PROBLEM SET 2 (DUE JAN. 26/2019)

1. Problem 1

Let $f \in C([a,b])$, and $\varphi \in C(\mathbb{R})$. Assume that φ is periodic function with period $T > 0$, i.e., $\varphi(x+T) = \varphi(x)$ for all $x \in \mathbb{R}$. Find the limit

$$\lim_{A\to\infty} \int_a^b f(t)\varphi(At)dt.$$

2. Problem 2

Suppose that both $f, g : [0,1] \to \mathbb{R}$ are increasing functions. Show that

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

3. Problem 3

Assume φ is strictly increasing and continuous on $[0,\infty)$, $\varphi(0) = 0$, and $\psi(x) = \varphi^{-1}(x)$ is the inverse function. Then for all $a \in [0,\infty)$, and any $b \in [0, \sup\varphi)$ one has

$$ab \leq \int_0^a \varphi(t)dt + \int_0^b \psi(t)dt.$$

4. Problem 4

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be $n+1$ times continuously differentiable function, i.e., $f, f', f'', ..., f^{(n+1)}$ exist and are continuous. Show that for any $a \in \mathbb{R}$ the following Taylor's formula with integral reminder holds true

$$f(x) = \sum_{0\leq k\leq n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^{n+1}}{n!}\int_0^1 (1-t)^n f^{(n+1)}(a+t(x-a))dt$$

for all $x \in \mathbb{R}$.

Another way of writing it is (after change of variables)

$$f(x) = \sum_{0\leq k\leq n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^{n+1}}{(n+1)!} \cdot \int_0^1 f^{(n+1)}(x+s(a-x))\, s^n(n+1)ds.$$

I like this second way because it says that you just continue adding the usual term in Taylor's formula $\frac{(x-a)^{n+1}}{(n+1)!}$ but with a certain factor $\int_0^1 f^{(n+1)}(x+s(a-x))\, s^n(n+1)ds$ which integrates $f^{(n+1)}(y)$ along the line $[x,a]$ with a "probability density" $s^n(n+1)ds$. The latter I call it probability density because its total mass is one, i.e., $\int_0^1 s^n(n+1)ds = 1$.

5. Problem 5

Exercise 10, Chapter 6, Rudin.