# SHELAH'S SINGULAR COMPACTNESS THEOREM

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ABSTRACT. We present Shelah's famous theorem in a version for modules, together with a self-contained proof and some examples. This exposition is based on lectures given at CRM in October 2006.

## 0. Introduction

The Singular Compactness Theorem is about an abstract notion of "free". The general form of the theorem is as follows:

> *If $\lambda$ is a singular cardinal and $M$ is a $\lambda$-generated module such that* enough $<$ *$\kappa$-generated submodules are "free" for sufficiently many regular $\kappa < \lambda$, then $M$ is "free".*

Of course, for this to have any chance to be a theorem (of ZFC) there need to be assumptions on the notion of "free". These are detailed in the next section along with a precise statement of the theorem (Theorem 1.4), including a precise definition of "enough". Another version of the Singular Compactness Theorem—with a different definition of "enough"—is given at the start of section 3. The proof of Theorem 1.4 is given in sections 3 and 4; some examples and applications are given in sections 2 and 5.

First a few words about the history of the theorem.

**0.1 History.** In 1973 Saharon Shelah proved that the Whitehead problem for abelian groups of cardinality $\aleph_1$ is undecidable in ZFC; in particular he showed under the assumption of the Axiom of Constructibility, V = L, that all Whitehead groups of cardinality $\aleph_1$ are free (see [13]). His argument easily extended, by induction, to prove that for all $n \in \omega$, Whitehead groups of cardinality $\aleph_n$ are free. But for Whitehead groups of cardinality $\aleph_\omega$, the first singular cardinal, a new ingredient was needed. In fact, that ingredient already existed for singular cardinals of cofinality $\omega$ or $\omega_1$, by theorems of Paul Hill (see [9] and [10]); these imply that if an abelian group has singular cardinality $\lambda$ where $\lambda$ is of cofinality $\omega$ (or $\omega_1$) and has the property that every subgroup of smaller cardinality is free, then the group itself is free. In 1974 Shelah proved a general theorem which applied not only to arbitrary singular cardinals but to a general notion of "free" defined axiomatically. The case of the ordinary notion of freeness for abelian groups, combined with the argument in his first paper on Whitehead's problem, led immediately to the conclusion that V = L implies that Whitehead groups of arbitrary cardinality are free. (See 5.4 below.)

Shelah's theorem was applicable to much more than abelian groups, or even modules; in fact, there was another application that Shelah had in mind when he proved his theorem in a general form: that of transversal theory; the parallels between results there and results about "almost free" abelian groups had already been noted. (See [12]; see also [5] for more on the history.)

Wilfrid Hodges [11] later wrote up and generalized another proof (due also to Shelah) of the Singular Compactness Theorem, one which is more user-friendly than the original. A version of this proof, adapted to modules, is given in [7].

Recently the Singular Compactness Theorem (for $\mathcal{Q}$-filtered modules, as in part III of section 2) has proved an essential tool in the study of Baer modules and tilting modules (see the references in section 5). So it seems useful to give a self-contained and streamlined exposition, based on the one in [7].

**0.2 Notation and terminology.** An infinite cardinal $\lambda$ is *singular* if it is the supremum of a set $S$ of fewer than $\lambda$ cardinals each less than $\lambda$; the smallest possible cardinality of such an $S$ is the *cofinality* of $\lambda$. If it is not singular, $\lambda$ is called *regular*. Every successor cardinal is regular. For any sets $X$ and $Y$, $X \setminus Y$ denotes their difference, i.e., $\{x \in X : x \notin Y\}$.

A chain of sets $\{X_\nu : \nu < \rho\}$ is *continuous* if for each limit ordinal $\sigma < \rho$, $X_\sigma = \bigcup_{\nu<\sigma} X_\nu$.

Throughout we consider left $R$-modules, where $R$ is an arbitrary ring. Given a module $M$ and a subset $Y$ of $M$, $\langle Y \rangle$ denotes the submodule of $M$ generated by $Y$. $M$ is $\lambda$-*generated* if it has a generating set of cardinality $\lambda$, and it is $\leq \lambda$-*generated* if it has a generating set of cardinality $\leq \lambda$.

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## 1. Statement of the theorem

Given a class $\mathcal{F}$ of modules containing the zero module, a module $M$ is called **$\mathcal{F}$-free** if and only if $M$ belongs to $\mathcal{F}$. Since $\mathcal{F}$ will be fixed, we will usually simply say $M$ is "free" when we mean $M$ is $\mathcal{F}$-free. Some examples of $\mathcal{F}$ are given in the next section.

The following is a precise version of enough $< \kappa$-generated submodules of $M$ being "free".

**1.1 Definition.** For any regular uncountable cardinal $\kappa$, define $M$ to be **$\kappa$-$\mathcal{F}$-free**, or simply **$\kappa$-"free"** if there is a set $\mathcal{C}$ of $< \kappa$-generated submodules of $M$ such that:
(1) every element of $\mathcal{C}$ is "free";
(2) every subset of $M$ of cardinality $< \kappa$ is contained in an element of $\mathcal{C}$; and
(3) $\mathcal{C}$ is closed under unions of well-ordered chains of length $< \kappa$.

Now we can state the general form of the Singular Compactness Theorem a little more precisely as follows:

> *If $\lambda$ is a singular cardinal and $M$ is a $\lambda$-generated module which is $\kappa$-"free" for sufficiently many regular $\kappa < \lambda$, then $M$ is "free".*

As was said in the introduction, some conditions must be imposed on the notion of "free", that is, on the class $\mathcal{F}$. So our next task is to introduce the assumptions on $\mathcal{F}$; the reader may want to study these in parallel with the examples given in section 2.

The hypotheses (specifically 1.2(iii)) involve a parameter $\mu$, an infinite cardinal which occurs in the statement of Theorem 1.4 below. They also involve two other primitive notions. One is that of a "basis" of an $\mathcal{F}$-free module. More precisely, we are given for each $M \in \mathcal{F}$, a non-empty set, $\mathcal{B}(M)$, of sets of subsets of $M$ (so if $Y \in \mathfrak{X} \in \mathcal{B}(M)$, then $Y$ is a subset of $M$). Each member $\mathfrak{X}$ of $\mathcal{B}(M)$ is called a "basis" of $M$.

We say that a submodule $A$ of $M$ is a **"free" factor** of $M$ if $A = \langle Y \rangle$ for some member $Y$ of a "basis" $\mathfrak{X}$ of $M$. For each "free" factor $A$ of a "free" module $M$, we are given a set $\mathcal{D}(A, M)$ of pairs of bases of $A$ and $M$ respectively; we write $\mathfrak{X}' = \mathfrak{X} \restriction A$ if $(\mathfrak{X}', \mathfrak{X}) \in \mathcal{D}(A, M)$.

**1.2 Hypotheses on $\mathcal{F}$.** For each $M \in \mathcal{F}$ and each "free" factor $A$ of $M$, there are non-empty sets $\mathcal{B}(M) \subseteq \mathcal{P}(\mathcal{P}(M))$ and $\mathcal{D}(A, M) \subseteq \mathcal{B}(A) \times \mathcal{B}(M)$ satisfying for some infinite cardinal $\mu$ the following conditions for all $\mathfrak{X} \in \mathcal{B}(M)$:

(i) $\emptyset \in \mathfrak{X}$; if $Y \in \mathfrak{X}$, then $\langle Y \rangle$ is "free";

(ii) $\mathfrak{X}$ is closed under unions of chains, i.e., if $C \subseteq \mathfrak{X}$ such that for all $Y$, $Y' \in C$, $Y \subseteq Y'$ or $Y' \subseteq Y$, then $\bigcup C \in \mathfrak{X}$;

(iii) if $Y \in \mathfrak{X}$ and $a \in M$, there exists $Y' \in \mathfrak{X}$ such that $Y \subseteq Y'$, $a \in \langle Y' \rangle$ and $|Y'| \leq |Y| + \mu$;

(iv) if $Z, Y \in \mathfrak{X}$ and $Z \subseteq Y$, then $Z$ is a member of a "basis" of $\langle Y \rangle$, so $\langle Z \rangle$ is a "free" factor of $\langle Y \rangle$;

(v) if $A$ is a "free" factor of $M$, then for any "basis" $\mathfrak{X}'$ of $A$, there is a "basis" $\mathfrak{X}$ of $M$ such that $\mathfrak{X}' = \mathfrak{X} \restriction A$, i.e., $(\mathfrak{X}', \mathfrak{X}) \in \mathcal{D}(A, M)$;

(vi) if $\{M_\alpha \colon \alpha < \rho\}$ is a continuous chain of "free" modules and for each $\alpha + 1 < \rho$, $M_\alpha$ is a "free" factor of $M_{\alpha+1}$, then $\bigcup_{\alpha<\rho} M_\alpha$ is "free";

(vii) given a chain $\{M_n : n \in \omega\}$ of "free" modules s.t. for each $n \in \omega$ $M_n$ is a "free" factor of $M_{n+1}$, and given a "basis" $\mathfrak{X}_n$ of each $M_n$ such that $\mathfrak{X}_n = \mathfrak{X}_{n+1} \restriction M_n$ for all $n \in \omega$, then $\bigcup_{n\in\omega} \mathfrak{X}_n$ is contained in some "basis" of $\bigcup_{n\in\omega} M_n$.

**1.3 Proposition.** *If $\mathcal{F}$ satisfies 1.2 for $\mu$ and $M$ is a $\lambda$-generated $\mathcal{F}$-free module where $\lambda$ is an uncountable cardinal, then for any regular cardinal $\kappa$ such that $\mu < \kappa \leq \lambda$, $M$ is $\kappa$-$\mathcal{F}$-free.*

*Proof.* Let $\mathfrak{X}$ be a "basis" of $M$. Let $\mathcal{C} = \{\langle Y\rangle : Y \in \mathfrak{X}$ and $|Y| < \kappa\}$. One can check easily that Definition 1.1 is satisfied for this $\mathcal{C}$. □

This Proposition shows that the hypothesis in the following theorem is necessary for $M$ to be free. The theorem says that the condition is sufficient when $\lambda$ is singular; it will follow immediately from the two theorems (3.1 and 4.1) proved in sections 3 and 4.

**1.4 The Singular Compactness Theorem.** *Suppose that $\mathcal{F}$ satisfies 1.2 for $\mu$. Let $\lambda$ be a singular cardinal $> \mu$ and let $M$ be a $\lambda$-generated module such that $M$ is $\kappa$-$\mathcal{F}$-free for all regular cardinals $\kappa$ such that $\mu < \kappa < \lambda$.*
*Then $M$ is $\mathcal{F}$-free.*

## 2. **Some examples**

We give three different types of examples; there is a non-empty intersection between the different classes of examples.

**I. The usual notion of free.** $\mathcal{F}$ is the class of free modules, that is, modules which have a linearly independent generating set, or, equivalently, are isomorphic to a direct sum of copies of $R$. Here $\mu = \aleph_0$.

If $M \in \mathcal{F}$, we let $\mathcal{B}(M)$ consist of all $\mathfrak{X}$ such that there is a basis $B$ of $M$ (in the usual sense) such that $\mathfrak{X}$ is the set of all subsets of $B$. If $A$ is an $\mathcal{F}$-free factor of $M$, let $(\mathfrak{X}', \mathfrak{X}) \in \mathcal{D}(A, M)$ if and only if $\mathfrak{X}' = \{Z \in \mathfrak{X} \colon Z \subseteq A\}$. It is easy to verify the conditions in 1.2.

**II. Modules defined by direct sum decompositions.** For a given set $\mathcal{L}$ of $\leq \mu$-generated modules, let $\mathcal{F}$ consist of all modules which are isomorphic to a direct sum of the form

$$(\dagger) \qquad \bigoplus_{i\in I} L_i$$

where each $L_i \in \mathcal{L}$, and $I$ is an arbitrary (possibly empty) set. (When $I$ is empty, we obtain the zero module.) In particular, when $\mathcal{L}$ is the set of countably-generated projective modules (and $\mu = \aleph_0$), $\mathcal{F}$ is the class of all projective modules, by a theorem of Kaplansky.

For each $L \in \mathcal{L}$, fix a generating set $S_L$ of cardinality $\leq \mu$ for $L$. For $M \in \mathcal{F}$, let $\mathcal{B}(M)$ consist of all sets $\mathfrak{X}$ such that there is an isomorphism $\varphi$ of $M$ with a direct sum of the form (†) and

$$(*) \qquad \mathfrak{X} = \{Y \colon \exists J \subseteq I \text{ s.t. } Y = \varphi^{-1}[\bigcup_{i \in J} S_{L_i}]\}.$$

(Here we abuse notation by identifying $S_{L_i}$ with its image under the canonical embedding of $L_i$ as the $i$th summand of $M$.) Note that if $Y$ is as in $(*)$, then $\varphi$ induces an isomorphism of $\langle Y \rangle$ with $\bigoplus_{i \in J} L_i$.

If $A$ is an $\mathcal{F}$-free factor of $M$, let $\mathcal{D}(A, M)$ consist of all pairs $(\mathfrak{X}', \mathfrak{X}) \in \mathcal{B}(A) \times \mathcal{B}(M)$ such that

$$\mathfrak{X}' = \{Z \in \mathfrak{X} : Z \subseteq A\}$$

.

Then one can check that 1.2 is satisfied for the parameter $\mu$. Indeed, 1.2 (i), (ii) and (iii) are clear; regarding 1.2(iv), note that if $Y$ is as above, $Z \in \mathfrak{X}$, and $Z \subseteq Y$, then $Z = \varphi^{-1}[\bigcup_{i \in J'} S_{L_i}]$ for some $J' \subseteq J$. So $Z$ is a member of the "basis" of $\langle Y \rangle$ determined by the isomorphism $\varphi$ of $\langle Y \rangle$ with $\bigoplus_{i \in J} L_i$.

Regarding 1.2(v), if $A = \langle Y \rangle$ is a "free" factor of $M$, then there is an isomorphism $\theta$ of $M$ with

$$A \oplus \bigoplus_{i \in (I \setminus J)} L_i$$

which is the identity on $A$; if $\mathfrak{X}'$ is a "basis" of $A$, we can define a "basis" $\mathfrak{X}$ of $M$ to consist of all $W$ such that

$$W = Z \cup \theta^{-1}[\bigcup_{i \in K} S_{L_i}]$$

for some $Z \in \mathfrak{X}'$ and some subset $K$ (possibly empty) of $I \setminus J$. The last two parts of 1.2 are also easy to check.

**III. $\mathcal{Q}$-filtered modules** For a set of modules $\mathcal{Q}$ such that $0 \in \mathcal{Q}$, define $M$ to be **$\mathcal{Q}$-filtered** if $M$ is the union of a continuous chain $\{M_\alpha : \alpha < \sigma\}$ s.t. $M_0 = 0$ and $M_{\alpha+1}/M_\alpha \in \mathcal{Q}$ for all $\alpha + 1 < \sigma$. For a fixed set $\mathcal{Q}$, let $\mathcal{F}$ consist of all $\mathcal{Q}$-filtered modules. A continuous chain $\{M_\alpha : \alpha < \sigma\}$ with $M_0 = 0$ which demonstrates that $M$ is "free", i.e., $\mathcal{Q}$-filtered, will be called a *"free" chain for $M$*.

We claim that if $\mathcal{Q}$ consists of $\leq \mu$-presented modules, then $\mathcal{F}$ satisfies 1.2 for $\mu$. (We follow the proof in [7].)

First we need to define the auxiliary notions $\mathcal{B}(M)$ and $\mathcal{D}(A, M)$. For $M \in \mathcal{F}$, let $\mathcal{B}(M)$ consist of all sets $\mathfrak{X}$ such that there is a "free" chain $\{M_\nu : \nu < \sigma\}$ for $M$ such that $Y \in \mathfrak{X}$ if and only if $Y \subseteq M$ and for all $\nu + 1 < \sigma$, $\langle Y \rangle \cap (M_{\nu+1} \setminus M_\nu) \neq \emptyset$ implies $M_{\nu+1} \subseteq M_\nu + \langle Y \rangle$. If $A$ is an $\mathcal{F}$-free factor of $M$, let $\mathcal{D}(A, M)$ consist of all pairs $(\mathfrak{X}', \mathfrak{X})$ such that there is a "free" chain $\{M_\nu : \nu < \sigma\}$ for $M$ with $A = M_{\nu_0}$ for some $\nu_0$, $\mathfrak{X}$ is the "basis" for $M$ determined by this chain, and $\mathfrak{X}' = \{Z \in \mathfrak{X} : Z \subseteq A\}$.

To prove 1.2(i), assume $Y \in \mathfrak{X}$ and let $A = \langle Y \rangle$. Suppose that $\mathfrak{X}$ is determined by the "free" chain $\{M_\nu : \nu < \sigma\}$; for all $\nu < \sigma$, let $A_\nu = A \cap M_\nu$. Since $Y \in \mathfrak{X}$, for all $\nu < \sigma$, either $A_{\nu+1}/A_\nu = 0$ or $A_{\nu+1}/A_\nu$ is isomorphic to $M_{\nu+1}/M_\nu$; in either case, the quotient belongs to $\mathcal{Q}$, so $\{A_\nu : \nu < \sigma\}$ is a "free" chain witnessing that $A \in \mathcal{F}$. Notice also that the "basis" of $A$ determined by this chain is $\{Z \in \mathfrak{X} : Z \subseteq A\}$, so 1.2(iv) follows.

Condition 1.2(ii) is obvious. For 1.2(iii) we use the assumption that the members of $\mathcal{Q}$ are $\leq \mu$-presented. Suppose that $\mathfrak{X}$ is determined by the "free" chain $\{M_\nu : \nu < \sigma\}$, as in the definition of "basis". Let $M_\sigma = M$. We prove by induction on $\nu \leq \sigma$ that for any $Y \in \mathfrak{X}$ and any subset $S$ of $M_\nu$ of cardinality $\leq \mu$, there is an element $Y'$ of $\mathfrak{X}$ such that $Y \subseteq Y'$, $|Y'| \leq |Y| + \mu$ and $S \subseteq \langle Y' \rangle$, and such that $Y' = Y$ if $S \subseteq \langle Y \rangle$. If $\nu = 0$, there is nothing to prove. If $\nu$ is a limit ordinal, define by induction on $\beta < \nu$ a chain of sets $Y_\beta$ in $\mathfrak{X}$ of cardinality $\leq |Y| + \mu$ such that $Y \subseteq Y_0$ and $\langle Y_\beta \rangle$ contains $S \cap M_{\beta+1}$; since $\mathfrak{X}$ is closed under unions of chains, we can do this by the inductive hypothesis, and $Y' = \bigcup_{\beta<\nu} Y_\beta$ will be the desired set. If $\nu = \beta + 1$, suppose first that $\langle Y \rangle \cap (M_{\beta+1} \setminus M_\beta) \neq \emptyset$; then $M_{\beta+1} \subseteq M_\beta + \langle Y \rangle$ by the definition of a "basis". For each $a \in S$ $(\subseteq M_{\beta+1})$ such that $a \notin \langle Y \rangle$, choose $y_a \in \langle Y \rangle$ such that $a - y_a \in M_\beta$. By induction there exists $Y' \in \mathfrak{X}$ such that $Y \subseteq Y'$, $|Y'| \leq |Y| + \mu$ and $\{a - y_a : a \in S\} \subseteq \langle Y' \rangle$; hence $S \subseteq \langle Y' \rangle$. If $\langle Y \rangle \cap (M_{\beta+1} \setminus M_\beta) = \emptyset$ and $S \nsubseteq \langle Y \rangle$, it suffices to show that there exists $\tilde{Y} \supseteq Y$ in $\mathfrak{X}$ such that $|\tilde{Y}| \leq |Y| + \mu$ and $\left\langle \tilde{Y} \right\rangle \cap (M_{\beta+1} \setminus M_\beta) \neq \emptyset$, for then we are reduced to the first case. Since $M_{\beta+1}/M_\beta$ is isomorphic to a member of $\mathcal{Q}$, there exists a generating set, $G$, of $M_{\beta+1}$ over $M_\beta$ of cardinality $\leq \mu$ such that $\langle G\rangle \cap M_\beta$ is $\leq \mu$-generated. By induction we can choose $Y''$ in $\mathfrak{X}$ containing $Y$ such that $|Y''| \leq |Y| + \mu$ and $\langle G\rangle \cap M_\beta \subseteq \langle Y''\rangle$. If $\langle Y''\rangle \cap (M_{\beta+1} \setminus M_\beta) \neq \emptyset$, let $\tilde{Y} = Y''$. Otherwise, let $\tilde{Y} = Y'' \cup G$; in this case we must show that $\tilde{Y} \in \mathfrak{X}$, in other words, for all $\nu < \sigma$, $\left\langle \tilde{Y} \right\rangle \cap (M_{\nu+1} \setminus M_\nu) \neq \emptyset$ implies $M_{\nu+1} \subseteq M_\nu + \left\langle \tilde{Y} \right\rangle$. For $\nu = \beta$ the conclusion follows by construction. In general suppose that $y + g \in M_{\nu+1} \setminus M_\nu$ where $y \in \langle Y''\rangle$ and $g \in \langle G\rangle$. If $\nu < \beta$, then $y + g \in M_\beta$ so $y$ belongs to $M_{\beta+1}$ (since $g \in M_{\beta+1}$) and hence $y \in \langle Y''\rangle \cap M_{\beta+1} \subseteq M_\beta$; but then $g \in M_\beta \cap \langle G\rangle \subseteq \langle Y''\rangle$; thus $y + g$ shows that $\langle Y''\rangle \cap (M_{\nu+1} \setminus M_\nu) \neq \emptyset$ and we are done since $Y'' \in \mathfrak{X}$. The final case is when $\nu > \beta$; then $y \in M_{\nu+1} \setminus M_\nu$ since $g \in M_{\beta+1} \subseteq M_\nu$ and therefore $M_{\nu+1} \subseteq M_\nu + \langle Y''\rangle$ since $Y'' \in \mathfrak{X}$. This completes the proof of 1.2(iii).

As for 1.2(v), suppose that $A = \langle Y\rangle$ where $Y$ belongs to the "basis" determined by the "free" chain $\{M_\nu : \nu < \sigma\}$. Suppose that $\mathfrak{X}'$ is a "basis" for $A$ determined by a "free" chain $\{A'_\rho : \rho < \tau\}$ for $A$. We will define by induction an extension $\{A'_\rho : \rho < \tau'\}$ of the chain $\{A'_\rho : \rho < \tau\}$ which will be a "free" chain for $M$. The extension will be defined to have the property that for all $\rho \geq \tau$ and $\nu < \sigma$,

$$(\dagger) \qquad A'_\rho \cap (M_{\nu+1} \setminus M_\nu) \neq \emptyset \text{ implies } M_{\nu+1} \subseteq M_\nu + A'_\rho.$$

Let $A'_\tau = A$. If $A'_\rho$ has been defined for all $\rho \leq \beta$ for some $\beta \geq \tau$, choose $\gamma$ minimal such that $M_{\gamma+1} \nsubseteq A'_\beta$. (If there is none, then $A'_\beta = M$ and we stop the construction.) Then, by continuity, $M_\gamma \subseteq A'_\beta$. Set $A'_{\beta+1} = A'_\beta + M_{\gamma+1}$. It follows from the choice of $A'_{\beta+1}$ and the inductive property (†) for $\rho = \beta$ and $\nu = \gamma$ that the natural map $M_{\gamma+1}/M_\gamma \to A'_{\beta+1}/A'_\beta$ is an isomorphism, so $A'_{\beta+1}/A'_\beta \in \mathcal{Q}$. (Note that the map is one-one because otherwise (†) implies $M_{\gamma+1} \subseteq M_\gamma + A'_\beta \subseteq A'_\beta$, a contradiction.) One can then check that (†) holds for $\rho = \beta + 1$ and all $\nu$. Finally, if $\mathfrak{X}$ is the basis determined by the chain $\{A'_\rho : \rho < \tau'\}$, then $(\mathfrak{X}', \mathfrak{X}) \in \mathcal{D}(A, M)$.

The proof of 1.2(v) shows that whenever $A$ is a "free" factor of $M$, any "free" chain for $A$ can be extended to a "free" chain for $M$. This allows us, given $\{M_\alpha : \alpha < \rho\}$ as in 1.2(vi), to inductively define a continuous chain $\{B_\nu : \nu < \tau\}$ whose union is $\bigcup_{\alpha<\rho} M_\alpha$ and such that for every $\alpha < \rho$, some initial segment of the chain $\{B_\nu : \nu < \tau\}$ is a "free" chain for $M_\alpha$.

Finally, for 1.2(vii), we will show that there is a chain $\{B_\nu : \nu < \tau\}$ such that for all $n \in \omega$, some initial segment $\{B_\nu : \nu < \alpha_n\}$ is a "free" chain for $M_n$ which determines $\mathfrak{X}_n$. It is then clear that the "basis" determined by this chain contains $\bigcup_{n\in\omega} \mathfrak{X}_n$. Suppose that we have constructed $\{B_\nu : \nu < \alpha_n\}$ for some $n \in \omega$; by assumption, there is a "free" chain $\{K_\nu : \nu < \sigma\}$ for $M_{n+1}$ which determines $\mathfrak{X}_{n+1}$ and is such that $M_n = K_{\nu_0}$ for some $\nu_0$ and $\mathfrak{X}_n = \{Z \in \mathfrak{X}_{n+1} : Z \subseteq M_n\}$. Let

$$B_{\alpha_n+\ell} = K_{\nu_0+\ell}$$

for all $\ell \geq 0$ such that $\nu_0 + \ell < \sigma$.

## 3. Proof of Theorem 1.4, part 1

In this section we will prove the following version of a singular compactness theorem:

**3.1 Theorem.** *Suppose that $\mathcal{F}$ satisfies 1.2 for $\mu$. Let $\lambda$ be a singular cardinal $> \mu$ and let $M$ be a $\lambda$-generated module such that $M$ is strongly $\kappa^+$-$\mathcal{F}$-free for all cardinals $\kappa$ such that $\mu < \kappa < \lambda$.*
*Then $M$ is $\mathcal{F}$-free.*

This theorem involves the following notion:

**3.2 Definition.** For a cardinal $\kappa$, define $M$ to be **strongly $\kappa^+$-$\mathcal{F}$-free**, or simply **strongly $\kappa^+$-"free"** if there is a family $\mathcal{S}$ of $\leq \kappa$-generated "free" submodules of $M$ containing 0 and such that for any subset $X$ of $M$ of cardinality $\kappa$ and any $N \in \mathcal{S}$, there exists $N' \in \mathcal{S}$ such that $N' \supseteq N \cup X$ and $N$ is a "free" factor of $N'$.

**Remark.** A module which is strongly-$\kappa^+$-$\mathcal{F}$-free is not necessarily $\kappa^+$-$\mathcal{F}$-free. (See [17] for a counterexample.) The terminology originally arose in the context of abelian groups, where the implication does hold.

The rest of this section is devoted to the proof of the theorem. Let $\tau = \mathrm{cf}(\lambda)$; so $\tau < \lambda$ since $\lambda$ is singular. Choose and fix a continuous increasing sequence of cardinals $\langle \kappa_\nu : \nu < \tau \rangle$, each strictly less than $\lambda$, whose supremum is $\lambda$ and such that $\kappa_0 > \max\{\tau, \mu\}$. Choose a generating set $G$ for $M$ of cardinality $\lambda$ and a continuous increasing chain $\{G_\nu : \nu < \tau\}$ of subsets of $G$ whose union is $G$ and such that the cardinality of $G_\nu$ equals $\kappa_\nu$.We will define by induction on $n \in \omega$ simultaneously, for all $\nu < \tau$, the following:

- a subset $C_\nu^n$ of $M$ of cardinality $\kappa_\nu$;
- a "free" submodule $F_\nu^n$ of $M$ which is $\leq \kappa_\nu$-generated;
- $\mathfrak{X}_\nu^n \in \mathcal{B}(F_\nu^n)$;
- $Y_\nu^n \in \mathfrak{X}_{\nu+1}^n$ of cardinality $\kappa_\nu$.

We require the following for all $n \in \omega$ and $\nu < \tau$:

**3.3 Properties**

(0) $G_\nu \subseteq F_\nu^n \subseteq \langle C_\nu^n \rangle \subseteq F_\nu^{n+1}$;

(1) $F_\nu^n$ is a "free" factor of $F_\nu^{n+1}$, and $\mathfrak{X}_\nu^n = \mathfrak{X}_\nu^{n+1} \restriction F_\nu^n$;

(2) $C_\rho^{n-1} \subseteq C_\nu^n$ for all $\rho \leq \nu$.

(3) $Y_\nu^n \subseteq Y_\nu^{n+1} \subseteq C_\nu^{n+1} \subseteq \langle Y_\nu^{n+3} \rangle$;

If we let $C_\nu = \bigcup_{n\in\omega} \langle C_\nu^n \rangle$, (2) implies that the $C_\nu$ form a chain. We require also that

(4) $\{C_\nu : \nu < \tau\}$ is a *continuous* chain.

Assuming, for the moment, that we can do the inductive construction, we will finish the proof.

By 3.3(0), $C_\nu = \bigcup_{n\in\omega} F_\nu^n$ and $\bigcup_{\nu<\tau} C_\nu = M$. By 1.2 (vi) and (vii) and 3.3(1), $C_\nu$ is "free" and $\bigcup_{n\in\omega} \mathfrak{X}_\nu^n$ is contained in a "basis" of $C_\nu$; call this "basis" $\mathfrak{X}_\nu$. By 3.3(3), $C_\nu$ is generated by $\bigcup_{n\in\omega} Y_\nu^n$ and by 1.2(ii), $\bigcup_{n\in\omega} Y_\nu^n \in \mathfrak{X}_{\nu+1}$. Therefore, $C_\nu$ is a "free" factor of $C_{\nu+1}$. But then, by 1.2(vi) and 3.3(4), $M = \bigcup_{\nu<\tau} C_\nu$ is "free".

It remains to do the inductive construction. For each $\nu < \tau$, fix a set $\mathcal{S}_\nu$ of $\leq \kappa_\nu$-generated "free" submodules of $M$ which witness that $M$ is strongly $\kappa_\nu^+$-"free", as in Definition 3.2; we will choose $F_\nu^n$ to be a member of $\mathcal{S}_\nu$. At stage $n$ we choose $F_\nu^n$, $\mathfrak{X}_\nu^n$, $C_\nu^{n-1}$, and $Y_\nu^n$ as well as a set $\{u_{\nu,\alpha}^n : \alpha < \kappa_\nu\}$ of generators of $F_\nu^n$. We begin with $n = 0$:

Pick $F_\nu^0 \in \mathcal{S}_\nu$ so that it contains $G_\nu$ and is $\leq \kappa_\nu$-generated. Pick $\mathfrak{X}_\nu^0 \in \mathcal{B}(F_\nu^0)$. Let $C_\nu^{-1} = \emptyset = Y_\nu^0$.

Now suppose $n \geq 0$ and $F_\nu^k$, $\mathfrak{X}_\nu^k$, $C_\nu^{k-1}$, and $Y_\nu^k$ have been chosen for all $k \leq n$ and all $\nu < \tau$, along with a set of generators $\{u_{\nu,\alpha}^k : \alpha < \kappa_\nu\}$

for $F^k_\nu$. Define

$$C^n_\nu = Y^n_\nu \cup \bigcup_{\rho \le \nu} C^{n-1}_\rho \cup \{u^n_{\rho,\alpha} : \rho < \tau,\, \alpha < \kappa_\nu\}.$$

Note that $\langle C^n_\nu \rangle$ contains $F^n_\nu$ because $C^n_\nu$ contains $\{u^n_{\nu,\alpha} : \alpha < \kappa_\nu\}$. Now choose $F^{n+1}_\nu \in \mathcal{S}_\nu$ containing $F^n_\nu \cup C^n_\nu$ which is $\le \kappa_\nu$-generated and such that $F^n_\nu$ is a "free" factor of $F^{n+1}_\nu$; this is possible by 3.2. By 1.2(v) we can select $\mathfrak{X}^{n+1}_\nu \in \mathcal{B}(F^{n+1}_\nu)$ such that the second part of 3.3(1) holds. We can choose $Y^{n+1}_\nu \in \mathfrak{X}^{n+1}_{\nu+1}$ containing $Y^n_\nu$ and such that $\left\langle Y^{n+1}_\nu \right\rangle$ contains $C^n_\nu \cap F^{n+1}_{\nu+1}$. (Add one element of $C^n_\nu \cap F^{n+1}_{\nu+1}$ at a time using 1.2(iii) and take unions at limit stages, using 1.2(ii).) It is easy to see that 3.3(0), (1), and (2) are satisfied. The first two inclusions in 3.3(3) are clear from construction. For the last one, note that $C^{n+1}_\nu \subseteq C^{n+2}_{\nu+1} \subseteq F^{n+3}_{\nu+1}$ and $C^{n+1}_\nu \subseteq C^{n+2}_\nu$, so $C^{n+1}_\nu \subseteq C^{n+2}_\nu \cap F^{n+3}_{\nu+1} \subseteq \left\langle Y^{n+3}_\nu \right\rangle$.

It remains to check 3.3(4). Suppose that $\gamma$ is a limit ordinal less than $\tau$. We must prove that $C_\gamma \subseteq \bigcup_{\nu<\gamma} C_\nu$. But

$$C_\gamma = \bigcup_{n\in\omega} C^n_\gamma = \bigcup_{n\in\omega} F^n_\gamma$$

by 3.3(0), which, by construction, equals

$$\bigcup_{n\in\omega} \left\langle u^n_{\gamma,\alpha} : \alpha < \kappa_\gamma \right\rangle = \bigcup_{n\in\omega} \bigcup_{\nu<\gamma} \left\langle u^n_{\gamma,\alpha} : \alpha < \kappa_\nu \right\rangle$$

since $\kappa_\gamma = \sup\{\kappa_\nu : \nu < \gamma\}$. But the latter is contained in

$$\bigcup_{n\in\omega} \bigcup_{\nu<\gamma} \langle C^n_\nu \rangle$$

by construction of $C^n_\nu$. (Note that in defining $C^n_\nu$ we include $u^n_{\rho,\alpha}$ for *all* $\rho$, as long as $\alpha < \kappa_\nu$.) Finally

$$\bigcup_{n\in\omega} \bigcup_{\nu<\gamma} \langle C^n_\nu \rangle = \bigcup_{\nu<\gamma} C_\nu$$

by definition of $C_\nu$. This completes the proof of Theorem 3.1.

## 4. **Proof of Theorem 1.4, part 2**

The proof of Theorem 1.4 will be complete once we prove the following result.

**Theorem 4.1** *For any infinite cardinal $\kappa > \mu$, if $M$ is $\kappa^{++}$-$\mathcal{F}$-free, then $M$ is strongly $\kappa^{+}$-$\mathcal{F}$-free.*

We begin the proof of Theorem 4.1. Fix a cardinal $\kappa$ such that $M$ is $\kappa^{++}$-"free". For any $\leq \kappa$-generated "free" submodule $N$ of $M$ define the *$N$-Shelah game*. This is a game between two players, I and II, who take turns making moves. For each $n \in \omega$, player I plays first a subset $X_n$ of $M$ of cardinality $\kappa$; Player II replies with a $\leq \kappa$-generated submodule $N_n$ of $M$ (containing $N$). So after $n+1$ moves by each player, we have a sequence

$$X_0, N_0, X_1, N_1, ..., X_n, N_n$$

The game may go on for $\omega$ moves by each player. Player II wins if at each move he plays a "free" submodule $N_n$ containing $N_{n-1} \cup X_n$ (where $N_{-1} = N$) such that $N_{n-1}$ is a "free" factor of $N_n$. Otherwise player I wins; that is, I wins if and only if after some move $X_n$, player II is unable to respond with a legal move.

A *winning strategy* for player I in the $N$-Shelah game is a function $s_N$ which gives a first move $s_N(N)$ for player I, and then for all $n \in \omega$ gives a move $s_N(N_0, ..., N_{n-1})$ to follow the play

$$s_N(N), N_0, s_N(N_0), N_1, s_N(N_0, N_1)..., s_N(N_0, ...N_{n-2}), N_{n-1}$$

such that player I will eventually win the game played according to the strategy, i.e., player II will be unable to move at some stage.

We claim that player I does not have a winning strategy in the 0-Shelah game. Assuming this is the case for a moment, we can complete the proof. Let $\mathcal{S}$ consist of all $\leq \kappa$-generated "free" submodules $N$ of $M$ such that I does not have a winning strategy for the $N$-Shelah game. We must check that $\mathcal{S}$ satisfies the conditions in Definition 3.2. By the claim, 0 belongs to $\mathcal{S}$. Suppose that $N \in \mathcal{S}$ and $X$ is a subset of $M$ of cardinality $\kappa$. Consider a play of the $N$-Shelah game where player I's first move is $X$. Since I does not have a winning strategy for the $N$-Shelah game, player II must be able to respond with a legal move $N'$ such that I does not have a winning strategy for the $N'$-Shelah game, for otherwise player I would have a winning strategy for the $N$-Shelah game (whose first move is $X$). But then $N'$ belongs to $\mathcal{S}$ and (because $N'$ is a legal move) $N \cup X \subseteq N'$ and $N$ is a "free" factor of $N'$.

So it remains to prove the claim. Given a strategy $s = s_0$ for player I in the 0-Shelah game, we show how player II can defeat the strategy. Let $\mathcal{C}$ be a set of $\leq \kappa^+$-generated submodules as in the definition of $\kappa^{++}$-"free" (cf. Definition 1.1). We will construct by induction on $\nu$ a continuous chain $\{N_\nu : \nu < \kappa^+\}$ consisting of $\leq \kappa$-generated submodules of $M$. At each stage we will also pick an element $F_\nu$ of $\mathcal{C}$ which contains $N_\nu$, and a set $\{u^\nu_\tau : \tau < \kappa^+\}$ of generators of $F_\nu$. We also fix a bijection $\psi$ of $\kappa^+$ with $\kappa^+ \times \kappa^+$ such that for all $\nu$, if $\psi(\nu) = (\alpha, \tau)$ then $\alpha \leq \nu$. Let $N_0 = 0$ and let $F_0$ be any member of $\mathcal{C}$. Suppose that $N_\alpha$, $F_\alpha$, and $\{u^\alpha_\tau : \tau < \kappa^+\}$ have been chosen for each $\alpha < \nu$ for some $\nu$. If $\nu$ is a limit ordinal we simply take unions. If $\nu$ is a successor, choose $N_\nu$ so that it contains $u^\alpha_\tau$ where $\psi(\nu - 1) = (\alpha, \tau)$, and such that it also contains $s(0)$ and $s(N_{\alpha_1}, \ldots, N_{\alpha_k})$ whenever $k \geq 1$ and $\alpha_1 < \cdots < \alpha_k < \nu$ and $s(N_{\alpha_1}, \ldots, N_{\alpha_k})$ is defined. (This is possible since there are at most $\kappa$ such sequences.) Choose $F_\nu$ in $\mathcal{C}$ to contain $N_\nu \cup F_{\nu-1}$. This completes the inductive step in the construction.

Now let $F = \bigcup_{\nu<\kappa^+} N_\nu$. Then $F = \bigcup_{\nu<\kappa^+} F_\nu$ by construction; so $F \in \mathcal{C}$ since $\mathcal{C}$ is closed under unions of well-ordered chains of length $< \kappa^{++}$, so $F$ is "free"; let $\mathfrak{X}$ be a "basis" of $F$.

Let $D$ be the subset of $\kappa^+$ defined by $\alpha \in D$ if and only if $N_\alpha$ is generated by an element $Y_\alpha$ of $\mathfrak{X}$. Then $D$ is an unbounded subset of $\kappa^+$. Indeed, given any $\gamma < \kappa^+$, we can choose an increasing sequence

$$\gamma = \nu_0 < \nu_1 < .... < \nu_n < ...$$

of elements of $\kappa$, and a chain of elements of $\mathfrak{X}$

$$Y_0 \subseteq Y_1 \subseteq ... \subseteq Y_n \subseteq ...$$

such that for all $n$, $Y_n$ ($\subseteq F$) has cardinality $\kappa$ and

$$N_{\nu_n} \subseteq \langle Y_n \rangle \subseteq N_{\nu_{n+1}}.$$

This is possible by properties of a "basis". Then $\alpha = \sup\{\nu_n : n \in \omega\}$ is an element of $D$ where $Y_\alpha = \bigcup_{n\in\omega} Y_n$.

Now player II's strategy to defeat $s$ is to play $N_\alpha$ where $\alpha \in D$; thus a play of the game according to this strategy will look like

$$s(0), N_{\alpha_1}, s(N_{\alpha_1}), N_{\alpha_2}, s(N_{\alpha_1}, N_{\alpha_2}), \ldots$$

where each $\alpha_k \in D$ and $\alpha_1 < \alpha_2 < \ldots$. Player II will win because for each $k$, $N_{\alpha_k}$ is a "free" factor of $N_{\alpha_{k+1}}$ by 1.2(iv) because $Y_{\alpha_k} \subseteq Y_{\alpha_{k+1}}$. Thus the claim is proved, and the proof of 4.1 is finished.

**4.2 Remark.** Examination of the proofs of Theorems 3.1 and 4.1 will show that the following weaker notion of "sufficiently many" suffices for the conclusion of Theorem 1.4: there is a continuous increasing sequence of cardinals $\langle \kappa_\nu : \nu < \tau \rangle$, each strictly less than $\lambda$, whose supremum is $\lambda$, such that $\kappa_0 > \max\{\tau, \mu\}$ and such that $M$ is $\kappa_\nu^{++}$-$\mathcal{F}$-free for all $\nu < \tau$.

## 5. **Applications to deconstruction**

The purpose of this section is to illustrate the role of the Singular Compactness Theorem in some applications. Complete proofs will not be given. Recall that the notion of $\mathcal{Q}$-filtered is defined in III of section 2.

**5.1 Definition.** A class $\mathcal{A}$ of modules is $\kappa$**-deconstructible** if every module in $\mathcal{A}$ is $\mathcal{Q}$-filtered, where $\mathcal{Q}$ is the set of $\leq \kappa$-generated elements of $\mathcal{A}$. $\mathcal{A}$ is **deconstructible** (or **bounded**) if it is $\kappa$-deconstructible for some $\kappa$.

We want to explain the role of the Singular Compactness Theorem in proving the deconstructibility of certain classes $\mathcal{A}$. The Singular Compactness Theorem will be applied for $\mathcal{F}$ the class of $\mathcal{Q}$-filtered modules, where $\mathcal{Q}$ is as above. The classes $\mathcal{A}$ we consider will be of the form

$$^{\perp}\mathcal{B} = \{N \mid \mathrm{Ext}^1_R(N, M) = 0 \text{ for all } M \in \mathcal{B}\}$$

for some class $\mathcal{B}$ (which could be a proper class or a set).

The proof that every member $M$ of $\mathcal{A}$ is $\kappa$-deconstructible (for a fixed $\kappa$) proceeds by induction on the minimal number of generators, $\lambda$, of $M$. When $\lambda$ is regular, a result of the following type is used:

**5.2** *If $M$ is $\lambda$-generated and is the union of a continuous chain $\{M_\alpha : \alpha < \lambda\}$ of $< \lambda$-generated submodules belonging to $\mathcal{A} = {}^\perp\mathcal{B}$, then there is a continuous increasing $f : \lambda \to \lambda$ such that the continuous chain $\{M_{f(\alpha)} : \sigma < \lambda\}$ has the property that $M_{f(\alpha+1)}/M_{f(\alpha)} \in \mathcal{A}$ for all $\alpha < \lambda$.*

Such a result can be obtained under either a set-theoretic hypothesis (the so-called "diamond" principles which are consequences of the Axiom of Constructibility, V = L) or (in ZFC) under a hypothesis on $\mathcal{B}$ (that $\mathcal{B}$ is closed under direct sums). Once one has the conclusion of 5.2, one can apply the inductive hypothesis to $M_{f(\alpha+1)}/M_{f(\alpha)}$ and "fill-in" between $M_{f(\alpha)}$ and $M_{f(\alpha+1)}$ with a chain whose successive quotients are $\leq \kappa$-generated.

When $\lambda$ is singular, the Singular Compactness Theorem is applied. The conclusion sought is exactly that $M$ is $\mathcal{F}$-free (where $\mathcal{F}$ is as described above) but some argument must be made to obtain the hypothesis of Theorem 1.4. We give an example where it is easy to verify the hypothesis of 1.4.

**5.3 Theorem.** *Assume $V = L$. Suppose $N$ is an $R$-module of injective dimension* 1*. Then ${}^\perp N$ is deconstructible.*

*Proof.* Let $\mu = |R| + |N| + \aleph_0$. We will prove that ${}^\perp N$ is $\mu$-deconstructible. It is a fact, which we will not prove here, that 5.2 holds when $\lambda$ is a regular cardinal $> \mu$ (under the hypothesis V = L). The proof that a $\lambda$-generated $M \in {}^\perp N$ is $\mu$-deconstructible proceeds by induction on $\lambda$. For $\lambda \leq \mu$, there is nothing to prove. When $\lambda > \mu$ is regular, we use 5.2 as discussed immediately after 5.2. Suppose that $\lambda$ is singular. We apply the Singular Compactness Theorem, 1.4, with $\mathcal{F}$ the class of $\mathcal{Q}$-filtered modules, where $\mathcal{Q}$ is the set of $\leq \mu$-generated elements of ${}^\perp N$. (Note that since $\mu \geq |R|$, a $\leq \mu$-generated module is $\leq \mu$-presented.) Since $N$ has injective dimension 1, every submodule of $M$ also belongs to ${}^\perp N$. By inductive hypothesis, every $< \lambda$-generated submodule is $\mathcal{F}$-free; so the hypothesis of 1.4 is satisfied and we conclude that $M$ is $\mathcal{F}$-free. $\square$

As a special case, we have the conclusion about Whitehead groups mentioned in 0.1. The Whitehead groups are, by definition, the members of ${}^\perp\mathbb{Z}$. (For more on Whitehead groups, see [7, Chaps XII and XIII].)

**5.4 Corollary** *Assume $V = L$. Then every Whitehead group is $\aleph_0$-deconstructible. Hence, since every countable Whitehead group is (provably in ZFC) free, every Whitehead group is free.*

*Proof.* The first assertion is a special case of Theorem 5.3 and its proof. It is a classical theorem of K. Stein that every countable Whitehead group is free, so the last assertion follows because whenever $\{M_\nu : \nu < \sigma\}$ is a continuous chain such that $M_0 = 0$ and every quotient of successive members is free, we can inductively find a basis for the union of the chain. □

Theorem 5.3 has been extended by J. Saroch and J. Trlifaj ([15]) to the more general assumption that ${}^\perp N$ is closed under pure submodules. Other applications of the Singular Compactness Theorem in a deconstructibility argument can be found in:

(1) [6], for Baer modules over arbitrary domains;
(2) [2], for 1-tilting modules;
(3) [16], for n-tilting modules.

In all of these, the deconstructibility is a theorem of ZFC, and not all submodules of smaller cardinality are "free", so some effort is involved in showing that there are enough "free" submodules.

These deconstructibility results are a key step in obtaining structural information about the modules in question. In particular, the first result implies that every Baer module (over an arbitrary integral domain) is projective provided that the countably-generated Baer modules are projective. The latter has been proved by Angeleri-Hugel, Bazzoni and Herbera (see [1]). The second and third results are used to prove that all tilting modules are of finite type. The case of 1-tilting modules was settled by Bazzoni and Herbera [3] and the general case by Bazzoni and Šťovíček [4].

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