HW I, DUE APRIL 14

1. PROBLEM

Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that $\sup_{|x|\leq 1} |Ax| = \sup_{|x|=1} |Ax|$.

2. PROBLEM 1 (SPIVAK, 1-7)

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *norm preserving* if $|Tx| = |x|$, and *inner product preserving* if $\langle Tx, Ty\rangle = \langle x, y\rangle$ (here $\langle x, y\rangle$ means dot product).

(a) Prove that T is norm preserving if and only if T is inner-product preserving.

(b) Prove that such a linear transformation T is $1-1$ and T^{-1} is of the same sort.

3. PROBLEM 2 (SPIVAK, 1-8)

If $x, y \in \mathbb{R}^n$ are non-zero, the *angle* between x and y is defined as $\arccos\left(\frac{\langle x,y\rangle}{|x||y|}\right)$. The linear transformation T is *angle preserving* if T is 1-1, and for $x, y \neq 0$, we have

$$\arccos\left(\frac{\langle x, y\rangle}{|x||y|}\right) = \arccos\left(\frac{\langle Tx, Ty\rangle}{|Tx||Ty|}\right).$$

(a) Prove that if T is norm preserving, then T is angle preserving.

(b) If there is a basis $x_1, \ldots, x_n$ of $\mathbb{R}^n$ and numbers $\lambda_1, \ldots, \lambda_n$ such that $Tx_j = \lambda_j x_j$, prove that T is angle preserving if and only if all $|\lambda_j|$ are equal (here we are also assuming that T is *symmetric*, i.e., $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathbb{R}^n$).

(c) What are all angle preserving linear maps $T : \mathbb{R}^n \to \mathbb{R}^n$?

4. PROBLEM 3 (SPIVAK, 1-9)

If $0 \leq \theta < \pi$, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ have the matrix

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Show that T is angle preserving transformation and if $x \neq 0$, then

$$\arccos\left(\frac{\langle x, Tx\rangle}{|x||Tx|}\right) = \theta.$$

5. PROBLEM 4

(a) The linear map $P : X \to Y$ is called projection if $P^2 = P$, here P^2 means PP. If X is a finite dimensional vector space, and Y is a vector space in X, then there is a projection P such that $P(X) = Y$.

(b) Let $A : X \to Y$ be a linear map, here X, Y are an arbitrary vector spaces. Let us call the set $\{x \in X : Ax = 0\}$ *null space* of A. Show that the null space of A is a vector space.

6. PROBLEM 5

See problem 5, Chapter 9, Rudin.