1. Problem 1

a) Show that if $f \in C([0,1])$, i.e., f is continuous on $[0,1]$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f dx.$$

where $\int_0^1 f dx$ is the Riemann integral.
b) In particular, show that

$$\lim_{n\to\infty} \sum_{0\leq k\leq 2n} \frac{k}{k^2+n^2} = \int_0^1 \frac{4x}{4x^2+1} dx$$

2. Problem 2

a) Let f be a decreasing function on $[0,1]$. Show that

$$\frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \leq \int_0^1 f dx \leq \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)$$

for all $n \geq 1$.
b) Show that

$$\sum_{1\leq k\leq N} \frac{1}{k} \geq \int_1^{N+1} \frac{dx}{x}$$

for all $N \geq 1$.

3. Problem 3

See problem 2, Chapter 6, Rudin.

4. Problem 4

See problem 5, Chapter 6, Rudin.

5. Problem 5

See problem 8, Chapter 6, Rudin.