardinality $\aleph_1$ is Whitehead if and only if $\Gamma(A) \leq \tilde{S}$ i.e., it has an ω_1-filtration $\{A_\nu : \nu < \omega_1\}$ such that

$$\{\nu < \omega_1 : \exists \mu > \nu \text{ s.t. } A_\mu / A_\nu \text{ is not free}\} \subseteq S.$$

It suffices to prove that $\Gamma(A \otimes B) \leq \tilde{S}$ if $A, B \in \mathcal{W}$. Fix ω_1-filtrations $\{A_\nu : \nu < \omega_1\}$ and $\{B_\nu : \nu < \omega_1\}$ as above.
Then $\{A_\nu \otimes B_\nu : \nu < \omega_1\}$ is an ω_1-filtration of $A \otimes B$. For each $\mu > \nu$ there is an exact sequence

$$0 \to A_\nu \otimes B_\nu \to A_\mu \otimes B_\mu \to (A_\mu \otimes (B_\mu / B_\nu)) \oplus ((A_\mu / A_\nu) \otimes B_\mu)$$

If $\nu \notin S$, then the two summands on the right are free, and hence the quotient $A_\mu \otimes B_\mu / A_\nu \otimes B_\nu$ is free. Thus we have proved that $\Gamma(A \otimes B) \leq \tilde{S}$.

Question

Is it provable in ZFC ($+$ GCH) that the tensor product of two Whitehead groups is always a Whitehead group?

Dual Groups and Reflexive Groups

A dual group is one of the form $A^* = \mathrm{Hom}(A, \mathbb{Z})$.

B is reflexive if the natural map $B \to B^{**}$ is an isomorphism.

Theorem. (Huber) If B is $\aleph_1$-coseparable, i.e., $\mathrm{Ext}(B, \mathbb{Z}^{(\omega)}) = 0$, then B is reflexive.

Model (III)

Theorem. (Eklof-Shelah 2002) There is a model of ZFC + GCH with a non-free strongly $\aleph_1$-free Whitehead group B of cardinality $\aleph_1$ such that B^* is free of infinite rank.
Hence B is not reflexive.

W-test groups

A W^* group is one of the form A^* where A is a Whitehead group.
If A is a W-group, A^* is non-zero and separable, so $\mathrm{Ext}(B, A^*) = 0$ implies B is a Whitehead group.

Definition and Question

G is a **W-test group** if $\mathrm{Ext}(B, G) = 0$ if and only if B is a Whitehead group.
Is it provable in ZFC (+ GCH) that every W^* group is a W-test group?

Equivalence of the questions

Lemma

For any Whitehead groups A and B,
$\mathrm{Ext}(B, A^*) \cong \mathrm{Ext}(A, B^*) \cong \mathrm{Ext}(A \otimes B, \mathbb{Z})$.
Hence $A \otimes B$ is a Whitehead group if and only if $\mathrm{Ext}(A, B^*) = 0$.

Proof We use the fact (Cartan-Eilenberg) that

$$\mathrm{Ext}(A, \mathrm{Hom}(B, \mathbb{Z})) \oplus \mathrm{Hom}(A, \mathrm{Ext}(B, \mathbb{Z})) \cong \mathrm{Ext}(A \otimes B, \mathbb{Z}) \oplus \mathrm{Hom}(\mathrm{Tor}(A, B), \mathbb{Z})$$

In our case this reduces to

$$\mathrm{Ext}(A, B^*) \cong \mathrm{Ext}(A \otimes B, \mathbb{Z})$$

Since $A \otimes B \cong B \otimes A$, this suffices. $\square$

Another model of set theory

Conclusion

The tensor product of two Whitehead groups is always Whitehead if and only if every W^* group is a W-test group

Models (I) and (II): these equivalent statements are true.

Model (III)

A model of ZFC + GCH with a non-free Whitehead group B of cardinality $\aleph_1$ such that B^* is free of infinite rank.
Claim: In this model, B^* is not a W-test group,
In fact, $\text{Ext}(B, B^*) \neq 0$, and hence $B \otimes B$ is not Whitehead.

Proof. Since B is a non-reflexive W-group, the theorem of Huber implies that B is not $\aleph_1$-coseparable, i.e., $\text{Ext}(B, \mathbb{Z}^{(\omega)}) \neq 0$. Thus $\text{Ext}(B, B^*) \neq 0$. □

Open Questions

Is it consistent with ZFC (+ GCH) that there is a Whitehead group B such that

- B is $\aleph_1$-coseparable, but $\mathrm{Ext}(B, B^*) \neq 0$?
- B is $\aleph_1$-coseparable, but B^* is not W-test?
- B is reflexive, but B^* is not W-test?

Open Questions rephrased

Is it provable in ZFC (+ GCH) that for every Whitehead group B

- B $\aleph_1$-coseparable implies $\mathrm{Ext}(B, B^*) = 0$?
- B $\aleph_1$-coseparable implies B^* is W-test?
- B reflexive implies B^* is W-test?

One final model

Model (IV) A model of MA $+ 2^{\aleph_0} = \aleph_2$ plus
$\diamondsuit_\lambda(E)$ holds for every regular cardinal $\lambda > \aleph_1$ and every stationary subset E of λ.

Theorem

In Model (IV) every W^*-group is a W-test group and hence the tensor product of any two W-groups is a W-group.

Martin's Axiom

For every $\kappa < 2^{\aleph_0}$ and for every c.c.c. poset P and every family $\mathcal{D} = \{D_\alpha \colon \alpha \in \kappa\}$ of dense subsets of P, there is a directed subset $\mathcal{G}$ of P such that for all $\alpha \in \kappa$, $\mathcal{G} \cap D_\alpha \neq \emptyset$.

Theorem. (Shelah) Assuming MA $+ 2^{\aleph_0} > \aleph_1$, the following are equivalent for an $\aleph_1$-free group A of cardinality $\aleph_1$

- A is Whitehead;
- A is a Shelah group;
- A is $\aleph_1$-coseparable.

Model (IV)

By Consequence 3, it suffices to prove the result for groups of cardinality $\leq \aleph_1$.
We will prove that if A and B are W-groups of cardinality $\leq \aleph_1$, then $\mathrm{Ext}(A, B^*) = 0$

Fix a short exact sequence

$$0 \to B^* \overset{\iota}{\hookrightarrow} N \overset{\pi}{\to} A \to 0$$

We may assume that ι is the inclusion map. We also fix a set function $\gamma : A \to N$ such that for all $a \in A$, $\pi(\gamma(a)) = a$. We will show that the short exact sequence splits by proving the existence of a function $h : A \to B^*$ such that the function

$$\gamma - h : A \to N : a \mapsto \gamma(a) - h(a)$$

is a homomorphism. The function h will be obtained via a directed subset $\mathcal{G}$ of a c.c.c. poset P.

The partial order

P is the set of all triples $p = (A_p, B_p, h_p)$ where

- A_p (resp. B_p) is a pure and finitely-generated summand of A (resp. B);
- h_p is a function from A_p to B_p^*.such that the function which takes $a \in A_p$ to $\gamma(a) - h_p(a)$ is a homomorphism from A_p into $N/\{f \in B^* : f \restriction B_p \equiv 0\}$.
- $p = (A_p, B_p, h_p) \leq p' = (A'_p, B'_p, h'_p)$ if and only if $A_p \subseteq A'_p$, $B_p \subseteq B'_p$, and $h'_p(a) \restriction B_p = h_p(a)$ for all $a \in A_p$.

The dense sets

Let

$$D_a^1 = \{p \in P : a \in A_p\}$$

for all $a \in A$ and

$$D_b^2 = \{p \in P : b \in B_p\}$$

for all $b \in B$.

If these sets are dense and P is c.c.c., then MA + $2^{\aleph_0} > \aleph_1$ yields a directed subset $\mathcal{G}$ which has non-empty intersection with each dense subset.
We can then define h by: $h(a)(b) = h_p(a)(b)$ for some (all) $p \in \mathcal{G}$ such that $a \in A_p$ and $b \in B_p$. It is easy to check that h is well-defined and has the dcsired properties.

verification of denseness

Claim

Given a basis $\{x_i : i = 1, ..., n\}$ of a finitely-generated pure subgroup A_0 of A, a basis $\{y_j : j = 1, ..., m\}$ of a finitely-generated pure subgroup B_0 of B, and an indexed set $\{e_{ij} : i = 1, ..., n, j = 1, ..., m\}$ of elements of $\mathbb{Z}$, there is one and only one $p \in P$ such that $A_p = A_0$, $B_p = B_0$ and $h_p(x_i)(y_j) = e_{ij}$ for all $i = 1, ..., n$, $j = 1, ..., m$.

Given p, there is a finitely-generated pure subgroup A' of A which contains A_p and a. Now A_p is a summand of A' so we can choose a basis $\{x_i : i = 1, ..., n\}$ of A' which includes a basis $\{x_i : i = 1, ..., k\}$ of A_p; choose a basis $\{y_j : j = 1, ..., m\}$ of B_p. Then by the Claim there is an element p' of P such that $A_{p'} = A'$, $B_{p'} = B_p$, and $h_{p'}(x_i)(y_j) = h_p(x_i)(y_j)$ for all $i = 1, ..., k$. Clearly p' extends p and belongs to D^1_a.

verification of c.c.c.

Lemma

If G is a Shelah group of cardinality $\aleph_1$ and $\{S_\alpha : \alpha \in \omega_1\}$ is a family of finitely-generated pure subgroups of G, then there is an uncountable subset I of ω_1 and a pure free subgroup G' of *G such that $S_\alpha \subseteq G'$ for all $\alpha \in I$.*

We must prove that if $\{p_\nu = (A_{\nu,} B_\nu, h_\nu) : \nu \in \omega_1\}$. then $\exists \mu \neq \nu$ such that p_μ and p_ν are compatible. By the Lemma we can assume that there are a pure free subgroup A' of A and a pure free subgroup B' of B such that $A_\nu \subseteq A'$ and $B_\nu \subseteq B'$ for all $\nu \in \omega_1$. Choose a basis X of A' and a basis Y of B'. By density we can assume that each A_ν is generated by a finite subset, $X_{\nu,}$ of X and each B_ν is generated by a finite subset, Y_ν, of Y. Moreover we can assume that there is a (finite) subset T of X (resp. W of Y) which is contained in each X_ν (resp. each Y_ν) and is maximal w.r.t. the property that it is contained in uncountably many X_ν (resp. Y_ν). Passing to a subset, we can assume that $h_\nu(x)(y)$ has a value independent of ν for each $x \in T$ and $y \in W$.

verification of c.c.c., continued

By a counting argument we can find $\nu > 0$ such that $X_\nu \cap X_0 = T$ and $Y_\nu \cap Y_0 = W$.
We define $q \in P \geq p_0, p_\nu$.
Let $A_q = \langle X_0 \cup X_\nu \rangle$ and $B_q = \langle Y_0 \cup Y_\nu \rangle$. Clearly these are pure subgroups of A (resp. B).
Use the Claim to define $h_q : A_q \longrightarrow B_q^*$ such that for $x \in T$

$$h_q(x)(y) = \begin{cases} \text{the common value} & \text{if } y \in W \\ h_0(x)(y) & \text{if } y \in Y_0 - W \\ h_\nu(x)(y) & \text{if } y \in Y_\nu - W \end{cases}$$

and for $x \in X_0 - T$

$$h_q(x)(y) = \begin{cases} h_0(x)(y) & \text{if } y \in Y_0 \\ \text{arbitrary} & \text{if } y \in Y_\nu - W \end{cases}$$

and similarly for $x \in X_\nu - T$.

Proof of Claim

Claim

Given a basis $\{x_i : i = 1, \ldots, n\}$ *of a finitely-generated pure subgroup* A_0 *of* A*, a basis* $\{y_j : j = 1, \ldots, m\}$ *of a finitely-generated pure subgroup* B_0 *of* B*, and an indexed set* $\{e_{ij} : i = 1, \ldots, n,\ j = 1, \ldots, m\}$ *of elements of* $\mathbb{Z}$*, there is one and only one* $p \in P$ *such that* $A_p = A_0$*,* $B_p = B_0$ *and* $h_p(x_i)(y_j) = e_{ij}$ *for all* $i = 1, \ldots, n,\ j = 1, \ldots, m$*.*

Proof of uniqueness. Suppose there are p_1 and p_2 in P such that for $\ell = 1, 2$, $h_{p_\ell}(x_i)(y_j) = e_{ij}$ for all $i = 1, \ldots, n\ \ j = 1, \ldots, m$. It suffices to prove that for all $a \in A_0$,

$$h_{p_1}(a)(y_j) = h_{p_2}(a)(y_j)$$

for all $j = 1, \ldots, m$. Let $a = \sum_{i=1}^{n} d_i x_i$. Then

$$\gamma(a) - \sum_{i=1}^{n} d_i \gamma(x_i) = h_{p_\ell}(a) - \sum_{i=1}^{n} d_i h_{p_\ell}(x_i).$$

for $\ell = 1, 2$.

Proof of Claim continued

Claim

Given a basis $\{x_i : i = 1, ..., n\}$ of a finitely-generated pure subgroup A_0 of A, a basis $\{y_j : j = 1, ..., m\}$ of a finitely-generated pure subgroup B_0 of B, and an indexed set $\{e_{ij} : i = 1, ..., n, j = 1, ..., m\}$ of elements of $\mathbb{Z}$, there is one and only one $p \in P$ such that $A_p = A_0$, $B_p = B_0$ and $h_p(x_i)(y_j) = e_{ij}$ for all $i = 1, ..., n$, $j = 1, ..., m$.

$$\gamma(a) - \sum_{i=1}^{n} d_i \gamma(x_i) = h_{p_\ell}(a) - \sum_{i=1}^{n} d_i h_{p_\ell}(x_i).$$

Applying both sides to $y_j \in B_0$, we obtain that

$$h_{p_\ell}(a)(y_j) = \sum_{i=1}^{n} d_i e_{ij} + (\gamma(a) - \sum_{i=1}^{n} d_i \gamma(x_i))(y_j)$$

for $\ell = 1, 2$. Since the right-hand side is independent of ℓ, we can conclude the desired identity.